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Tsuji functions with segments of Julia*

Dedicated to Helmut Grunsky on his 60th Birthday, 11th July 1964

By

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1. Introduction

Let D denote the unit disk $|z|<1$, C the unit circle $|z|=1$, and C_r the circle $|z|=r$. Corresponding to any function w meromorphic in D we denote by w^* the spherical derivative:

$$w^*(z)=\frac{|w'(z)|}{1+|w(z)|^2}.$$

We say that w is a *Tsuji function* provided the spherical length of the curve $w(C_r)$ is a bounded function in $0<r<1$, in other words, provided

$$\sup_{r<1}\int_0^{2\pi} w^*(r e^{i\vartheta})\, r\, d\vartheta<\infty.$$

A rectilinear segment S lying in D except for one endpoint $e^{i\vartheta}$ on C is called a *segment of Julia* for w, provided in each open triangle in D having one vertex at $e^{i\vartheta}$ and meeting S, the function w assumes all values on the Riemann sphere except possibly two. A point $e^{i\vartheta}$ is a *Julia point* for w provided each rectilinear segment lying in D except for one endpoint at $e^{i\vartheta}$ is a segment of Julia for w.

Corresponding to each ϑ and each $\alpha(|\alpha|<\pi/2)$, let $S(\vartheta,\alpha)$ be the segment that joins the points $e^{i\vartheta}$ and $(1-e^{i\alpha}\cos\alpha)\, e^{i\vartheta}$; in other words, let $S(\vartheta,\alpha)$ denote the chord of the circle with diameter $[0, e^{i\vartheta}]$ that forms a directed angle α with $[0, e^{i\vartheta}]$ at $e^{i\vartheta}$. In case $w(z)$ approaches a limit as $z\to e^{i\vartheta}$ on $S(\vartheta,\alpha)$, we denote this limit by $w(\vartheta,\alpha)$.

The present note answers a question that W. Seidel raised concerning a theorem of M. Tsuji [3]. In terms of the notation introduced in the preceding paragraph, we can state Tsuji's theorem as follows.

Let w be a Tsuji function, and let $\Lambda(\vartheta,\alpha)$ denote the spherical length of the image under w of $S(\vartheta,\alpha)$. Then, for each α in $|\alpha|<\pi/2$, $\Lambda(\vartheta,\alpha)$ is an integrable function of ϑ; and for almost all ϑ, $\Lambda(\vartheta,\alpha)$ is an integrable function of α. Moreover, for all ϑ in a set of measure 2π on $[0,2\pi)$, the relation $w(\vartheta,\alpha)=w(\vartheta,\beta)$ holds whenever both limits exist, while $S(\vartheta,\gamma)$ is a segment of Julia if $\Lambda(\vartheta,\gamma)=\infty$. In particular, the theorem implies that if w is a Tsuji function, then $\Lambda(\vartheta,\alpha)<\infty$ except for a set of points (ϑ,α) of two-dimensional measure 0.

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In a review of [3], SEIDEL asked whether the segments of Julia mentioned in the theorem can actually occur. We shall display several relevant examples.

2. Meromorphic Tsuji functions

Lemma. *If $\{z_n\}$ is a sequence of points in the unit disk D such that $|z_{n+1}|>|z_n|$ for all n and $z_n \to 1$ as $n \to \infty$, then the function*

(1) $$w(z)=\sum a_n/(z-z_n)$$

is a Tsuji function provided the a_n are small enough.

Proof. We chose a sequence $\{\rho_n\}$ of positive numbers such that $\rho_{n+1}+\rho_n<|z_{n+1}|-|z_n|$ for all n, and we denote by D_n the disk $|z-z_n|<\rho_n$. Clearly, $\sum \rho_n<1$.

Now let $\{a_n\}$ denote any sequence subject to the restriction that $0<a_n<\rho_n^3$ for all n. Then

(2) $$\sum{}^* \frac{a_n}{|z-z_n|}<\sum \frac{a_n}{\rho_n}<\sum \rho_n^2<1$$

and

(3) $$\sum{}^* \frac{a_n}{|z-z_n|^2}<\sum_n \frac{a_n}{\rho^2}<\sum \rho_n<1,$$

where the asterisk indicates that if z lies in D_m, the m-th term is to be omitted.

If a circle C_r meets none of the disks D_n, it follows from (3) that $|w'(z)|<1$ on C_r, and hence that $w(C_r)$ has Euclidean length less than 2π.

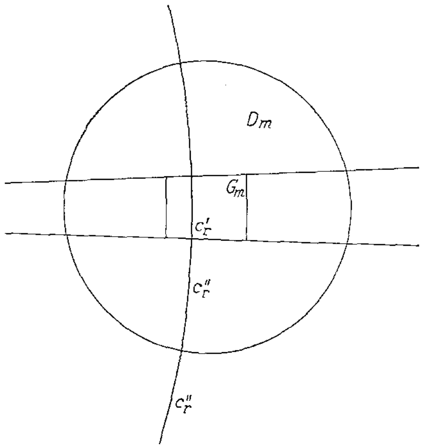

Fig. 1

If C_r meets the set $\cup D_n$, it meets precisely one of the disks, say D_m. For the sake of typographical convenience we suppose that z_m is real and that $\frac{1}{2}<z_m<1$ (so that $a_m<\frac{1}{8}$). We denote by G_m the rectangle

$$\{z \mid z=r e^{i\vartheta},\ |r-z_m| \leqq a_m/3,\ |\vartheta| \leqq a_m/3\};$$

we write $C_r'=C_r \cap G_m$ and $C_r''=C_r \setminus C_r'$ (Fig. 1 shows the relation – not to scale – between D_m, G_m, and C_r; note that C_r' is empty if $z_m-\rho_m<r<z_m-a_m/3$ or $z_m+a_m/3<r<z_m+\rho_m$); and we estimate separately the spherical length of $w(C_r')$ and the Euclidean length of $w(C_r'')$.

On C_r', we use the relations

$$|z-z_m|^2=r^2+z_m^2-2 r z_m \cos \vartheta=(r-z_m)^2+4 r z_m \sin^2 \vartheta/2<2 a_m^2/9.$$

They imply that $a_m/|z-z_m|>2$; together with (2), this gives the inequalities

$$|w(z)|>\frac{a_m}{|z-z_m|}-\sum{}^* \frac{a_n}{|z-z_n|}>\frac{a_m}{|z-z_m|}-\sum \frac{a_n}{\rho_n}>\frac{a_m}{2|z-z_m|}.$$

Also, for points z on $C_r' \setminus \{z_m\}$ it follows from (3) that

$$|w'(z)| < \frac{a_m}{|z-z_m|^2} + \sum{}^* \frac{a_n}{|z-z_n|^2} < \frac{a_m}{|z-z_m|^2} + 1 < \frac{2a_m}{|z-z_m|^2},$$

and we deduce that

$$w^*(z) < |w'(z)| \cdot |w(z)|^{-2} < 8/a_m$$

on C_r'. Since C_r' has length less than $2a_m/3$, the curve $w(C_r')$ has spherical length less than $\frac{16}{3}$.

The arc C_r'' may contain a subarc of points $z = re^{i\vartheta}$ with $-a_m/3 < \vartheta < a_m/3$. If that is the case, the inequalities

$$|w'(z)| < 1 + a_m/|z-z_m|^2 < 1 + 1/9\,a_m$$

hold on the subarc, and therefore w maps the subarc onto an arc of Euclidean length less than

$$\frac{2}{3}a_m\left(1 + \frac{1}{9a_m}\right) = \frac{2}{3}a_m + \frac{2}{27} < \frac{1}{6}.$$

For the remainder of C_r'', we use the relations

$$|re^{i\vartheta} - z_m|^2 = (r-z_m)^2 + 4rz_m \sin^2\vartheta/2 \geqq 4rz_m\vartheta^2/\pi^2 > 2r\vartheta^2/\pi^2.$$

They yield the upper bound

$$2\int_{a_m/3}^{\pi} |w'(re^{i\vartheta})|\, r\, d\vartheta < 2\int_{a_m/3}^{\pi}\left(1 + \frac{a_m}{|re^{i\vartheta} - z_m|^2}\right) r\, d\vartheta$$

$$< 2\pi + a_m\pi^2 \int_{a_m/3}^{\pi} \vartheta^{-2}\, d\vartheta < 2\pi + 3\pi^2.$$

In summary: the spherical length of $w(C_r)$ is less than $2\pi + 3\pi^2 + \frac{17}{3}$, and the lemma is established.

Theorem 1. *There exists a Tsuji function for which each point $e^{i\vartheta}$ is a Julia point.*

Proof. Let $z_n = (1 - n^{-\frac{1}{2}})\, e^{i \log n}$ $(n = 2, 3, \ldots)$, and choose the constants a_n as in the proof of the lemma. Then the function (1) is a Tsuji function.

Since the right member of (1) converges uniformly in the complement H (relative to the plane) of the set $\bigcup D_n$, it defines a function w that is continuous on H. Now let S denote a line segment in D, with an endpoint $e^{i\vartheta}$, and let A denote a Stolz angle containing S. Then there exist infinitely many integers n_k such that the disk D_{n_k} lies in A. For large k, the set of values omitted by w in D_{n_k} lies in a small neighborhood of the point $w(e^{i\vartheta})$, and therefore S is a segment of Julia. This completes the proof of Theorem 1.

The following theorem shows that segments of Julia may occur even if all segmental limits $w(\vartheta, \alpha)$ exist.

Theorem 2. *There exists a Tsuji function w with the following two properties:*

(i) *If S is a chord of the unit disk, then the spherical length of the arc $w(S)$ is less than some constant independent of S.*

(ii) *The radius of the point 1 is a segment of Julia for w.*

Proof. First we choose the points z_n so that they lie on the parabola $y=(x-1)^2$; then we select the constants ρ_n small enough so that no line meets more than two of the disks D_n. The remainder of the proof follows the pattern that we have already established.

Theorem 3. *If E is a set of measure* 0 *on C, then there exists a Tsuji function of bounded characteristic for which every point of E is a Julia point.*

Proof. Since E has measure 0, we can choose a sequence of arcs A_m on C, of lengths σ_m and with midpoints t_m, such that each point of E lies in infinitely many of the arcs A_m and such that $\sum \sigma_m < \infty$. For each m, we denote by J_m the intersection of D with the circle $|z-t_m|=\sigma_m$.

There exists a sequence $\{k_m\}$ of positive integers such that $k_m \to \infty$ and $\sum k_m \sigma_m < \infty$. If on each arc J_m we choose k_m equally spaced points ζ_{mn} (in such a way that the angular distance between ζ_{mn} and $\zeta_{m,n+1}$ is approximately π/k_m), then, at each point of E, every Stolz angle contains infinitely many of the points ζ_{mn}. We can easily choose the ζ_{mn} in such a way that $|\zeta_{m_1 n_1}| \neq |\zeta_{m_2 n_2}|$ except when $m_1 = m_2$ and $n_1 = n_2$, and therefore we can choose disks D_{mn} with centers ζ_{mn} in such a way that no circle C_r meets more than one of the disks. We now form two Blaschke products $B_1(z)$ and $B_2(z)$, the first with zeros $b_{mn} = (1+\varepsilon_{mn})\,\zeta_{mn}$, the second with zeros $c_{mn}=(1-\varepsilon_{mn})\,\zeta_{mn}$. If the ε_{mn} are sufficiently small, then each pair of zeros lies close to the center of the corresponding disk. The convergence of the two products follows from the inequality

$$\sum_{m,n} (1-|\zeta_{mn}|) < \sum_{m=1}^{\infty} k_m \sigma_m .$$

Now let $w(z)=B_1(z)/B_2(z)$. If the ε_{mn} are sufficiently small, the product

$$\prod_{m,n} \frac{b_{mn}-z}{1-\bar{b}_{mn} z} \cdot \frac{1-\bar{c}_{mn} z}{c_{mn}-z} \tag{4}$$

converges uniformly in $\bar{D} \setminus \bigcup D_{mn}$, and therefore the symbol $w(e^{i\vartheta})$ has a meaning. Moreover, if $\varepsilon_{mn} \to 0$ fast enough as $m \to \infty$, then for any sequence of disks D_{mn} tending to a point $e^{i\vartheta}$, the set of values omitted by w in D_{mn} lies in a small neighborhood of $w(e^{i\vartheta})$ when m is large. Therefore every segment in D terminating at a point of E is a segment of Julia for w.

To see that w is a Tsuji function, we note that if the ε_{mn} are small enough, then $w'(z)$ is bounded in the set $D \setminus \bigcup D_{mn}$, and that in D_{mn} the function w is the product of the factor with index (m, n) in (4) and a function whose values lie in an annulus $R_1 < |z| < R_2$ (R_1 and R_2 positive, independent of m and n) and whose derivative is bounded.

We observe that since the function is of bounded characteristic, the set of its Fatou points (at each of which it has a uniform limit in every Stolz angle) is of measure 2π on C, so that the set of its Plessner points on C (at each of which the cluster set of the function in every Stolz angle is total) is of zero measure, by Plessner's theorem [*2*, p. 70]. Since the Julia points form a subset of the Plessner points, the property of the set E in Theorem 3 is best possible.

3. The Tsuji set of a meromorphic function

Let w denote any meromorphic function in the unit disk D; corresponding to each point α in D, we write

$$w_\alpha(z) = w\left(\frac{z-\alpha}{1-\bar{\alpha} z}\right),$$

and we define the *Tsuji set* of w to be the set of values α for which w_α is a Tsuji function. Since the quantity

$$\sup_{r<1} \int_0^{2\pi} w_\alpha^*(r e^{i\vartheta})\, r\, d\vartheta$$

is a lower-semicontinuous function of α, the Tsuji set of a meromorphic function is a point set of type F_σ.

Theorem 4. *The Tsuji set of a function may be the point set* $D\setminus\{0\}$.

Proof. For $k=2, 3, \ldots$, we choose k equally spaced points z_{nk} on the circle $|z| = 1-b_k$, where $\{b_k\}$ is a strictly decreasing sequence with $b_1<1$, $b_k \to 0$. We construct the function (1) as in the proof of the lemma, except that now some circles C_r meet several of the disks D_{nk}. If the b_k and the a_{nk} are small enough, then the spherical length of the curve $w(C_r)$, where $r=1-b_k$, has the order of magnitude πk, and therefore w is not a Tsuji function. On the other hand, if $b_{k+1}/b_k \to 0$ fast enough, then for each α in $D\setminus\{0\}$ the number of disks that meet the circle $|(z-\alpha)/(1-\bar{\alpha}z)| = r$ is a bounded function of r (in fact, it is 1 or 0, for $r>r_\alpha$), and therefore w_α is a Tsuji function for all α except $\alpha=0$.

4. Holomorphic Tsuji functions

Theorem 5. *There exist holomorphic Tsuji functions with segments of Julia.*

If the function w constructed in the proof of Theorem 2 were to omit the value $w(1)$, in D, then the function $1/[w(z)-w(1)]$ would provide a proof of Theorem 5. However, it is not obvious that the a_n can always be chosen so that w omits the value $w(1)$. In particular, if the z_n and the a_n are real, then $w(1)$ is real, and w assumes this value in each interval (z_n, z_{n+1}).

We shall prove that the function

$$w(z) = \exp\left(\frac{1+z}{1-z}\right)^2$$

is a Tsuji function with two segments of Julia. First we observe that if r is near to 1, the mapping

$$f(z)=\frac{1+z}{1-z}$$

carries the circle C_r onto a large circle a long arc of which lies near the imaginary axis; that the function $\exp f(z)$ carries this arc onto an arc making many turns around the unit circle C, not far from C; and that $\exp f(z)$ is therefore not a Tsuji function. On the other hand, the mapping $g(z)=[f(z)]^2$ carries C_r onto a reniform curve Γ_r that meets the imaginary axis in four points, each time at an angle of approximately $\pi/4$ (Fig. 2 shows approximately the upper half of the circle $f(C_r)$ and the curve Γ_r, for $r=\frac{1}{2}$). The function $w(z)=\exp g(z)$ therefore carries C_r onto a curve making few turns around the origin, except quite near the origin and quite far from the origin; that is, the majority of the turns of the image of C_r make only small contributions to the spherical length of the image. We shall now show that w is indeed a Tsuji function.

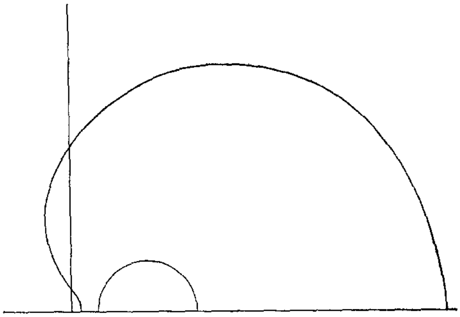

Fig. 2

Since

$$|w'(z)|=\left|\frac{4(1+z)\,w}{(1-z)^3}\right| \quad\text{and}\quad w^*(z)<\left|\frac{4(1+z)}{(1-z)^3\,w}\right|,$$

it will be convenient to integrate the first or the second expression, on subarcs of C_r, according as the real part of $g(z)$ is negative or positive on these subarcs. Now, at $z=re^{i\vartheta}$,

$$\left|\frac{1+z}{(1-z)^3}\right|=\frac{[1+r^2+2r\cos\vartheta]^{\frac{1}{2}}}{[1+r^2-2r\cos\vartheta]^{\frac{3}{2}}}=\frac{[(1+r)^2-4r\sin^2\vartheta/2]^{\frac{1}{2}}}{[(1-r)^2+4r\sin^2\vartheta/2]^{\frac{3}{2}}}$$

and

$$|w(z)|=\exp\Re\left(\frac{1+z}{1-z}\right)^2=\exp\Re\left(\frac{1-r^2+2ir\sin\vartheta}{(1-r)^2+4r\sin^2\vartheta/2}\right)^2$$

$$=\exp\frac{(1-r^2)^2-4r^2\sin^2\vartheta}{[(1-r)^2+4r\sin^2\vartheta/2]^2}.$$

In view of the symmetry of $w(C_r)$, it will be sufficient to show that the integral

$$\int_0^\pi \frac{4[(1+r)^2-4r\sin^2\vartheta/2]^{\frac{1}{2}}}{[(1-r)^2+4r\sin^2\vartheta/2]^{\frac{3}{2}}}\exp\left\{-\frac{|(1-r^2)^2-4r^2\sin^2\vartheta|}{[(1-r)^2+4r\sin^2\vartheta/2]^2}\right\}d\vartheta$$

is a bounded function of r, for $\frac{1}{2}<r<1$. Over the interval $\pi/4\leqq\vartheta\leqq\pi$, the integrand has a bound independent of r, and therefore we may restrict our attention to the range $0\leqq\vartheta\leqq\pi/4$.

We deal first with the range $0\leqq\vartheta\leqq\sin^{-1}(1-r^2)/r$. Since $\cos\vartheta$ is bounded away from 0, on this range, the substitution

$$\sin\vartheta=\frac{1-r^2}{2r}\lambda, \qquad \cos\vartheta\, d\vartheta=\frac{1-r^2}{2r}\,d\lambda$$

allows us to replace the integral in question with

$$K_1\int_0^2 (1-r)^{-2}\exp\left\{-\frac{(1-r^2)^2\,|1-\lambda^2|}{K_2(1-r)^4}\right\}d\lambda$$

(here K_1 and K_2 denote positive constants independent of r), and if we write $1-r=\mu$, we obtain the upper bounds

$$K_1\int_0^2 \mu^{-2}\exp\{-K_3\,|1-\lambda|\,\mu^{-2}\}\,d\lambda$$
$$=2K_1\int_0^1 \mu^{-2}\exp\{-K_3\,\lambda\,\mu^{-2}\}\,d\lambda<2K_1\int_0^\infty \exp(-K_3\,s)\,ds=2K_1/K_3.$$

For the integral from $\vartheta=\sin^{-1}(1-r^2)/r$ to $\vartheta=\pi/4$ we have the majorant

$$K_4\int_{\sin^{-1}(1-r^2)/r}^{\pi/4}(\sin^2\vartheta/2)^{-\frac{3}{2}}\exp\left\{-\frac{3r^2\sin^2\vartheta}{K_5\sin^4\vartheta/2}\right\}d\theta,$$

and the substitution $\sin\vartheta=t$ shows that this is less than

$$K_6\int_0^\infty t^{-3}\exp(-K_7\,t^{-2})\,dt=K_6/2K_7.$$

This concludes the proof that w is a Tsuji function.

To see that the two segments $S(0,\pm\pi/4)$ (which make angles $\pm\pi/4$ with the real axis at $z=1$) are segments of Julia, we consider (for example) two segments $S(0,\pi/4\pm\varepsilon)$. The function f carries the Stolz angle between these segments into a certain infinite triangle in the right half-plane. The triangle is bounded by portions of two lines through the point $z=-1$ and by the segment of the imaginary axis that lies between them.

The function g carries the same Stolz angle into a domain containing a wedge that in turn contains the imaginary axis, and it follows immediately that the segment $S(0,\pi/4)$ is a segment of Julia for w. This concludes the proof of Theorem 5.

Conjecture 1. If w is a holomorphic Tsuji function, then at most finitely many points $e^{i\vartheta}$ are endpoints of segments of Julia in D, for w.

Conjecture 2. If w is a holomorphic Tsuji function, then at most finitely many segments in D are segments of Julia for w.

Let A denote one of the two circular arcs in D that meet the circle C at an angle $\pi/4$ at the two points $z=\pm 1$. Then the function w that we used in the proof of Theorem 5 has the property that $|w(z)|=1$ on A, and it follows further that $|w^*(z)|=2|1+z|/|1-z|^3$ on A. By a theorem of O. LEHTO and K. I. VIRTANEN [1, Section 12], we conclude that w is not a normal function in D.

Conjecture 3. If w is a holomorphic, normal Tsuji function, then w has no segments of Julia.

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