Numer. Math. 57, 435–451 (1990)

Numerische Mathematik

Domain Imbedding Methods for the Stokes Equations

Christoph Börgers

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

Received March 9, 1989/January 23, 1990

Summary. We study direct and iterative domain imbedding methods for the Stokes equations on certain non-rectangular domains in two space dimensions. We analyze a continuous analog of numerical domain imbedding for bounded, smooth domains, and give an example of a simple numerical algorithm suggested by the continuous analysis. This algorithm is applicable for simply connected domains which can be covered by rectangular grids, with uniformly spaced grid lines in at least one coordinate direction. We also discuss a related FFT-based fast solver for Stokes problems with physical boundary conditions on rectangles, and present some numerical results.

Subject Classifications: AMS(MOS) 65N20, 65F10; CR: G1.3, G1.8.

1 Introduction

Domain imbedding methods are elliptic solvers constructed in the following way. Let Ω be an open, bounded domain in two or three dimensions. Let R be a rectangular domain containing the closure $\bar{\Omega}$. To solve an elliptic problem on Ω, use in some way an auxiliary elliptic solver on R.

Methods of this kind have been studied extensively for second order scalar elliptic boundary value problems. We shall only give a very brief discussion of some of this work here. More extensive references can be found in [5] and [6].

The earliest papers on numerical domain imbedding methods which I know of are [8] and [13]. Domain imbedding is particularly simple for finite element discretizations of Neumann problems. For this case, there is a straightforward and quite efficient method, with a speed of convergence independent of the mesh width; see e.g. [17] and [6]. For Dirichlet and mixed boundary conditions, more sophisticated methods are needed to obtain mesh width independent convergence speed. Two such methods for Dirichlet problems were proposed and analyzed in [10] and [16]. These and a third method, closely related to one of the algorithms of

[4], were also studied in [5]. The method discussed in the present paper is an analog of this third method for scalar second-order Dirichlet problems.

One of the original motivations for studying domain imbedding was that fast elliptic solves were available for rectangles, but not for most other geometries. This reasoning has become invalid, at least with respect to iterative solvers, since multigrid methods have been shown to be quite efficient for complicated geometries; see e.g. [19], p. 147. For scalar second order elliptic operators, numerical evidence indicates that multigrid is more efficient than domain imbedding; compare e.g. [6] and [9]. In addition, domain imbedding methods have the disadvantage of requiring regular, uniform or nearly uniform grids.

Nevertheless I believe that domain imbedding is quite useful in some contexts. First, consider the domain Ω of Fig. 1, and assume that we wish to solve a boundary value problem for Laplace's equation on Ω. A reasonable numerical method could be constructed as follows. Cover Ω by the uniform square grid G^H shown in Fig. 2. Refine G^H by globally halving the mesh width H a few times, then by local refinement near re-entrant corners. Discretize on the finest grid, for instance using finite elements. Solve the discrete problem using a multigrid-like algorithm based on the hierarchy of grids constructed by refining G^H. One now needs an approximate solver for discrete boundary value problems on G^H. One could use multigrid for this purpose, too. However, this would require the use of irregular coarse grids, and therefore some analysis or experimentation to find an effective coarsening strategy. A domain imbedding method would be less efficient, but much more convenient.

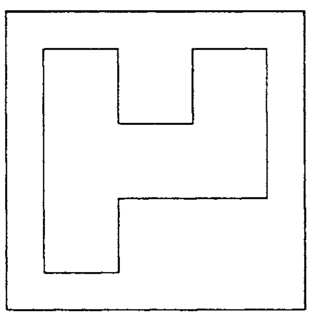

Fig. 1. The test domain of example 3

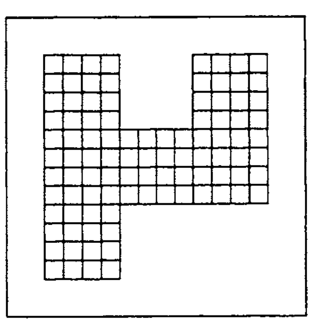

Fig. 2. The finite difference mesh

Since efficiency on the coarsest grid does not matter much if the number of grid levels is relatively large, domain imbedding is a reasonable choice here.

In the context of this application, the restriction to domains which can be covered by regular, uniform meshes is less severe than it would be if domain imbedding were to be used alone. More complicated domains can be approximated by domains which can be covered by regular, uniform meshes. This reduces the order of accuracy with which the continuous problem is approximated. However, in multigrid methods, there is in general no need to use the order of accuracy of the finest grid discretization on all grids; for a simple example illustrating this point, compare the second remark on p. 113 of [19].

Second, *direct* domain imbedding methods are, in a certain sense, quite efficient. They require an expensive preprocessing step. However, this step only requires knowledge of the domain and the discrete operator, not of the right-hand side and boundary values. Not including the data-independent preprocessing in the operation count, direct domain imbedding methods allow the exact solution of certain discrete elliptic problems on complicated domains in $O(n\log(n))$ operations, where n denotes the number of grid points. There are few direct elliptic solvers with this capability. Ignoring the cost of the preprocessing step may be justified in some situations, such as time dependent problems on a fixed domain, when many elliptic problems are to be solved on the same grid.

An additional advantage of domain imbedding techniques lies in the fact that by far the largest part of the computational work is spent on Fast Fourier Transforms, for which very efficient software exists.

It should also be mentioned that our somewhat pessimistic judgment of the efficiency of domain imbedding in comparison with multigrid methods is based on the assumption that standard fast solvers are used on the rectangle R. However, the problems on R are often of a very special nature: Almost all entries of the right-hand side are zero, and the solution is needed in only few grid points. This observation can be used to accelerate the fast solver; see [2, 3, 11, 14], and [15]. Techniques of this kind may broaden the class of problems for which domain imbedding is useful.

In this paper, we shall describe a mathematical background for domain imbedding methods for the Stokes equations, and show how the basic principle leads to simple numerical methods. We shall study the following idea for constructing an imbedding Stokes solver. Given a Stokes problem on a non-rectangular domain Ω, solve this problem using auxiliary Stokes problems on a larger rectangle, with an exterior force which equals the exterior force of the original problem plus a singular force distribution with support on the boundary of Ω, forcing the velocity to assume zero values, or other prescribed values, on $\partial\Omega$. In Sect. 3, we give an analysis of this idea in the continuous case, for general bounded domains with smooth boundaries. In Sects. 4–8, we study a numerical algorithm suggested by the continuous analysis, applicable to simply connected domains which can be covered by rectangular grids with uniformly spaced grid lines in at least one coordinate direction. There are two variants of this method: A fast direct Stokes solver, and a simple but not very fast iterative Stokes solver, with a speed of convergence which appears to be bounded uniformly in the mesh width h, judging from numerical results.

In Sect. 9, we discuss an FFT-based fast solver for Stokes problems with physical boundary conditions on rectangles. This algorithm is closely related to the imbedding method studied in this paper, and is quite efficient if one does not count a preprocessing phase which depends on the grid, but not on the right-hand side and boundary values.

Fortran codes implementing the methods describing in this paper are available from the author.

2 Formulation of the Stokes Problem

We consider the Stokes problem

$$-\Delta \underline{u} + \underline{\nabla} \underline{p} = \underline{0} \quad \text{in} \quad \Omega \tag{1}$$

$$\underline{\nabla} \cdot \underline{u} = 0 \quad \text{in} \quad \Omega \tag{2}$$

$$\underline{u} = \underline{g} \quad \text{on} \quad \partial\Omega . \tag{3}$$

These are the equations of two-dimensional incompressible fluid flow dominated by viscosity. $\underline{u}$ is the velocity, and p the pressure. $\underline{u}$ and p are functions of $\underline{x} \in \Omega$. We shall use the notation

$$\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix},$$

and

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Equations (1) are the momentum equations, and Eq. (2) the continuity equation. We assume that Ω is open and $\bar{\Omega} \subseteq (0,1)^2$. For the motivating discussion of Sect. 3, we shall also assume $\partial\Omega \in C^2$.

For $\underline{g} \in H^{\frac{1}{2}}(\partial\Omega)$, Eqs. (1)–(3) have a solution $(\underline{u}, p)$ if and only if

$$\int_{\partial\Omega} \underline{g} \cdot \underline{n} ds = 0 , \tag{4}$$

where $\underline{n}$ denotes the exterior unit normal vector on $\partial\Omega$, and the solution is unique up to a constant added to p; see [20]. We use the notation

$$\mathscr{H}^{\frac{1}{2}}(\partial\Omega) := \left\{ \underline{g} \in (H^{\frac{1}{2}}(\partial\Omega))^2 : \int_{\partial\Omega} \underline{g} \cdot \underline{n} ds = 0 \right\}.$$

3 The Continuous Background

Let

$$\underline{\mu} : \partial\Omega \to R^2$$

be given, and let $\delta_{\underline{\mu}}$ be the line δ-distribution on $\partial\Omega$ with strength $\underline{\mu}$. Consider the Stokes problem

$$-\Delta \underline{w} + \underline{\nabla} q = \delta_{\underline{\mu}} \quad \text{in} \quad (0,1)^2 \tag{5}$$

$$\underline{\nabla} \cdot \underline{w} = 0 \quad \text{in} \quad (0,1)^2 \tag{6}$$

$$\underline{w} = \underline{0} \quad \text{on} \quad \partial(0,1)^2 . \tag{7}$$

Let $\underline{g} \in \mathscr{H}^{\frac{1}{2}}(\partial\Omega)$ be given. One can then find $\underline{\mu}$ such that

$$\underline{w} = \underline{g} \quad \text{on} \quad \partial\Omega . \tag{8}$$

$\underline{\mu}$ can be described as follows. Let $(\underline{u}_{\text{int}}, p_{\text{int}})$ and $(\underline{u}_{\text{ext}}, p_{\text{ext}})$ be the solutions of

$$-\Delta \underline{u}_{\text{int}} + \underline{\nabla} p = \underline{0} \quad \text{in} \quad \Omega \tag{9}$$

$$\underline{\nabla} \cdot \underline{u}_{\text{int}} = 0 \quad \text{in} \quad \Omega \tag{10}$$

$$\underline{u}_{\text{int}} = \underline{g} \quad \text{on} \quad \partial\Omega \tag{11}$$

and

$$-\Delta \underline{u}_{\text{ext}} + \underline{\nabla} p = \underline{0} \quad \text{in} \quad (0,1)^2 - \bar{\Omega} \tag{12}$$

$$\underline{\nabla} \cdot \underline{u}_{\text{ext}} = 0 \quad \text{in } (0,1)^2 - \bar{\Omega} \tag{13}$$

$$\underline{u}_{\text{ext}} = \underline{g} \quad \text{on} \quad \partial\bar{\Omega} \tag{14}$$

$$\underline{u}_{\text{ext}} = \underline{0} \quad \text{on} \quad \partial(0,1)^2 . \tag{15}$$

Then

$$\underline{\mu} = \left[\frac{\partial \underline{u}}{\partial \underline{n}} - p\underline{n} \right] := \left(\frac{\partial \underline{u}_{\text{int}}}{\partial \underline{n}} - p_{\text{int}} \underline{n} \right) - \left(\frac{\partial \underline{u}_{\text{ext}}}{\partial \underline{n}} - p_{\text{ext}} \underline{n} \right). \tag{16}$$

Consider now the mapping

$$A : \underline{\mu} \to \underline{g} ,$$

defined by Eqs. (5)–(8). Properties of A are given in Theorem 1. Note that the exterior unit normal vector $\underline{n}$ on $\partial\Omega$ is an element of $(H^{-\frac{1}{2}}(\partial\Omega))^2$, so that the quotient

$$(H^{-\frac{1}{2}}(\partial\Omega))^2 / span\{\underline{n}\}$$

referred to in Theorem 1 is well-defined. We assume that $\partial\Omega$ is a simple, closed curve of class C^2. In particular, this implies that Ω must be simply connected, an assumption which will also be needed, for different reasons, in later sections.

Theorem 1. (i) *A is a topological isomorphism*

$$(H^{-\frac{1}{2}}(\partial\Omega))^2 / span\{\underline{n}\} \to \mathscr{H}^{\frac{1}{2}}(\partial\Omega) .$$

(ii) *A is symmetric:*

$$(\forall \underline{\varrho}, \underline{\mu} \in (H^{-\frac{1}{2}}(\partial\Omega))^2) \ (A\underline{\varrho}, \underline{\mu}) = (A\underline{\mu}, \underline{\varrho}) .$$

(iii) *A^{-1} is positive definite:*

$$(\exists \lambda, \Lambda \text{ with } 0 < \lambda < \Lambda < \infty)(\forall \underline{g} \in \mathscr{H}^{\frac{1}{2}}(\partial\Omega)) \ \lambda \|\underline{g}\|^2_{H^{\frac{1}{2}}} \leqq (A^{-1}\underline{g}, \underline{g}) \leqq \Lambda \|\underline{g}\|^2_{H^{\frac{1}{2}}} .$$

(iv) *A is positive definite:*

$$(\exists \lambda, \Lambda \text{ with } 0 < \lambda < \Lambda < \infty)(\forall \underline{\mu} \in (H^{-\frac{1}{2}}(\partial\Omega))^2) \ \lambda \|\underline{\mu}\|^2_{H^{-\frac{1}{2}}} \leqq (A\underline{\mu}, \underline{\mu}) \leqq \Lambda \|\underline{\mu}\|^2_{H^{-\frac{1}{2}}} .$$

Proof. (i) For $\underline{\mu} \in (H^{-\frac{1}{2}}(\partial\Omega))^2$, $A\underline{\mu} \in \mathscr{H}^{\frac{1}{2}}(\partial\Omega)$, and the mapping A is continuous. This follows from standard theorems about the Stokes equations; see [20]. A is a mapping onto $\mathscr{H}^{\frac{1}{2}}(\partial\Omega)$, since for any $\underline{g} \in \mathscr{H}^{\frac{1}{2}}(\partial\Omega)$, one can construct $\underline{\mu} \in (H^{-\frac{1}{2}}(\partial\Omega))^2$ with $A\underline{\mu} = \underline{g}$ by Eqs. (9)–(16). $A\underline{\mu} = \underline{0}$ if and only if $\delta_{\underline{\mu}}$ is, in the distributional sense, a gradient. This is the case if and only if $\underline{\mu}$ is a constant multiple of $\underline{n}$. (Here we have used the assumption that Ω is simply connected.) Thus, A as a mapping

$$(H^{-\frac{1}{2}}(\partial\Omega))^2/span\{\underline{n}\} \to \mathscr{H}^{\frac{1}{2}}(\partial\Omega)$$

is continuous, one-to-one, and onto. By the open mapping theorem ([18], p. 47), its inverse is continuous, too.

(ii) Let $\underline{\varrho}, \underline{\mu} \in (H^{-\frac{1}{2}}(\partial\Omega))^2$. Let $(\underline{w}_{\mu}, p_{\mu})$ denote the solution of Eqs. (5)–(7), and let $(\underline{w}_{\varrho}, p_{\varrho})$ be defined similarly. Then

$$(17) \qquad (A\underline{\varrho}, \underline{\mu}) = \int_{(0,1)^2} \underline{w}_{\varrho} \cdot \delta_{\mu} d\underline{x} = \int_{(0,1)^2} \underline{w}_{\varrho} \cdot (-\Delta \underline{w}_{\mu} + \underline{\nabla} p) d\underline{x}$$

$$= \int_{(0,1)^2} \underline{\nabla w}_{\varrho} \cdot \underline{\nabla w}_{\mu} - (\underline{\nabla} \cdot \underline{w}_{\mu}) p d\underline{x} = \int_{(0,1)^2} \underline{\nabla w}_{\varrho} \cdot \underline{\nabla w}_{\mu} d\underline{x}.$$

This shows the symmetry of A.

To prove (iii), let $\underline{g} \in \mathscr{H}^{\frac{1}{2}}(\partial\Omega)$. Define $\underline{\varrho} := A^{-1}\underline{g}$. Then, by Eq. (17),

$$(A^{-1}\underline{g}, \underline{g}) = (\underline{\varrho}, A\underline{\varrho}) = \int_{(0,1)^2} \underline{\nabla w}_{\varrho} \cdot \underline{\nabla w}_{\varrho} d\underline{x}.$$

We now wish to show that there are constants $\lambda > 0$ and $\Lambda > 0$ independent of $\underline{g} \in \mathscr{H}^{\frac{1}{2}}$ such that

$$(18) \qquad \lambda \|\underline{g}\|^2_{H^{\frac{1}{2}}} \leq \int_{(0,1)^2} \underline{\nabla w}_{\varrho} \cdot \underline{\nabla w}_{\varrho} d\underline{x} \leq \Lambda \|\underline{g}\|^2_{H^{\frac{1}{2}}}.$$

The existence of Λ follows from the continuous dependence of the solutions of (9)–(11) and (12)–(15) on $\underline{g}$; see [20]. The existence of λ follows from the Poincaré inequality for $\underline{w}_{\varrho}$ and the trace theorem for functions in $H^1((0,1)^2)$.

(iv) is an immediate consequence of (i) and (iii). □

Next we shall describe a mapping A_0 which has similar properties, but is simpler from a numerical point of view. A discrete analog of A_0 will serve as a preconditioner for a discrete analog of A.

Define

$$A_0 : (H^{-\frac{1}{2}}(\partial\Omega))^2 \to (H^{\frac{1}{2}}(\partial\Omega))^2$$

as follows. Given

$$\underline{\mu} \in H^{-\frac{1}{2}}(\partial\Omega))^2,$$

let $(\hat{\underline{\mu}}(k))_{k \in Z}$ be the Fourier coefficients. More precisely, since $\partial\Omega$ is assumed to be a simple, closed, smooth curve, distributions defined on $\partial\Omega$ can be identified via the arc-length parametrization of $\partial\Omega$ as distributions on the real line with period $l(\partial\Omega)$ = length of $\partial\Omega$. The Fourier coefficients of $\underline{\mu}$ are then defined to be the Fourier coefficients of the periodic distribution on the real line corresponding to $\underline{\mu}$. Define $A_0\underline{\mu} \in (H^{\frac{1}{2}}(\partial\Omega))^2$ by

$$(A_0\underline{\mu})\hat{}(k) := \frac{\hat{\underline{\mu}}(k)}{\sqrt{k^2+c}},$$

with $c > 0$. A_0 has the following properties.

Theorem 2. (i) A_0 *is a topological isomorphism*

$$(H^{-\frac{1}{2}}(\partial\Omega))^2 \to (H^{\frac{1}{2}}(\partial\Omega))^2.$$

(ii) A_0 *is symmetric:*

$$(\forall \underline{\varrho}, \underline{\mu} \in (H^{-\frac{1}{2}}(\partial\Omega))^2)(A_0\underline{\varrho}, \underline{\mu}) = (A_0\underline{\mu}, \underline{\varrho}).$$

(iii) A_0 *is positive definite:*

$$(\exists \lambda, \Lambda \text{ with } 0 < \lambda < \Lambda < \infty)(\forall \underline{\mu} \in (H^{-\frac{1}{2}}(\partial\Omega))^2)\, \lambda \|\underline{\mu}\|^2_{H^{-\frac{1}{2}}} \leqq (A_0\underline{\mu}, \underline{\mu}) \leqq \Lambda \|\underline{\mu}\|^2_{H^{-\frac{1}{2}}}$$

Proof. (i) is an immediate consequence of the definitions of the spaces $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$: $H^{\frac{1}{2}}(\partial\Omega)$ is the space of all distributions ϕ on $\partial\Omega$ for which

$$\sum_{k=-\infty}^{\infty} (1+k^2)^{1/2} |\hat{\phi}(k)|^2 < \infty,$$

$H^{-\frac{1}{2}}(\partial\Omega)$ is the space of all distributions ψ on $\partial\Omega$ for which

$$\sum_{k=-\infty}^{\infty} (1+k^2)^{-1/2} |\hat{\psi}(k)|^2 < \infty,$$

and the norms on $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ are

$$\left[\sum_{k=-\infty}^{\infty} (1+k^2)^{1/2} |\hat{\phi}(k)|^2\right]^{1/2}$$

and

$$\left[\sum_{k=-\infty}^{\infty} (1+k^2)^{-1/2} |\hat{\psi}(k)|^2\right]^{1/2}.$$

The following computation proves (ii):

$$(A_0\underline{\varrho}, \underline{\mu}) = \sum_{k=-\infty}^{\infty} \frac{\hat{\underline{\varrho}}_k}{\sqrt{k^2+c}} \cdot \hat{\underline{\mu}}_k = \sum_{k=-\infty}^{\infty} \frac{\hat{\underline{\mu}}_k}{\sqrt{k^2+c}} \cdot \hat{\underline{\varrho}}_k = (A_0\underline{\mu}, \underline{\varrho}).$$

Letting $\underline{\mu} = \underline{\varrho}$, we obtain (iii), with

$$\lambda = \min_{k \in Z} \frac{\sqrt{k^2+1}}{\sqrt{k^2+c}} > 0$$

and

$$\Lambda = \max_{k \in Z} \frac{\sqrt{k^2+1}}{\sqrt{k^2+c}} < \infty. \quad \square$$

4 Discretization of the Stokes Equations

We use a discretization of Eqs. (1)–(3) based on a square grid as in Fig. 2; compare, e.g., [12]. The mesh width is $h = 1/N$, N integer. The velocity $\underline{u}$ is approximated in the cell vertices, and the pressure p in the cell centers. We assume from now on that $\bar{\Omega}$ is a union of cells

$$[ih, (i+1)h] \times [jh, (j+1)h].$$

We use central, second order differencing. The discrete momentum equations are centered in the cell vertices, and the discrete continuity equation in the cell centers.

We write the discretization in the form

(19) $$-\underline{\Delta}^h \underline{u}^h + \underline{\nabla}^h p^h = \underline{0} \quad \text{in cell vertices in} \quad \Omega$$

(20) $$\underline{\nabla}^h \cdot \underline{u}^h = 0 \quad \text{in cell centers in} \quad \Omega$$

(21) $$\underline{u}^h = \underline{g} \quad \text{in cell vertices on} \quad \partial\Omega .$$

Here

$$\underline{\Delta}^h \underline{u}^h(ih, jh) := \frac{1}{h^2}\,(-4\underline{u}^h_{i,j} + \underline{u}^h_{i-1,j} + \underline{u}^h_{i+1,j} + \underline{u}^h_{i,j-1} + \underline{u}^h_{i,j+1})$$

with

$$\underline{u}^h_{k,l} := \underline{u}^h(kh, lh)$$

and

$$\underline{\nabla}^h p^h(ih, jh) := \begin{pmatrix} \partial^h_x \\ \partial^h_y \end{pmatrix} p^h(ih, jh)$$

with

(22) $$(\partial^h_x)(ih, jh) := \frac{1}{2h}\,(p^h_{i+1/2,j+1/2} + p^h_{i+1/2,j-1/2} - p^h_{i-1/2,j+1/2} - p^h_{i-1/2,j-1/2})$$

and

(23) $$(\partial^h_y)(ih, jh) := \frac{1}{2h}\,(p^h_{i+1/2,j+1/2} + p^h_{i-1/2,j+1/2} - p^h_{i+1/2,j-1/2} - p^h_{i-1/2,j-1/2}) .$$

$\underline{\nabla}^h \cdot \underline{u}^h$ is defined similarly. A more general discrete Stokes problems, with non-zero right-hand sides in the discrete momentum and continuity equations, can be reduced to one of the form (19)–(21) at the expense of one call to a fast Stokes solver on the unit square.

The kernel of the discrete gradient operator consists of all functions p^h defined in the cell centers with the following property. If the four points $((i \pm \frac{1}{2})h, (j \pm \frac{1}{2})h)$ are centers of cells belonging to Ω, then

(24) $$p^h((i-\tfrac{1}{2})h, (j-\tfrac{1}{2})h) = p^h((i+\tfrac{1}{2})h, (j+\tfrac{1}{2})h)$$

and

(25) $$p^h((i+\tfrac{1}{2})h, (j-\tfrac{1}{2})h) = p^h((i-\tfrac{1}{2})h, (j+\tfrac{1}{2})h) .$$

Assume from now on that Ω is not only a union of h by h cells, but even a union of cells of the $2h$-grid. In this case, the kernel of the discrete gradient operator $\underline{\nabla}^h$ is spanned by the elements

$$p((i+\tfrac{1}{2})h, (j+\tfrac{1}{2})h) \equiv 1$$

and

$$p((i+\tfrac{1}{2})h, (j+\tfrac{1}{2})h) \equiv (-1)^{i+j} .$$

Corresponding to these elements, there are two discrete compatibility conditions:

1) The discrete analog of Eq. (4) obtained by discretizing the integral $\int_{\partial\Omega} \underline{g} \cdot \underline{n} ds$ using the trapezoidal rule.

2) A "non-physical" compatibility condition, requiring that a certain alternating sum of the tangential components of $\underline{g}$ in the cell vertices lying on $\partial\Omega$ be zero.

Precisely, these compatibility conditions can be stated as follows. Eliminate the boundary conditions in (19)–(21), obtaining a problem of the form

(26) $$-\Delta^h \underline{u}^h + \underline{\nabla}^h p^h = \underline{f}^h \quad \text{in cell vertices in } \Omega$$

(27) $$\underline{\nabla}^h \cdot \underline{u}^h = s^h \quad \text{in cell centers in } \Omega$$

(28) $$\underline{u}^h = \underline{0} \quad \text{in cell vertices on } \partial\Omega .$$

Then s^h should be orthogonal, in the euclidean sense, to the kernel of the discrete gradient operator $\underline{\nabla}^h$.

It is easy to see that the dimension of $\ker(\underline{\nabla}^h)$, and therefore the number of discrete compatibility conditions, can be larger than 2 if Ω is not a union of cells of the $2h$-grid.

5 The Discrete Analogs of the Mappings A and A_0

Given a vector field $\underline{\mu}^h$ defined in the cell vertices on $\partial\Omega$, let $\underline{f}^h(ih, jh) = \underline{\mu}^h(ih, jh)$ if $(ih, jh) \in \partial\Omega$, and $\underline{f}^h(ih, jh) = \underline{0}$ otherwise. Solve

(29) $$-\Delta^h \underline{w}^h + \underline{\nabla}^h q^h = \underline{f}^h \quad \text{in cell vertices in } [0,1) \times (0,1)$$

(30) $$\underline{\nabla}^h \cdot \underline{w}^h = 0 \quad \text{in cell centers in } (0,1)^2$$

(31) $$\underline{w}^h = \underline{0} \quad \text{in} \quad (ih, jh) \quad \text{with} \quad jh = 0 \quad \text{or} \quad jh = 1 ,$$

with periodicity in the x-direction. The use of periodicity conditions in the x-direction, instead of $\underline{w} \equiv 0$ for $x = 0$, $x = 1$, allows the discrete problems on the unit square to be solved using the Fast Fourier Transform; see Sect. 7. Let A^h be the mapping

$$\underline{\mu}^h \to (\underline{w}^h(ih, jh))_{(ih, jh) \in \partial\Omega} .$$

To study the properties of A^h, we shall use the following lemma, which states that algebraic analogs of the mapping A of Sect. 3 are symmetric and positive semi-definite under general assumptions.

Lemma 1. *Let p and q be integers, $B \in R^{q \times p}$ and $C \in R^{q \times q}$. Assume*

$$B = B^T > 0 .$$

Let

$$M := \begin{pmatrix} B & C \\ C^T & 0 \end{pmatrix} \in R^{(p+q) \times (p+q)},$$

and let $M^{\oplus}$ be the Moore-Penrose pseudo-inverse of M. Partition $M^{\oplus}$ in the form

$$M^{\oplus} = \begin{pmatrix} D & E \\ E^T & F \end{pmatrix},$$

with

$$D \in R^{p \times p}, \qquad E \in R^{p \times q}, \qquad F \in R^{q \times q} .$$

Then

$$D = D^T \geqq 0$$

and

$$\ker(D) = \operatorname{range}(C) .$$

Proof. We shall compute D. First note that

(32) $$\ker(M)=\left\{\begin{pmatrix}\underline{0}\\ \underline{y}\end{pmatrix} : C\underline{y}=\underline{0}\right\}.$$

Proof of Eq. (32): "$\supseteq$" is obvious. To prove "$\subseteq$", suppose that

$$B\underline{x}+C\underline{y}=\underline{0}$$

and

$$C^T\underline{x}=\underline{0}.$$

Then

$$\underline{x}=-B^{-1}C\underline{y},$$

and thus

(33) $$\underline{0}=C^T\underline{x}=-C^TB^{-1}C\underline{y}.$$

Equation (33) implies

$$C\underline{y}=\underline{0},$$

and thus also $B\underline{x}=\underline{0}$, i.e. $\underline{x}=\underline{0}$. The proof of Eq. (32) is complete.

From Eq. (32), it follows that

$$\begin{pmatrix}\underline{f}\\ \underline{0}\end{pmatrix}\in\text{range}(M)$$

for every $\underline{f}\in R^p$. We now solve

(34) $$B\underline{x}+C\underline{y}=\underline{f}$$

and

(35) $$C^T\underline{x}=\underline{0}.$$

Using Eq. (34), we rewrite Eq. (35) as follows:

$$C^TB^{-1}C\underline{y}=C^TB^{-1}\underline{f}.$$

Thus

(36) $$\underline{y}=(C^TB^{-1}C)^{\oplus}C^TB^{-1}\underline{f}+\text{some element of }\ker(C^TB^{-1}C).$$

Inserting (36) into (34) and solving for $\underline{x}$, we find:

$$\underline{x}=B^{-1}\underline{f}-B^{-1}C(C^TB^{-1}C)^{\oplus}C^TB^{-1}\underline{f}.$$

This shows:

$$D=B^{-1}-B^{-1}C(C^TB^{-1}C)^{\oplus}C^TB^{-1}.$$

Next we observe that every $\underline{x}\in R^p$ has a decomposition in the form

$$\underline{x}=\underline{\xi}+C\underline{\eta}$$

with $\underline{\eta}\in R^q$, $\underline{\xi}\in R^p$, and

$$\underline{\xi}\perp\text{range}(B^{-1}C).$$

It is easy to verify that such a decomposition is given by

$$\underline{\eta}=(C^TB^{-1}C)^{\oplus}C^TB^{-1}\underline{x}$$

and

$$\underline{\xi}=\underline{x}-C(C^TB^{-1}C)^{\oplus}C^TB^{-1}\underline{x}.$$

We use this decomposition to study the quadratic form associated with D:

$$\begin{aligned}\underline{x}^T D\underline{x} &= (\underline{\xi}+C\underline{\eta})^T(B^{-1}-B^{-1}C(C^TB^{-1}C)^{\oplus}C^TB^{-1})(\underline{\xi}+C\underline{\eta})\\ &= \underline{\xi}^T B^{-1}\underline{\xi}+\underline{\eta}^T C^T B^{-1}C\underline{\eta}-\underline{\eta}^T C^T B^{-1}C(C^TB^{-1}C)^{\oplus}C^TB^{-1}C\underline{\eta}\\ &= \underline{\xi}^T B^{-1}\underline{\xi}.\end{aligned}$$

This implies the statement of the lemma. □

We return now to the study of the discrete analog A^h of A.

Theorem 3. *A^h is symmetric and positive semi-definite. If Ω is simply connected, and if $\bar{\Omega}$ and $[0,1]^2-\Omega$ are unions of cells of the $2h$*-grid, *e.g. unions of squares of the form*

$$[i\cdot 2h,(i+1)\cdot 2h]\times[j\cdot 2h,(j+1)\cdot 2h], \qquad 0\leqq i,j\leqq \frac{N}{2}-1,$$

then $\dim\ker(A^h)=2$, *and the range of A^h is exactly the space of vector fields defined in cell vertices on $\partial\Omega$ which satisfy the discrete compatibility conditions.*

Proof. From Lemma 1, we conclude that A^h is symmetric and positive semi-definite, and that $\ker(A^h)$ consists of those vector fields $\underline{\mu}^h$ for which the vector field $\underline{f}^h$ defined by $\underline{f}^h(ih,jh)=\underline{\mu}^h(ih,jh)$ for $(ih,jh)\in\partial\Omega$, $\underline{f}^h(ih,jh)=\underline{0}$ otherwise, is a discrete gradient, i.e. an element of the range of the discrete gradient operator $\underline{\nabla}^h$.

Consider an arbitrary element $\underline{\mu}^h$ of $\ker(A^h)$, and let q^h be a grid function with

$$\underline{f}^h=\underline{\nabla}^h q^h.$$

By subtracting a grid function on $(0,1)^2$ with zero gradient, we can assume that $q^h\equiv 0$ outside Ω. Since the space of grid functions defined in Ω with zero discrete gradients has dimension two, it follows that $\ker(A^h)$ has dimension two.

Denote by W the space of all discrete vector fields on $\partial\Omega$, i.e. vector fields defined in grid points on $\partial\Omega$. Denote by V the space of discrete vector fields on $\partial\Omega$ satisfying the discrete compatibility conditions. V and range(A^h) are subspaces of W. It is clear that range$(A^h)\subseteq V$. The co-dimension of V is two, since there are two discrete compatibility conditions. The co-dimension of range(A^h) is the dimension of $\ker(A^h)$, hence it is also two. Thus range(A^h) and V are equal. □

If Ω is not simply connected, then the number of discrete compatibility conditions is still two, but the kernel of A^h has the dimension $2+2g$, where g is the number of holes in Ω. There are then vector fields defined in the cell vertices on $\partial\Omega$ which satisfy the discrete compatibility condition, but do not lie in the range of A^h, and the methods of Sect. 6 are not applicable.

To define a discrete analog of A_0, assume as in Sect. 3 that $\partial\Omega$ is a simple, closed curve. Define the matrix

$$D:=\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1\\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0\\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot\\ \cdot & 0 & -1 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0\\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & -1\\ -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{pmatrix}+\frac{c}{l^2}I$$

with $c>0$. D is of size l by l, where l is the number of cell vertices lying on $\partial\Omega$.

A_0^h acts on vector fields $\underline{\mu}^h$ defined in the cell vertices which lie on $\partial\Omega$. Identifying such vector fields with pairs of l-vectors, with the first l-vector corresponding to the first component of $\underline{\mu}^h$, and the second to the second component of $\underline{\mu}^h$, we define

(37) $$A_0^h := \begin{pmatrix} D^{-\frac{1}{2}} & 0 \\ 0 & D^{-\frac{1}{2}} \end{pmatrix}.$$

For the numerical experiments of Sect. 8, we have always used $c=\frac{1}{4}$.

6 The Direct and Iterative Imbedding Methods

Suppose that $\underline{g}^h$ is a given vector field defined in the cell vertices on $\partial\Omega$, satisfying the discrete compatibility conditions. Solve

(38) $$A^h \underline{\mu}^h = \underline{g}^h.$$

Then let $\underline{f}^h(ih, jh) = \underline{\mu}^h(ih, jh)$ if $(ih, jh) \in \partial\Omega$, and $\underline{f}^h(ih, jh) = \underline{0}$ otherwise. Solve Eqs. (29)–(31). Then $(\underline{u}^h, p^h) := (\underline{w}^h, q^h)$, restricted to Ω, solve Eqs. (1)–(3).

Note that matrix vector products of the form $A^h \underline{\mu}^h$ can be computed at the expense of one call to a fast Stokes solver on $(0,1)^2$. Thus, there are the following two possibilities for solving the system (38):

1) Direct method: Compute A^h and its Cholesky decomposition.

2) Iterative method: Solve (38) using the conjugate gradient method, with the preconditioner A_0^h.

The iterative method does not require the computation of A^h. Note that

$$\ker(A_0^h) = \{\underline{0}\} \subseteq \ker(A^h).$$

This ensures that the conjugate gradient iteration for (38) can indeed be preconditioned by A_0^h.

7 A Fast Stokes Solver on a Rectangle with Periodicity in one Coordinate Direction

In this section, we describe an FFT-based method for solving Eqs. (29)–(31) with periodicity conditions in the x-direction.

Lemma 2. *If N is even, the discrete Stokes operator on the square with periodicity conditions in the x-direction has a two-dimensional kernel, spanned by*

$$\begin{pmatrix} \underline{u}^h \\ p^h \end{pmatrix} = \begin{pmatrix} \underline{0} \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \underline{u}^h \\ p^h \end{pmatrix} = \begin{pmatrix} \underline{0} \\ (-1)^{i+j} \end{pmatrix}.$$

Proof. This immediately follows from the characterization (24), (25) of the grid functions with zero discrete gradient. □

We shall assume from now on that N is even. Let $\underline{u}^h$ be a vector field defined in the cell vertices

$$(ih, jh), \quad 0 \leqq i \leqq N-1, \quad 0 \leqq j \leqq N,$$

with

$$\underline{u}^h(ih, jh) = \underline{0} \quad \text{for} \quad j=0 \quad \text{and} \quad j=N.$$

Let p^h be a scalar function defined in the cell vertices

$$((i-\tfrac{1}{2})h, (j-\tfrac{1}{2})h), \quad 1 \leqq i, j \leqq N.$$

The functions u^h, v^h and p^h have expansions of the form

$$u^h(x,y) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \alpha_u(k,y) e^{2\pi i k x}, \tag{39}$$

$$v^h(x,y) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \alpha_v(k,y) e^{2\pi i k x}, \tag{40}$$

$$p^h(x,y) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \alpha_p(k,y) e^{2\pi i k x}. \tag{41}$$

For a fixed $k \in \left\{-\frac{N}{2}, \dots \frac{N}{2}-1\right\}$, we define

$$\underline{\alpha}_k := \begin{bmatrix} \alpha_p\left(k, \frac{h}{2}\right) \\ \alpha_u(k,h) \\ \alpha_v(k,h) \\ \alpha_p\left(k, \frac{3h}{2}\right) \\ \alpha_u(k,2h) \\ \alpha_v(k,2h) \\ \vdots \\ \alpha_u(k,(N-1)h) \\ \alpha_v(k,(N-1)h) \\ \alpha_p\left(k, \frac{(2N-1)h}{2}\right) \end{bmatrix}.$$

Equations (29)–(31) are equivalent with linear systems of equations for the vectors $\underline{\alpha}_k$, of the form

$$\mathscr{M}_k \underline{\alpha}_k = \underline{\beta}_k, \tag{42}$$

where $\mathscr{M}_k$ is a matrix of dimension $3N-2$ by $3N-2$ with zero entries everywhere except in a band of seven diagonals around the main diagonal, and $\underline{\beta}_k \in R^{3N-2}$. From Lemma 2, we immediately obtain:

Lemma 3. *$\mathcal{M}_k$ is non-singular for* $-\frac{N}{2}+1 \leqq k \leqq \frac{N}{2}-1$, $k \neq 0$. $\dim\ker(\mathcal{M}_{-\frac{N}{2}}) = \dim\ker(\mathcal{M}_0) = 1$, *and*

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ \cdot \\ \cdot \end{bmatrix} \in \ker(\mathcal{M}_{-\frac{N}{2}}),$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ \cdot \\ \cdot \end{bmatrix} \in \ker(\mathcal{M}_0).$$

Equations (29)–(31) can be solved by expanding the right-hand side $\underline{f}^h$ into its Fourier series in the x-direction, analogous to Eqs. (39) and (40), solving the systems (42), and computing $\underline{u}^h$ and p^h through inverse Fourier transform. The total cost of solving (29)–(31) is $O(N^2 \log(N))$.

8 Numerical Results

We define

$$\mathrm{cond}_2(A^h) := \frac{\max\{\lambda : \lambda \text{ eigenvalue of } A^h\}}{\min\{\lambda : \lambda \text{ eigenvalue of } A^h,\ \lambda > 0\}},$$

and compute $\mathrm{cond}_2(A^h)$ and the analogously defined condition number of the preconditioned matrix

$$(A_0^h)^{-\frac{1}{2}} A^h (A_0^h)^{-\frac{1}{2}},$$

for several domains and values of h.

Example 1:

$$\Omega = (\tfrac{1}{4}, \tfrac{3}{4})^2.$$

This example is not completely artificial, since Stokes problems are not easy to solve even on rectangles, unless there is a periodicity condition in at least one coordinate direction. Compare, however, Sect. 9.

Table 1. Condition numbers for imbedding method without preconditioning

N	8	16	32
Rectangle	148	566	2202
L-shaped domain	181	678	2620
Domain of Fig. 1	–	3169	11752

Table 2. Condition numbers for imbedding method with preconditioning

N	8	16	32
Rectangle	118	99	93
L-shaped domain	99	84	80
Domain of Fig. 1	–	1352	1147

Example 2: The L-shaped domain

$$\Omega=(\tfrac{1}{4},\tfrac{3}{4})\times(\tfrac{1}{4},\tfrac{1}{2})\cup(\tfrac{1}{4},\tfrac{1}{2})\times[\tfrac{1}{2},\tfrac{3}{4}).$$

Example 3: The domain of Fig. 1

Tables 1 and 2 contain our results. It appears that

$$\mathrm{cond}_2(A^h)$$

grows proportionally to $\frac{1}{h^2}$, while

$$\mathrm{cond}_2(A_0^{-\frac{1}{2}}A^h A_0^{-\frac{1}{2}})$$

remains bounded as h decreases.

9 A Fast Stokes Solver on a Rectangle with Physical Boundary Conditions

The methods of Sect. 6 can be used to solve Stokes problems on rectangles with physical boundary conditions, i.e. with the velocity $\underline{u}$ prescribed on the boundary. However, we shall now outline a simpler, less expensive and more natural variant of the method for this special case.

Consider a problem of the form

$$-\Delta^h\underline{u}^h+\underline{\nabla}^h p^h=\underline{f}^h \quad \text{in cell vertices in } (0,1)^2 \tag{43}$$

$$\underline{\nabla}^h\cdot\underline{u}^h=0 \quad \text{in cell centers in } (0,1)^2 \tag{44}$$

$$\underline{u}^h=\underline{0} \text{ in } (ih,jh) \text{ with } jh=0 \text{ or } jh=1 \text{ or } ih=0 \text{ or } ih=1\,. \tag{45}$$

This problem differs from Eqs. (29)–(31) only in the boundary conditions at $x\equiv 0$ and $x\equiv 1$. Note that the right-hand side $\underline{f}^h$ for Eq. (29) is defined in the cell vertices (x,y) with $x=0$, while the right-hand side for Eq. (43) is only defined in cell vertices (x,y) with $0<x<1$.

Equations (43)–(45) can be solved in the following way. In a first step, we reduce the problem to the form

(46) $$-\Delta^h \underline{u}^h + \underline{\nabla}^h p^h = 0 \quad \text{in cell vertices in } (0,1)\times(0,1)$$

(47) $$\underline{\nabla}^h \cdot \underline{u}^h = 0 \quad \text{in cell centers in } (0,1)^2$$

(48) $$\underline{u}^h(ih, jh) = \underline{0} \quad \text{in} \quad (ih, jh) \quad \text{with} \quad jh = 0 \quad \text{or} \quad jh = 1$$

(49) $$\underline{u}^h(ih, jh) = \underline{g}^h(jh) \quad \text{in} \quad (ih, jh) \quad \text{with} \quad ih = 0 \quad \text{or} \quad ih = 1\,.$$

This can easily be accomplished by one call to the fast solver of Sect. 7. We then determine $\underline{r}^h(jh)$, $j = 1, \ldots, N-1$, such that the solution of Eqs. (29)–(31) with

(50) $$\underline{f}^h(ih, jh) = \underline{0} \quad \text{for } 1 \leqq i, j \leqq N-1\,,$$

(51) $$\underline{f}^h(ih, jh) = \underline{r}^h(jh) \quad \text{for } i = 0\,, \quad 1 \leqq j \leqq N-1\,,$$

solves Eqs. (46)–(49).

This requires inverting the matrix $\tilde{A}^h$ representing the mapping

$$\underline{r}^h \to \underline{g}^h\,.$$

Theorem 4. *$\tilde{A}^h$ is symmetric and positive semi-definite. If N is even, then $\tilde{A}^h$ is positive definite.*

Proof. The symmetry and semi-definiteness of $\tilde{A}^h$ follows from Lemma 1. If $\tilde{A}^h \underline{r}^h = \underline{0}$, then the force field $\underline{f}^h$ given by Eqs. (50) and (51) is a discrete gradient. Thus,

$$\underline{f}^h = \underline{\nabla}^h q^h\,.$$

Away from the line $i = 0$, i.e. the line $i = N$, q^h has zero gradient, and is thus either constant, or a multiple of $(-1)^{i+j}$. If q^h is constant, then clearly $\underline{\nabla}^h q^h = \underline{0}$ everywhere, even on the line $i = 0$. Similarly, if q^h is a multiple of $(-1)^{i+j}$, and if N is even, then $\underline{\nabla}^h q^h = \underline{0}$ everywhere. Thus, for even N, $\tilde{A}^h \underline{r}^h = \underline{0}$ implies $\underline{f}^h = \underline{0}$, i.e. $\underline{r}^h = \underline{0}$. □

As before, there are a direct and an iterative version of the method, and the direct version solves Eqs. (43)–(45) in $O(N^2 \log(N))$ arithmetic operations, provided that the data-independent work needed for computing and factoring $\tilde{A}^h$ is not counted.

10 Summary and Discussion

We summarize our conclusions. The operator which is inverted numerically in our method can, briefly and symbolically, be described as follows:

(52) $$\delta\text{-distribution of force along } \partial\Omega \to \text{velocity along } \partial\Omega\,.$$

It seems natural to attempt to invert this operator numerically. We have shown that this idea can be carried out, at least for simply connected domains which can be covered by uniform square grids, or a little more generally, by rectangular grids with uniformly spaced grid lines in at least one coordinate direction. (The condition that the grid be uniform in at least one coordinate direction allows the construction of FFT-based fast Stokes solvers on the rectangle.) It is also natural to precondition the operator (52) by a symmetric, positive definite operator which raises the order of differentiability by one. We have described a numerical implementation of this idea, leading to a substantial reduction of the condition number.

The numerical experiments of Sect. 8 indicate that the iterative version of the method is inefficient in comparison with what can be obtained, for suitable discretizations of the Stokes problem, with multigrid methods; compare e.g. [1] and [7]. This confirms the comments of the introduction: Domain imbedding can be used to construct simple but not very fast iterative solvers, or efficient direct solvers.

Acknowledgements. This work was supported in part by NSF grant DMS-8801991, and by a Rackham Research Grant and Faculty Fellowship at the University of Michigan.

References

1. Abdalass, E.M., Maitre, J.F., Musy, F.: A multigrid solver for a stabilized finite element discretization of the Stokes problem. In: Hackbusch, W., Trottenberg, U., Lecture Notes in Mathematics, vol. 1228. (eds.) Berlin Heidelberg New York: Springer 1986
2. Bakhvalov, N.S., Orekhov, M.Y.: On fast methods of solving Poissons's equation. U.S.S.R. Comput. Math. Math. Phys. **22**, 107–114 (1982)
3. Banegas, A.: Fast Poisson solvers for problems with sparsity. Math. Comput. **32**, 441–446 (1978)
4. Bjørstad, P., Widlund, O.B.: Iterative methods for the solution of elliptic problems on regions partitioned into substructures. SIAM J. Numer. Anal. **23**, 1097–1120 (1986)
5. Börgers, C., Widlund, O.B.: Finite Element Capacitance Matrix Methods, Technical Report 261, Computer Science Department, New York University, and LBL-Report 22583, Lawrence Berkeley Laboratory (1986)
6. Börgers, C., Widlund, O.B.: On finite element domain imbedding methods. (submitted to) SIAM J. Numer. Anal. **27** (1990) (to appear)
7. Brandt, A., Dinar, N.: Multi-grid solutions to elliptic flow problems. In: Parter, S.V. (ed.) Numerical Methods for Partial Differential Equations. pp. 53–147. New York: Academic Press 1979
8. Buzbee, B.L., Dorr, F.W., George, J.A., Golub, G.H.: The direct solution of the discrete Poisson equation on irregular regions. SIAM J. Numer. Anal. **8**, 722–736 (1971)
9. Chan, T.F., Saied, F.: A comparison of elliptic solvers for general two-dimensional regions. SIAM J. Sci. Stat. Comput. **6**, 742–760 (1985)
10. Dryja, M.: A finite element capacitance matrix method for the elliptic problem. SIAM J. Numer. Anal. **20**, 671–680 (1983)
11. Finogenov, S.A., Kuznetsov, Y.A.: Two-stage fictitious component method for solving the Dirichlet boundary value problem. Sov. J. Numer. Anal. Math. Modelling **3**, 301–324 (1988)
12. Fortin, M., Peyret, R., Temam, R.: Résolution numérique des équations de Navier-Stokes pour un fluide incompressible. J. Mec. **10**, 357–390 (1971)
13. Hockney, R.: The potential calculation and some applications. Methods Comput. Phys. **9**, 135–211 (1970)
14. Marchuk, G.I., Kuznetsov, Y.A., Matsokin, A.M.: Fictitious domain and domain decomposition methods. Sov. J. Numer. Anal. Math. Modelling **1**, 3–36 (1986)
15. Proskurowski, W.: Numerical solution of Helmholtz's equation by implicit capacitance matrix methods. ACM Trans. Math. Software **5**, 36–49 (1979)
16. Proskurowski, W., Widlund, O.B.: On the numerical solution of Helmholtz's equation by the capacitance matrix method. Math. Comput. **30**, 433–468 (1976)
17. Proskurowski, W., Widlund, O.B.: A finite element capacitance matrix method for the Neumann problem for Laplace's equation. SIAM J. Sci. Stat. Comput. **1**, 410–425 (1980)
18. Rudin, W.: Functional Analysis. New York: McGraw-Hill 1973
19. Stüben, K., Trottenberg, U.: Multigrid methods: Fundamental algorithms, model problem analysis and applications. In: Lecture Notes in Mathematics, vol. 960. Hackbusch, W., Trottenberg, U. (eds.) Berlin Heidelberg New York: Springer 1982
20. Temam, R.: Navier-Stokes Equations. North-Holland 1984