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A Nonlinear Instability for 3 × 3 Systems of Conservation Laws

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Abstract: The phenomenon of nonlinear resonance provides a mechanism for the unbounded amplification of small solutions of systems of conservation laws. We construct spatially 2π-periodic solutions $u^N \in C^\infty([0,\ t_N] \times \mathbb{R})$ with t_N bounded, satisfying

$$\|u^N\|_{L^\infty([0,\ t_N]\times\mathbb{R})} \to 0, \quad \int_0^{2\pi} |\partial_x u^N(0,x)|\, dx \leqq C\,,$$

$$\int_0^{2\pi} |\partial_x u^N(t_N,x)|\, dx \geqq N, \ \ \|u^N(t_N,x)\|_{L^p(\mathbb{R})} \geqq N \|u^N(0,x)\|_{L^p(\mathbb{R})} \quad 1 \leqq p \leqq \infty\,.$$

The variation grows arbitrarily large, and the sup norm is amplified by arbitrarily large factors.

Outline.

1. Main Result

The main existence theorems for $k \times k$ systems of conservation laws [G, CS, GL, NS, D, Y], have a common feature: either the systems under consideration have $k \leqq 2$, or the initial data are of small total variation. In the latter cases, the variation is uniformly bounded by a fixed multiple of the initial variation. In this note we explain that these restrictions are essential. When $k \geqq 3$, nonlinear resonance is a mechanism which can produce unbounded amplification of the total variation in finite time.

We construct 3×3 genuinely nonlinear systems such that each system has a sequence of spatially periodic smooth solutions u^N with the property that the L^∞ norm tends to zero, the initial variation is bounded above and there are uniformly bounded times t_N such that the variations at time t_N tend to infinity.

In order to produce unbounded amplification as above we need two things. First there must be a resonance relation involving three distinct modes of propagation. Second for i, j, k distinct the interaction coefficient of i, j waves on the k wave must be large compared to the k-selfinteraction coefficient. For the 3×3 Euler equations of compressible gas dynamics the interaction coefficient of two acoustic waves on shear waves is zero so our analysis does not apply in that case.

The idea of the construction is to use solutions of the profile equations of resonant nonlinear geometric optics with the property that the profiles explode in finite time. For linearly degenerate systems explicit explosive profiles were constructed by Majda and Rosales [MR, Remark 4.5]. For genuinely nonlinear systems J. Hunter [H] constructed explicit explosive profiles which are discontinuous, in fact of sawtooth form. The idea is to use the recent justifications of nonlinear geometric optics to conclude that exact solutions behave similarly. The important recent result of Schochet [S] for discontinuous profiles does not suffice in the present context for two reasons. First the L^1 error estimate is not strong enough and second the result does not allow one to approach the blowup time for the profiles. We construct smooth profiles for genuinely nonlinear systems which explode in finite time. The strategy is to show that the explosive profiles of Majda–Rosales persist under genuinely nonlinear perturbations. Then our results [JMR1] apply and the program is complete.

For the Euler equations one knows that the analogous profile equations do not explode in finite time [MRS]. Whether Euler profiles have unbounded growth as t tends to infinity is not known.

We would like to thank Blake Temple for calling our attention to this question and the work of Hunter and also for illuminating conversations.

The examples are 3×3 systems of conservation laws

$$\partial_t u + \partial_x(F(u)) = 0 , \tag{1.1}$$

$$u(t,x) \equiv (u_1(t,x),\ u_2(t,x),\ u_3(t,x)) , \tag{1.2}$$

$$F(u) = (F_1(u),\ F_2(u),\ F_3(u)) , \tag{1.3}$$

$F \in C^\infty(\mathbb{R}^3 : \mathbb{R}^3)$ with $F(0) = 0$. We make several choices so that the final analysis is simple.

The systems are

$$(\partial_t + \partial_x)u_1 + c(u_1^2)_x - (bu_2u_3)_x = 0, \tag{1.4$_1$}$$

$$(\partial_t)u_2 - c(u_2^2/2)_x + (bu_1u_3/2)_x = 0, \tag{1.4$_2$}$$

$$(\partial_t - \partial_x)u_3 + c(u_3^2)_x - (bu_1u_2)_x = 0 \tag{1.4$_3$}$$

with constants c and $b > 0$ to be specified below. Near $u = 0$, the systems are strictly hyperbolic and are genuinely nonlinear as soon as $c \neq 0$.

Introduce the phases

$$\varphi_1(t,x) \equiv x - t, \quad \varphi_2(t,x) \equiv -2x, \quad \varphi_3(t,x) \equiv x + t . \tag{1.7}$$

The φ_j are solutions of the eikonal equation for the linearization of (1.4) at $u = 0$.

The explosive examples are the solutions of (1.4) with initial data

$$u^\varepsilon(0,x) = \varepsilon(\sin(x/\varepsilon),\ \sin(-2x/\varepsilon),\ \sin(x/\varepsilon))\,. \tag{1.8}$$

Introduce

$$r_1 \equiv (1,\,0,\,0), \quad r_2 \equiv (0,\,1,\,0), \quad \text{and} \quad r_3 \equiv (0,\,0,\,1)\,. \tag{1.9}$$

Linearization yields a first approximation

$$u^\varepsilon(t,x) \approx \varepsilon \sum \sin(\varphi_j(t,x)/\varepsilon) r_j$$

uniformly valid for times $t = o(1)$ as ε tends to zero.

Nonlinear geometric optics provides an approximation valid for times of order 1 as ε tends to zero. In Sect. 2 we show that

$$\sum \varepsilon v(t, \varphi_j(t,x)/\varepsilon) r_j \tag{1.10}$$

is the leading term in a nonlinear geometric optics expansion for (1.4), (1.8) provided v is the real valued solution of

$$\partial_t v + c\partial_\theta(v^2) - bv * \partial_\theta v = 0\,, \tag{1.11}$$

$$v(0,\theta) = \sin(\theta)\,. \tag{1.12}$$

Notice that the same v occurs in each component of the solution. The key is to show that if c is small compared to b, then the solution v to (1.11), (1.12) explodes in finite time.

Note that u^ε and (1.10) are $4\pi\varepsilon$ periodic in x. In particular if $\varepsilon = 1/2K$ with $K \in \mathbb{N}$, then u^ε is 2π periodic with respect to x.

The initial variation, $\int_0^{2\pi} |\partial_x u^\varepsilon(0,x)|\, dx$, is independent of $\varepsilon = 1/2K$.

Main Theorem. *For $b > 0$ there are positive constants $c(b)$ and T so that for any $|c| \leqq c(b)$, $N > 0$, and $\delta > 0$, there is an $\varepsilon_0 > 0$ so that if $\varepsilon = 1/2K < \varepsilon_0$ with $K \in \mathbb{N}$, then there is a $t_\varepsilon \in]0, T]$ so that the solutions u^ε of* (1.4), (1.8) *belongs to $C^\infty([0, t_\varepsilon] \times \mathbb{R}/2\pi\mathbb{Z})$ and satisfy*

i. $|u^\varepsilon(t,x)| \leqq \delta$ for all $(t,x) \in [0, t_\varepsilon] \times \mathbb{R}$,

ii. $\int_0^{2\pi} |\partial_x u^\varepsilon(t_\varepsilon, x)|\, dx \geqq N$,

iii. $\|u^\varepsilon(t_\varepsilon, x)\|_{L^\infty(\mathbb{R})} \geqq N \|u^\varepsilon(0,x)\|_{L^\infty(\mathbb{R})}$.

Remarks. 1. The above result should be contrasted with shock formation. To show that shocks form one shows that sup $|u_x|$ blows up in finite time while u remains bounded [Jo]. This usually comes about when a single wave becomes steep. The total variation need not and usually does not become large.

2. The examples are also very different than the 3 × 3 linearly degenerate non-conservative example of Jeffrey [Je]. There the sup norm of a smooth solution explodes in finite time. For such an explosion the global behavior of the nonlinear term is crucial while our construction depends only on the second order Taylor polynomial of F at $u = 0$.

3. There is also unbounded amplification of the $L^2(S^1)$ norm for our examples. Such growth of L^2 norm is not possible if the system has a strictly convex entropy

[L]. For (1.4), $u_1^2 - 2u_2^2 + u_3^2$ is a conserved quantity as one sees by multiplying the equation by u_1, $-2u_2$, and u_3, and summing. The key is the identity

$$u_1 \partial_x(u_2 u_3) + u_2 \partial_x(u_1 u_2) + u_3 \partial_x(u_1 u_2) = 2\partial_x(u_1 u_2 u_3)\,.$$

Changing the coefficient $b/2$ in (1.4_2) to $-b$ produces a system for which $\sum u_j^2$ is a conserved quantity so there will be no such explosion.

4. The result is valid for more general F. Assuming that $F'(0)$ is diagonal with distinct eigenvalues, what is important is that the self interaction coefficients are small compared to the others, that is

$$\max_j |\partial^2 F_j(0)/\partial u_j \partial u_j| \ll \min_{j,k,l \text{ distinct}} |\partial^2 F_j(0)/\partial u_k \partial u_l)|$$

and that the mutual interactions are explosive. Remark 3 shows that distinguishing explosive interactions from others is not an obvious affair. However, some insight is provided by computing explicit solutions of profile equations analogous to the examples in Sect. 3, (see [H], [JMR2]).

5. Since the solutions are small in sup norm, there is a bound $\sigma = 1 + O(\varepsilon)$ on the speed of propagation. Thus considering $u^\varepsilon(t,x)$ in a trapezoidal domain of determinacy $\{|x| \leqq A - \sigma|t|,\ 0 \leqq t \leqq I\}$ shows that the amplification of the variation occurs locally. This argument does not suffice to construct examples with initial data of compact support, since if one extends the initial data beyond the base of the trapezoid one will no longer be sure that there is a smooth solution of (1.4) up to time t_ε. It is likely that nonlinear geometric optics argument using profiles $U(t,x,\theta)$ which depend on x suffices in that case.

6. Suppose T and C are as in the theorem. If one wants to prove the existence of BV solutions for $0 \leqq t \leqq T$ with initial data small in L^∞ and with BV norm as large as C, our result shows that one cannot do this by deriving uniform BV estimates for such solutions. This suggests that such a BV existence theorem is likely to be false.

Proof. Suppose that $v \in C^\infty([0,t_1] \times S^1)$ satisfies (1.11), (1.12). As explained in Sect. 2, the results of [JMR1] imply that for ε small $u^\varepsilon \in C^\infty([0,t_1] \times \mathbb{R})$ and

$$u^\varepsilon = \sum \varepsilon v(t, \varphi_j(t,x)/\varepsilon) r_j + o(\varepsilon) \tag{1.13}$$

with $o(\varepsilon)$ measured in the $L^\infty([0,t_1] \times \mathbb{R})$ norm. In particular

$$\|u^\varepsilon\|_{L^\infty([0,t_1]\times\mathbb{R})} = \varepsilon \|v^\varepsilon\|_{L^\infty([0,t_1]\times\mathbb{R})} + o(\varepsilon)\,. \tag{1.14}$$

In addition,

$$\partial_x u^\varepsilon = \sum \partial_\theta v(t, \varphi_j(t,x)/\varepsilon)(\partial_x \varphi_j) r_j + o(1) \quad \text{in } L^\infty([0,t_1] \times \mathbb{R})\,.$$

$$\int_0^{2\pi} |\partial_x u^\varepsilon(t,x)|\,dx = \text{const.} \int_0^{2\pi} |\,\partial_\theta v(t,\theta)\,|\,d\theta + o(1)\,. \tag{1.15}$$

The heart of the proof is to show that the solution v of the initial value problem (1.11), (1.12) explodes in finite time. In Sect. 5 we prove a theorem stronger than the following.

Theorem . *Given* $b > 0$ *there are constants* $c(b) > 0$ *and* $T < \infty$ *so that for* $|c| \leqq c(b)$, *there is a* $T_*(f, b, c) < T$ *so that the solution of* (1.11), (1.12) *belongs to* $C^\infty([0, T_*[\times S^1)$ *and as* $t \to T_*$, $\int_0^{2\pi} |\partial_\theta v(t, \theta)| d\theta \to \infty$, *and* $\max_\theta |v(t, \theta)| \to \infty$.

Then for any $N > 0$ there is a $t_1(N) < T_*$ so that v is smooth on $[0, t_1] \times S^1$ and

$$\int_0^{2\pi} |\partial_\theta v(t_1, \theta)| \, d\theta > N \int_0^{2\pi} |\partial_\theta v(0, \theta)| \, d\theta \, , \tag{1.16}$$

$$\|v(t_1, \theta)\|_{L^\infty(S^1)} > N \|v(0, \theta)\|_{L^\infty(S^1)} \, . \tag{1.17}$$

The Main Theorem follows from estimates (1.13) to (1.17) upon taking $\varepsilon = 2/K$ with $K \in \mathbb{N}$ large. /////

2. Derivation of the Profile Equation

We sketch the derivation of profile equation (1.11). The system (1.4) is of the form

$$\partial_t u + A(u) \partial_x u = 0 \, , \tag{2.1}$$

where $A(u)$ is an affine function of u, $A = \text{diag}(1, 0, -1) + B(u)$, where $B(u)$ is the linear matrix valued function of u,

$$B(u) \equiv \begin{bmatrix} 2cu_1 & -bu_3 & -bu_2 \\ bu_3/2 & -cu_2 & bu_1/2 \\ -bu_2 & -bu_1 & 2cu_3 \end{bmatrix} . \tag{2.2}$$

The asymptotic solutions have the form

$$u^\varepsilon(t, x) \sim \varepsilon U(t, t/\varepsilon, x/\varepsilon) \, , \tag{2.3}$$

where the profile

$$U(t, T, X) = \sum U_\alpha(t) e^{i\alpha \cdot (T, X)} \tag{2.4}$$

is 2π-periodic in the fast variables T, X.

The equations for U depend on projection operators whose definitions we recall,

$$\forall \alpha \in \mathbb{R}^2 \quad L(\alpha) \equiv \alpha_0 I + \alpha_1 A(0) \, , \tag{2.5}$$

$$\forall \alpha \in \mathbb{R}^2 \; E_\alpha \text{ is the projection on Ker } (L(\alpha)) \text{ along } Rg(L(\alpha)) \, . \tag{2.6}$$

Then $E_\alpha = 0$ unless α belongs to the characteristic variety of $L(\partial_T, \partial_X)$. The E_α serve to define a projection operator on trigonometric series in T, X,

$$\mathbb{E} \left(\sum V_\alpha e^{i\alpha \cdot (T, X)} \right) \equiv \sum (E_\alpha V_\alpha) e^{i\alpha \cdot (T, X)} \, . \tag{2.7}$$

As a map of trigonometric series $\mathbb{E}$ projects on $\text{Ker} L(\partial_T \partial_X)$ along $Rg L(\partial_T, \partial_X)$ ([JMR2]).

The profile equations take the form (see [JMR2] for a short derivation in this case and also [MR, JMR1, 3, 4])

$$\mathbb{E} U = U \quad \text{and} \quad \partial_t U + \mathbb{E}(B(U) \partial_X U) = 0 \, . \tag{2.8}$$

The results of [JMR1] show that solutions of these profile equations yield correct asymptotic solutions of (2.1). In the current setting the space of phases, Φ, is the set of linear functions of t,x. The strong transversality assumption of [JMR1] requires that if Y is equal to one of the vector fields ∂_t, $\partial_t \pm \partial_x$ and $\varphi = t\tau + x\xi$ is a phase such that $Y\varphi$ is not identically zero, then $Y\varphi$ is nonzero at almost every point of each integral curve of Y. However, $Y\varphi$ is a constant. Thus, if it is not identically zero it is nowhere zero, and the assumption is satisfied. Theorems 2.9.1 and 2.10.4 of [JMR1] yield the following result.

Theorem. *Suppose that $U(t,x,T,X) \in C^\infty([0,t_*] \times S^1 \times S^1)$ is a smooth doubly periodic solution of the profile equations* (2.8), *and $\underline{t} \in]0,t_*[$. Then there is an $\varepsilon_0 > 0$ so that for all $\varepsilon \in [0,\varepsilon_0[$ the initial value problem*

$$\partial_t u^\varepsilon + A(u^\varepsilon)\partial_x u^\varepsilon = 0, \quad u^\varepsilon(0,x) = \varepsilon U(0,x,0,x/\varepsilon)$$

has a unique solution $u^\varepsilon \in C^\infty([0,\underline{t}] \times \mathbb{R})$ and as $\varepsilon \to 0$,

$$|u^\varepsilon(t,x) - \varepsilon U(t,x,t/\varepsilon,x/\varepsilon)\,| = o(\varepsilon)\,,$$
$$|\nabla_{t,x}(u^\varepsilon(t,x) - \varepsilon U(t,x,t/\varepsilon,x/\varepsilon))\,| = o(1)\,,$$

uniformly for $0 \leqq t \leqq \underline{t}$ and $x \in \mathbb{R}$.

Since $\mathbb{E}U = U$, (2.6) shows that

$$U(t,T,X) = \sum \sigma_j(t,\varphi_j(T,X))r_j \tag{2.9}$$

with φ_j and r_j as in (1.7), (1.9) and $\sigma_j(t,\theta)$ 2π-periodic in θ.

Define Γ_j on scalar valued trigonometric series by

$$\Gamma_j\left(\sum v_\alpha e^{i\alpha\cdot(T,X)}\right) \equiv \sum_{\alpha\in\mathbb{R}d\varphi_j} v_\alpha e^{i\alpha\cdot(T,X)}\,. \tag{2.10}$$

Then $\mathbb{E}_j \equiv |r_j\rangle\Gamma_j\langle r_j\,|$ are projectors which satisfy $\mathbb{E}_j\mathbb{E} = \mathbb{E}\mathbb{E}_j = \mathbb{E}_j$, $\mathbb{E} = \sum\mathbb{E}_j$, and $\mathbb{E}_jU = \sigma_j(t,\varphi_j(T,X))r_j$.

Define the interaction constants γ_j^{mn} as the scalar products

$$\gamma_j^{mn} \equiv \langle r_j\,|B(r_m)r_n\rangle\,. \tag{2.11}$$

With F from (1.1), Taylor expansion yields

$$\gamma_j^{mn} = (1+\delta_{mn})^{-1}\partial^2 F_j(0)/\partial u_m \partial u_n\,,$$

so the γ_j^{mn} are symmetric in m, n.

The profile equations (2.8) are equivalent to

$$\partial_t\sigma_j(t,\varphi_j(T,X)) + \sum_{m,n}\gamma_j^{mn}\Gamma_j\left[\sigma_m(t,\varphi_m(T,X))\partial_X\sigma_n(t,\varphi_n(T,X))\right] = 0 \tag{2.12$_j$}$$

for $1 \leqq j \leqq 3$. Since the γ_j^{mn} are symmetric in mn the sum is an X derivative. It follows that $\int_0^{2\pi}\sigma_j(t,\theta)d\theta$ is independent of time. We restrict attention to solutions for which these integrals vanish.

To analyse (2.12), introduce the Fourier expansions

$$\sigma_j(t,\theta) = \sum\hat{\sigma}_j(t,k)e^{ik\theta}\,. \tag{2.13$_j$}$$

Our solutions have mean zero so $\hat{\sigma}_j(t,0) = 0$. Inserting (2.13) in (2.12_j) one must compute

$$\Gamma_j(\exp(i[k_1\varphi_m(T,X) + k_2\varphi_n(T,X)])) \qquad (2.14)_j$$

for all j,m,n and $k = (k_1,k_2) \in \mathbb{Z}^2$, $k_i \neq 0$. From the definition 2.10 one gets

$\exp(i[k_1\varphi_j(T,X) + k_2\varphi_j(T,X)])$ when $m = n = j$,

0 if exactly one of m,n is equal to j ,

0 if $m = n \neq j$, or m,n,j are distinct and $k_1 \neq k_2$,

$\exp(-ik\varphi_j(T,X))$ if m,n,j are distinct and $k_1 = k_2 = k$.

The last is most important and is a result of the resonance relation $\sum \varphi_j = 0$. The sum of the $m = n = j$ terms in $(2.12)_j$ is equal to

$$\gamma_j^{jj}\sigma_j(t,\varphi_j(T,X))\partial_X\sigma_j(t,\varphi_j(T,X)) .$$

The sum of the terms with m,n,j distinct and $k_1 = k_2 = k$ is equal to

$$\gamma_j^{mn}(\partial_X\varphi_m + \partial_X\varphi_n)\sum_k \hat{\sigma}_m(t,k)(ik)\hat{\sigma}_n(t,k)\exp(-ik\varphi_j(T,X))$$
$$= \gamma_j^{mn}(\partial_X\varphi_m + \partial_X\varphi_n)\partial_\theta(\sigma_m * \sigma_n)(-\varphi_j(T,X)) ,$$

where the convolution of two 2π-periodic function is taken with respect to the normalized measure $d\theta/2\pi$. Summarizing, the functions $\sigma(t,\theta)$ satisfies

$$\partial_t\sigma_j + \gamma_j^{jj}\partial_X\varphi_j\partial_\theta(\sigma_j^2/2) + R\left[\gamma_j^{mn}(\partial_X\varphi_m + \partial_X\varphi_n)\partial_\theta(\sigma_m * \sigma_n)\right] = 0 \,, (2.15)_j$$

where in the last term m,n,j are distinct and R is the reflection operator $(Rf)(\theta) \equiv f(-\theta)$.

Since the convolution of two odd functions is even we see that if the σ_j are odd at time $t = 0$ they remain so. We restrict attention to such odd functions to find

$$\partial_t\sigma_j + \gamma_j^{jj}\partial_X\varphi_j\partial_\theta(\sigma_j^2/2) - \gamma_j^{mn}(\partial_X\varphi_m + \partial_X\varphi_n)\partial_\theta(\sigma_m * \sigma_n) = 0 \,. \qquad (2.16)_j$$

The system (1.4) and phases (1.7) were chosen exactly so that

$$\gamma_j^{jj}\partial_X\varphi_j/2 = c \quad \text{for} \quad j = 1,2,3, \quad \text{and}$$

$$\gamma_j^{mn}(\partial_X\varphi_m + \partial_X\varphi_n) = b \quad \text{for} \quad m,n,j \text{ distinct} \; .$$

In that case $(\sigma_1,\sigma_2,\sigma_3) = (v,v,v)$ is an odd solution of (2.16) if and only if v is an odd solution of (1.11).

3. Two Explosive Profiles

We present two explicit solutions which show that the triple interaction can lead to explosive growth.

Example 1 of Hunter [*H*]. Let $S(\theta)$ be the sawtooth function which has slope equal to one and which jumps down by 2π at $\theta = 0$ so $S(\theta) = \theta - \pi$ for $0 < \theta < 2\pi$. Then $\hat{S}(n) = i/n$ and in the sense of distributions on the circle,

$$S_\theta = -2\pi\delta_{\theta=0} + 1, \ (S*S)_\theta = S*S_\theta = -S, \ (S^2)_\theta = 2S\,. \tag{3.1}$$

Note that convolution is with respect to the normalized measure $d\theta/2\pi$ which implies that $\delta * S = S/2\pi$. Using the identities (3.1), one sees that $v = \alpha(t)S(\theta)$ satisfies (1.11) if and only if

$$\alpha_t + (2c+b)\alpha^2 = 0\,. \tag{3.2}$$

Shocks in S satisfy the entropy condition iff $c\alpha \geqq 0$, and in that case the effect of the Burger's term is dissipative. However if the sign of b is opposite to that of c, the contribution of the b term is explosive. For example if $c > 0$, $\alpha(0) > 0$, and b so negative that $2c + b < 0$, the amplitude α explodes in finite time.

Example 2 of Majda–Rosales [*MR*]. If $c = 0$, the equation has no mixing of Fourier components, that is

$$\partial_t\hat{v}(t,n) - ibn(\hat{v}(t,n))^2 = 0\,. \tag{3.3}$$

For example

$$v(t) = \beta(t)\sin(\theta) = \beta(t)(e^{i\theta} - e^{-i\theta})/2i \tag{3.4}$$

with $\hat{v}(t,\pm 1) = \pm\beta(t)/2i$ is a solution if and only if

$$\beta_t - b\beta^2/2 = 0\,. \tag{3.5}$$

The solution

$$\beta(t) = 2\beta(0)/(2 - b\beta(0)t)$$

explodes at $t = 2/b\beta(0)$.

When $c = 0$, the system (1.4) is not genuinely nonlinear at $u = 0$, which is a weakness of this example. Our main result in Sect. 5 proves that there exist analogous explosive profiles when c is small.

4. Existence for the Profile Equation

As for Burger's equation, the initial value problem for (1.11) has a smooth solution on a maximal interval $[0, T_*[$ and if $T_* < \infty$, $\|v(t)\|_{\text{Lip}} \equiv \|v(t)\|_\infty + \|v_\theta(t)\|_\infty$ must diverge to ∞ as $t \to T_*$. A much more general result is proved in Sect. 6.3 of [JMR1]. For completeness we give a simple proof in the present context.

Proposition. *Suppose that* $c, b \in \mathbb{R}$ *and* $f \in \text{Lip}(S^1)$.
i. *There is a* $T_*(f,c,b) > 0$ *and a unique* v *uniformly Lipshitzean on* $[0,T] \times S^1$ *for all* $T < T_*$ *satisfying*

$$v_t + c(v^2)_\theta - b(v*v)_\theta = 0, \ \ v(0,\theta) = f(\theta)\,. \tag{4.1}$$

ii. *If* $T_* < \infty$, *then* $\|v(t)\|_{\text{Lip}} \to \infty$ *as* $t \to T_*$.
iii. *If* $f \in H^s(S^1)$ *for some* $s > 3/2$, *then* $v \in C([0,T_*[:H^S(S^1))$. *In particular, if* $f \in C^\infty(S^1)$, *then* $v \in C^\infty([0,T_*[\times S^1)$.

Proof. The proof is exactly as for the Burger's equation which has $b = 0$. In a sense the nonlinear term $(v*v)_\theta$ is weaker than the term $(v^2)_\theta$. For example the sup norm of the former is bounded by $\|v\|_1\|v_\theta\|_\infty$, while the latter is bounded by $\|v\|_\infty\|v_\theta\|_\infty$, where $\|\cdot\|_p$ denotes the $L^p(S^1)$ norm.

For parts i and ii, it suffices to show that for any $M > 0$ there is an $\eta > 0$ such that for all f, b, c with $\|f\|_{\mathrm{Lip}}, |b|, |c| < M$, there is a v in $\mathrm{Lip}([0,\eta] \times S^1)$ satisfying (4.1). The solution is constructed as the limit of Picard iterates v^ν which solve the linear equations

$$v^1(t,\theta) = f(\theta)\,, \tag{4.2}$$

$$(\partial_t + cv^\nu \partial_\theta)v^{\nu+1} - b(v^\nu * v^{\nu+1})_\theta = 0,\ \ v^{\nu+1}(0,\theta) = f(\theta),\ \ \nu \geqq 1\,. \tag{4.3}$$

It is easy to show by induction that each v^ν is uniformly Lipshitzean on compact subsets of $[0,\infty[\times S^1$. It suffices to prove bounds uniform in ν for $0 \leqq t \leqq \eta$.

Suppose that $\|v^\nu(t)\|_{\mathrm{Lip}} < M'$ for $0 \leqq t \leqq \eta$. Integrating (4.3) along characteristics shows that

$$\|v^{\nu+1}(t)\|_\infty \leqq \|f\|_\infty + C\int_0^t \|v_\theta^\nu(t) * v^{\nu+1}(t)\|_\infty \, dt\,. \tag{4.4}$$

Estimating $\|v_\theta^\nu\|_\infty \leqq M'$ yields

$$\|v^{\nu+1}(t)\|_\infty \leqq \|f\|_\infty + CM'\int_0^t \|v^{\nu+1}(t)\|_\infty \, dt\,. \tag{4.5}$$

For the derivative of v we have the equation

$$(\partial_t + cv^\nu \partial_\theta)v_\theta^{\nu+1} + cv_\theta^\nu v_\theta^{\nu+1} - bv_\theta^\nu * v_\theta^{\nu+1} = 0\,.$$

Then

$$\|v_\theta^{\nu+1}(t)\|_\infty \leqq \|f_\theta\|_\infty + (C+c)M'\int_0^t \|v_\theta^{\nu+1}(t)\|_\infty \, dt\,. \tag{4.6}$$

It follows from Gronwall's inequality that

$$\|v^{\nu+1}(t)\|_{\mathrm{Lip}} \leqq \|f\|_{\mathrm{Lip}} e^{(C+c)M't}. \tag{4.7}$$

Fix $M' = 2M$ and choose η so that $e^{(C+c)M'\eta} < 2$. It then follows that for $\nu \geqq 1$ and $t \in [0,\eta]$, $\|v^{\nu+1}(t)\|_{\mathrm{Lip}} \leqq M'$. This proves the existence parts of i. and ii.

For uniqueness suppose that v_1 and v_2 are Lipshitzean solutions for $0 \leqq t \leqq T$ and set $w \equiv v_1 - v_2$. Then

$$(\partial_t + cv_1\partial_\theta)w + c(\partial_\theta v_2)w - b\partial_\theta(v_1 + v_2) * w = 0\,.$$

Integrating along characteristics shows that

$$\|w(t)\|_\infty \leqq C'\int_0^t \|w(t)\|_\infty \, dt$$

and Gronwall's inequality shows that $w \equiv 0$.

The proof of part iii. again resembles arguments for Burger's equation with modifications as in the previous parts. The key estimates are (5.5), (5.6), (5.7) in the next section. The details are left to the reader. /////

From uniqueness it follows that if f is odd in θ then so is v. The corresponding result with odd replaced by even is not true since the maps $\partial_\theta v^2$ and $v * \partial_\theta v$ do not preserve the even functions.

5. Blowup for the Profile Equation

The next result shows that for positive b, if c is sufficiently small the solution of (4.1) with $f = \sin(\theta)$ explodes in finite time. The notation $\|\cdot\|_p$ for $L^p(S^1)$ norms and T_* for lifetimes are taken from Sect. 4.

Theorem. *Suppose that* $f = \sin(\theta)$ *and* $b > 0$. *There are constants* $c(b) > 0$ *and* $T < \infty$ *so that* $T_*(f, b, c) < T$ *for* $|c| \leqq c(b)$. *In addition,*

$$\int_0^{2\pi} v(t,\theta)\sin(\theta)d\theta \to \infty \quad \textit{as } t \to T_* .$$

Remark. The last conclusion is much stronger than the conclusions of the theorem at the end of Sect. 1.

Proof. The idea is to view the system as a perturbation of the system with $c = 0$. The latter initial value problem is solved in Example 2 of Sect. 3. The Fourier coefficients with $n = \pm 1$ explode at $t = 2/b$ while the others are identically zero. The strategy is to show that for c small the coupling between the $n = \pm 1$ modes and the others is sufficiently weak that the coefficients ± 1 explode while the others are smaller

Suppose $|c| \leqq 1$.

Introduce $\alpha_n(t) = \hat{v}(t, n)/i$ so that the $\alpha_n = -\alpha_{-n} \in \mathbb{R}$. Equation (4.1) is equivalent to

$$\partial_t \alpha_n - c\sum_m n\alpha_m \alpha_{n-m} + bn\alpha_n^2 = 0 \tag{5.1}$$

for the Fourier components. The initial conditions are $\alpha_{\pm 1} = \mp 1/2$ and $\alpha_n = 0$ otherwise.

The equation for $n = 1$ yields

$$|\partial_t \alpha_1 + b\alpha_1^2| = |c\sum_m \alpha_m \alpha_{1-m}| \leqq c\|u\|_2^2 . \tag{5.2}$$

Equation (5.2) yields lower bounds for $|\alpha_{\pm 1}|$ at the same time as one gets an upper bound for the other coefficients.

We use the Sobolev $H^s(S^1)$ norms on odd functions.

$$|v|_s^2 \equiv \int \left(\partial_\theta^s v\right)^2 d\theta = 2\pi \sum |n|^{2s}\alpha_n^2 . \tag{5.3}$$

The estimate for the coefficients with $|n| \geqq 2$ follows the idea of the standard energy estimates for Sobolev norms. With $s \in \mathbb{N}$, $s > 3/2$ one multiplies the differential equation by $\partial_\theta^{2s} v$ and integrates over the circle. The three terms are estimated using

$$\int \partial_\theta^{2s} v\, \partial_t v d\theta = (1/2)\partial_t |v|_s^2 , \tag{5.4}$$

$$\begin{aligned} \left|\int \partial_\theta^{2s} v\, v * \partial_\theta v\, d\theta\right| &= \left|\int \partial_\theta^s v\, \partial_\theta^s v * \partial_\theta v\, d\theta\right| \\ &\leqq \|\partial_\theta^s v\|_2 \|\partial_\theta^s v * \partial_\theta v\|_2 \\ &\leqq \|\partial_\theta^s v\|_2 \|\partial_\theta^s v\|_2 \|\partial_\theta v\|_1 \leqq \Gamma(s)\, |v|_s^3 , \end{aligned} \tag{5.5}$$

$$\left|\int \partial_\theta^{2s} v\, v\partial_\theta v\, d\theta\right| \leqq \Gamma\, |v|_s^3 , \tag{5.6}$$

where Γ denotes a constant which may depend on s and b and may change from line to line in the proof.

A direct proof of the estimate (5.6) in our case starts with

$$\pm \int \partial_\theta^{2s} v \, v \partial_\theta v d\theta = \sum_{n,\ell} n^{2s} \alpha_n \alpha_{n-\ell} \ell \alpha_\ell = \sum_{n,\ell} n^s [(n-\ell)+\ell]^s \alpha_n \alpha_{n-\ell} \ell \alpha_\ell \,.$$

Using the binomial theorem yields

$$\sum_{n,\ell,0 \leqq a \leqq s} \begin{bmatrix} s \\ a \end{bmatrix} n^s (n-\ell)^a \ell^{s-a+1} \alpha_n \alpha_{n-\ell} \alpha_\ell \,. \tag{5.7}$$

Use $\alpha_n = -\alpha_{-n}$ to get

$$\sum_{n,\ell} n^s (n-\ell)^a \ell^{s-a+1} \alpha_n \alpha_{n-\ell} \alpha_\ell = (-1)^{s+1} \sum_{n+m+\ell=0} n^s m^a \ell^{s-a+1} \alpha_n \alpha_m \alpha_\ell \,.$$

The tricky case is when $a = 0$. By symmetry one has

$$\begin{aligned} &\sum_{n+m+\ell=0} [n^s \ell^{s+1} + n^{s+1} \ell^s] \alpha_n \alpha_m \alpha_\ell \\ &= \sum_{n+m+\ell=0} n^s \ell^s [n+\ell] \alpha_n \alpha_m \alpha_\ell = - \sum_{n+m+\ell=0} n^s \alpha_n m \alpha_m \ell^s \alpha_\ell \,. \end{aligned}$$

This term is bounded by $|u|_s^2 \, \|m\alpha_m\|_\ell^1 \leqq \Gamma \, |u|_s^3$ since $s > 3/2$. For the terms with $s \geqq a \geqq 1$ estimate

$$|\, n-\ell \,|^a |\ell|^{s-a+1} \leqq |n-\ell| |\ell|^s + |\, n-\ell \,|^s |\ell| \,.$$

The sums on n, ℓ are again estimated by $|u|_s^2 \|m\alpha_m\|_\ell^1$. Estimate (5.6) follows.

Inequality (5.6) expressed in terms of the Fourier coefficients reads

$$|\sum_{n,\ell} n^{2s} \alpha_n \alpha_{n-\ell} \ell \alpha_\ell \,| \leqq \Gamma \, |u|_s^3 \,. \tag{5.8}$$

Writing the left-hand side as the sum of two terms yields

$$|\sum_{\ell,|n| \geqq 2} n^{2s} \alpha_n \alpha_{n-\ell} \ell \alpha_\ell \,| \leqq \Gamma \, |u\,|_s^3 + |\sum_{\ell,n=\pm 1} \alpha_n \alpha_{n-\ell} \ell \alpha_\ell \,| \leqq \Gamma' \, |u|_s^3 \,. \tag{5.9}$$

Denote the high frequency part of the s-norm by

$$H^2 \equiv \sum_{|n| \geqq 2} n^{2s} \alpha_n^2 \,. \tag{5.10}$$

Equation (5.1) yields

$$\begin{aligned} (1/2) dH^2/dt &= \sum_{|n| \geqq 2} (1/2)(d/dt) n^{2s} \alpha_n^2 \\ &= -b \sum_{|n| \geqq 2} n^{2s+1} \alpha_n^3 + c \sum_{\ell,|n| \geqq 2,} n^{2s} \alpha_n \alpha_{n-\ell} \ell \alpha_\ell \,. \end{aligned}$$

The first sum on the right is dominated by H^3. The second is estimated in (5.9). Thus

$$|\, dH^2/dt \,| \leqq \Gamma(H^3 + |c| |\alpha_1|^3), \quad H(0) = 0 \,. \tag{5.11}$$

This estimate is used in tandem with (5.2) which implies that

$$|d\alpha_1^2/dt + b\alpha_1^3| \leqq |c|\,\Gamma(H^3 + |\alpha_1|^3), \quad \alpha_1(0) = -1/2\,. \tag{5.12}$$

The strategy is to show that α_1^2 diverges while H^2 remains smaller than α_1^2/λ^2, where the parameter λ is chosen below. So long as $\alpha_1^2 \geqq \lambda^2 H^2$, one has

$$\begin{aligned} d\alpha_1^2/dt &\geqq (b - |c|\,\Gamma)\,|\alpha_1|^3 - |c|\,\Gamma\,|H|^3 \\ &\geqq (b - |c|\,\Gamma(\lambda^{-3} + 1))\,|\alpha_1|^3\,, \end{aligned} \tag{5.13}$$

$$|dH^2/dt| \leqq \Gamma(\lambda^{-3} + |c|)\,|\alpha_1|^3\,. \tag{5.14}$$

Therefore

$$d(\alpha_1^2 - \lambda^2 H^2)/dt \geqq \left[b - |c|\,\Gamma(\lambda^{-3} + \lambda^2 + 1) - \Gamma/\lambda\right]|\alpha_1|^3\,. \tag{5.15}$$

First choose λ positive so that $\Gamma/\lambda < b/3$. Then choose $c(b) > 0$ so that $|c|\,\Gamma(\lambda^{-3} + \lambda^2 + 1) < b/3$. Then for $|c| \leqq c(b)$, the quantity on the right in (5.15) is $\geqq (b/3)\,|\alpha_1|^3 \geqq 0$ and it follows that $\alpha_1^2 \geqq \lambda^2 H^2$ throughout the domain of existence of v.

Since $s > 3/2$,

$$|\alpha_1| \leqq \|v\|_{\mathrm{Lip}} \leqq \mathrm{const.}|v|_s \leqq \mathrm{const.}(\alpha_1^2 + H^2)^{1/2}\,.$$

As $\alpha_1^2 \geqq \lambda^2 H^2$ it follows that the blowup time for v coincides with that for α_1.

Let $T < \infty$ be the blowup time for the initial value problem

$$dy/dt = (b/3)y^{3/2}, \quad y(0) = 1/4\,.$$

Since $\alpha_1^2 \geqq \lambda^2 H^2$, (5.13) and (5.14) imply that

$$d\alpha_1^2/dt \geqq (b/3)\,|\alpha_1|^3, \quad \alpha_1^2(0) = 1/4$$

throughout the interval of existence, so $\alpha_1^2 \geqq y$. Thus α_1^2 diverges to infinity no later than at time T. This completes the proof of the theorem. /////

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