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Embeddings of $U_3(8)$, $Sz(8)$ and the Rudvalis group in algebraic groups of type E_7

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Dedicated to Armand Borel

Summary. We prove that the nearly simple finite group $U_3(8){:}6$ is in the adjoint group of type E_7 over any algebraically closed field. We also prove that the group $Sz(8)$ is embeddable in $E_7(K)$, for an algebraically closed field K, iff $\mathrm{char}(K)=2$, 5; in characteristic 5, we prove that there is such an embedding which extends to an embedding of the slightly larger group $Sz(8){:}3$ and to an embedding of the sporadic simple group of Rudvalis. Our methods include p-local analysis in groups of Lie type, nonabelian cohomology, representation theory of finite groups and computer calculations.

1 Introduction and statement of results

During the last decade, there have been detailed investigations of embeddings of finite groups in exceptional type algebraic groups e.g. [B, CoGr1, CoGr2, CoWa1, CoWa2, CLSS, Gr4, KMR, KlRy, KlWi]. In this paper, we resolve the embedding questions for the finite simple groups $U_3(8)$, $Sz(8)$ and the sporadic group Ru of Rudvalis into groups of type E_7; see [CoGr1].

Let K be a field of characteristic not 2 and let $\bar{K}$ be its algebraic closure. For each of the groups Ru, $Sz(8)$ and $U_3(8)$ we shall explicitly identify a 2-local subgroup inside the K-points of the relatively accessible Lie type subgroup $2^4 A_1^7(\bar{K}).\, \mathrm{GL}_3(2)$ of $2E_7(\bar{K})$. We then construct all possible extensions of the 2-local subgroup to a copy of our original group inside $\mathrm{GL}_{56}(K)$. The three situations are sufficiently rigid for us to determine exactly which extensions remain inside our $2E_7(K)$ in $\mathrm{GL}_{56}(K)$. We remark that, when K has a primitive eighth root of unity, our 2-local subgroups can be located as subgroups of a fairly natural finite subgroup of shape $2^{4+14}.\,[\Sigma_3\, wr\, \mathrm{GL}_3(2)]$ in the K-points of the above $2^4 A_1^7(K).\, \mathrm{GL}_3(2)$-subgroup (the wreath products are taken with respect to a transitive action of degree 7 of the $\mathrm{GL}_3(2)$-factor). In the $U_3(8)$ and $Sz(8)$-cases, we may work in the even smaller subgroup $2^{4+14}{\cdot}[\Sigma_3 \times \mathrm{GL}_3(2)]$.

We have arranged our paper so that the different embedding theorems for Ru, $Sz(8)$, $U_3(8)$ and their 2-local subgroups can be read independently. In Sects. 3, 4 and 5 we construct embeddings of 2-local subgroups of $U_3(8)$, $Sz(8)$ and Ru, respectively. These constructions are mutually independent, but they all make use of the notation introduced in Sect. 2. In Sects. 6, 7 and 8 we construct embeddings of $Sz(8)$, Ru and $U_3(8)$, respectively. These sections are also self contained, apart from occasional references to earlier embedding theorems for subgroups. Thus, for example, Sect. 7 provides a complete proof that $2 \cdot Ru < 2 \cdot E_7(5)$ subject to the hypotheses (verified in Sects. 5 and 6) that certain proper subgroups of $2 \cdot Ru$ embed into $2 \cdot E_7(5)$.

In the $U_3(8)$-case, we need a computer calculation, and so we take K to be the finite field $\mathbb{F}_{25}$. We then lift the copy of $U_3(8){:}12$ (see remarks on notation below) thus constructed to $2E_7(\mathbb{C})$. Only for characteristics 2 and 5 is $Sz(8)$ in the group E_7. In characteristic 5, we may extend an embedding of $Sz(8)$ to one of $2 \cdot Ru$ in $2 \cdot E_7(K)$, giving a short proof of a main result in [KMR]. Our arguments for $Sz(8)$ and Ru are computer-free.

We shall make frequent reference to the two classes of involutions of an algebraic group of type E_8 (in characteristic other than 2). Following [CoGr1, Gr4], we denote these classes by $2A$ and $2B$ where the classes have respective centralizers $2 \cdot E_7 A_1$ and $2 \cdot D_8$. Any group $E = 2 \cdot E_7(K)$ is naturally contained in an algebraic group, G, of type E_8; we shall refer to involutions of E as type $2A$ or type $2B$ according to their class in G. Results of [CoGr1] carry over appropriately to the case of positive characteristic, as discussed in [Gr4]. In particular, [CoGr1, (3.8)] the conjugacy assertions apply to groups of type E_8 over any field of characteristic not 2; moreover, if the field is algebraically closed, the structure of the normalizer is determined analogously.

Our notation for simple groups is consistent with that of [Go1, Go2, Hup, CoGr1, Gr4]. We use 1, 3, 3′ and 8 to denote the four $\mathbb{F}_2[GL_3(2)]$-irreducible modules. Our use of extension theoretic notation is standard. We clarify one instance of an extension which occurs in this article: $U_3(8){:}12$ is the unique nonsplit central extension of the centerless group $U_3(8){:}6$ by a group of order 2. See Sect. 3 for more details. The group $U_3(8){:}6$ is not contained in $2E_7(\mathbb{C})$, as one can see by studying character values (notice that the Schur multiplier of $U_3(8)$ has order 3). We must work with the group $U_3(8){:}12$. The relevant 56-dimensional representation of this group is isogenous to the irreducible 56-dimensional representation of $U_3(8){:}6$ of determinant 1: we take the latter 56 dimensional linear group, multiply it by the group of scalar matrices of order 4, then take the unique subgroup of this group $4{:}U_3(8){:}6$ of the form $U_3(8){:}12$ which intersects the group of scalar matrices in a group of order 2.

2 Construction of the 2-local subgroups

We use the term *Borel subgroup* to extend the usual meaning of the normalizer of a maximal unipotent group in a finite group of Lie type of any finite group which is a central extension of a Chevalley group extended by a group of outer automorphisms. In particular, we shall be concerned with Borels with structures $2^{3+3'}{:}7$ in $Sz(8)$, $2^{3+3'}{:}[7{:}3]$ in $Sz(8){:}3$, $2^{3+6}{:}21$ in $U_3(8)$ and $2^{3+6}{:}[\mathrm{Frob}_{21} \times 2 \cdot \Sigma_3]$ in $U_3(8){:}12$.

2.1. Definition. A *Suzuki 2-group* is a finite nonabelian 2-group, T, such that $\mathrm{Aut}(T)$ contains a cyclic subgroup which acts transitively on the set of involutions in T.

We shall make use of the classification of Suzuki 2-groups [Hig] to identify appropriate subgroups of $2E_7(K)$. For each of the above mentioned Borel subgroups, B, $O_2(B)$ is a Suzuki 2-group.

Our Borel subgroups each contain normal elementary abelian subgroups of order 8, and it turns out that they can all be located in the normalizer, $E(\Delta)$, (see (2.3)) of a $2B$-pure 2^3 subgroup in $2\cdot E_7(K)$. Our aim in this section is to construct $E(\Delta)$ and its subgroups $Q\cong 2^{4+14}$ and $F\cong \mathrm{Frob}_{21}$ where F normalizes Q. The somewhat complicated-looking group Q actually has very nice and useful coordinates based on error correcting codes; indeed, a code-theoretic philosophy for describing 2-locals has been developed in [Gr2, Gr3]. Finally, in (2.8) we include a classification of subgroups of E isomorphic to $2^3{:}7$ with all involutions of type $2B$; this result applies to all groups of the form $2^3{:}7$ which occur within our Borel subgroups. We begin with the following standard notation for Chevalley groups of type E_7.

2.2. Notation. K is a field of characteristic not 2; for simplicity, we assume that K contains $i=\sqrt{-1}$.

E is the group $2E_7(K)$. Let Φ be the root system belonging to E, and let W be the Weyl group of E. Let T be a maximal torus, and let N be its normalizer, so that $N/T\cong W$.

X_r is the root group associated to the root r; $x_r(t)$, $h_r(t)$ $n_r(t)$ are elements of $\langle X_r, X_{-r}\rangle$, written in the usual Chevalley group notation; $h_r:=h_r(i)$ and $n_r:=n_r(1)$.

We define $Q_r:=\langle h_r, n_r\rangle\cong \mathrm{Quat}_8$ and let z_r be the central involution. Define $L_r:=N_{\langle X_r, X_{-r}\rangle}(Q_r)$. Thus, $L_r\cong 2\cdot\Sigma_4$ or $\mathrm{SL}_2(3)$, depending on whether K has or does not have a primitive eighth root of 1.

Note that $Q_r=Q_{-r}$, $h_{-r}=h_r^{-1}$, etc. The root system Φ has one W-orbit of subsystems of type A_1^7. These subsystems give rise to our group $E(\Delta)$ as follows:

2.3. Notation. Let $\tilde{\Delta}$ be a subsystem of type A_1^7 in Φ. In W, the stabilizer of $\tilde{\Delta}$ is a group of shape $2^7{:}\mathrm{GL}_3(2)\cong 2\times 2^{3+3'}{:}\,\mathrm{GL}_3(2)$.

When A is a subset of $\tilde{\Delta}$, write $x_A(t)$, Q_A, ..., for the product of pairwise commuting elements or groups $x_r(t)$, Q_r, ..., indexed by $r\in A$. Thus, $h_A^2=n_A^2=z_A$, $[h_A, n_B]=z_{A\cap B}$, etc.

Define $Q:=Q_{\Delta}$ (we shall see after (2.6) that Q is a special 2-group of shape 2^{4+14}) and $L:=L_{\tilde{\Delta}}$.

The Borel subgroups which we are aiming at all involve a Frobenius group of order 21.

2.4. Notation. Let F be a subgroup of N of shape $7{:}3$ which stabilizes $\tilde{\Delta}$, maps faithfully to W and acts on the sets $\{h_r | r\in\tilde{\Delta}\}$. Thus F acts on the set $\{L_r | r\in\tilde{\Delta}\}$.

The group F has two orbits of length 7 in its action on $\tilde{\Delta}$. We denote one of these orbits by Δ; the other orbits is $-\Delta$.

Let $E(\Delta)$ be the normalizer in E of the corresponding Lie type subgroup of type A_1^7, or what is the same, of the group of order 2^4 which occurs as the center of this subgroup of type A_1^7. (Its structure may be derived from

[CoGr1, (3.8.i)]). In case K is algebraically closed, the normalizer in G, the overgroup of type E_8, has the form $2^4\ A_1^8(K)\cdot 2^3{:}\mathrm{GL}(3,2)$; we get the normalizer in E by taking in this group the centralizer of a $2A_1$-factor; thus, if K is algebraically closed, it has shape $2^4\ A_1^7(K)\cdot \mathrm{GL}_3(2)$. When K is not algebraically closed, we may need to admit certain outer diagonal automorphisms at the A_1^7 section.)

Let θ_r be an element of order 3 in L_r, chosen for all $r\in\Delta$ to form an orbit of length 7 under conjugation by F. Similarly, when K has an eighth root of 1, let ϕ_r be an element in L_r which inverts θ_r and has order 4. We may choose ϕ_r for all $r\in\Delta$ to form an orbit of length 7 under conjugation by F. Let $\theta=\theta_\Delta$, let $\phi=\phi_\Delta$ whenever this is defined and otherwise let $\phi=1$.

We note that F, θ, and ϕ all normalize Q. Also θ and ϕ both centralize F. We now introduce the F-invariant Hamming codes on Δ. These codes will enable us to select certain F-invariant subgroups of Q.

2.5. Notation. $\mathscr{P}(X)$ denotes the power set of the finite set X, with symmetric difference as the sum. The codimension 1 subspace of even cardinality subsets is denoted $\mathscr{P}\mathscr{E}(X)$.

2.6. Definition. A *Hamming code* is a subspace S of $\mathscr{P}(X)$, the power set of a 7-set X, of dimension 3 and minimum weight 4. It lies in $\mathscr{P}\mathscr{E}(X)$ and its group is $\mathrm{GL}(3,2)$. A *dual Hamming code* (i.e., dual to S) is a Hamming code which meets S trivially; their common stabilizer in Σ_X is a Frobenius group of order 21.

In Q there are 7 involutions of the form z_r; they lie in Q'. The relations they satisfy form a Hamming code [CoGr1, (3.8.i)]. Therefore, $Q'\cong 2^4$ and $Q\cong 2^{4+14}$. The rule $[h_A, n_B]=z_{A\cap B}$ (2.3), implies that the group Q is a special 2-group, i.e., $Q'=Z(Q)=\Phi(Q)$.

2.7. *Choice of particular Hamming codes.* $\mathscr{P}(\Delta)$ has a 3-dimensional subspace, $\mathscr{H}'$ say, such that z_A is trivial iff $A\in\mathscr{H}'$; see [CoGr1, (3.8.i)]. The subspace $\mathscr{H}'$ is a Hamming code in $\mathscr{P}(\Delta)$. In $\mathscr{P}(\Delta)$, $\mathscr{H}'$ is stabilized by F; we let $\mathscr{H}$ be the dual F-invariant Hamming code.

On Q', $E(\Delta)$ acts as $\mathrm{GL}(3,2)$ via its action on the seven involutions z_r, for $r\in\Delta$. Therefore, as a module, $Q'\cong\mathscr{H}\times 2$. In Proposition 2.8, we shall classify $2B$-pure $2^3{:}7$ subgroups in groups of type E_7. Although this result will not be needed later in this paper, it does show that the 2-local subgroups of Sects. 3, 4 and 5 must all be found inside $E(\Delta)$ (if they are in a group of type E_7). Moreover, Proposition 2.8 serves as a useful starting point for other interesting embedding questions, for example, a classification of Alt_8 subgroups in groups of type E_7. A direct proof of 2.8 turns out to be rather tricky for non-algebraically closed fields (because the normalizer of a $2B$-pure 2^3 subgroup of E_8 becomes somewhat more complicated). We therefore begin by proving (2.8) for algebraically closed fields and then using the method of [SpSt] to pass to finite fields. In Theorem 4.7 we shall take a similar approach to a classification of subgroups of type $2^{3+3'}{:}7$ in E_7. For our application of [SpSt], we shall construct a homogeneous space M whose points correspond to $2^3{:}7$ subgroups of an algebraic group $2\cdot E_7(\bar{K})$. By construction, M admits action by conjugation of the group $2\cdot E_7(\bar{K}){:}\langle\sigma\rangle$, where σ is the Frobenius automorphism with fixed subfield K. The method of [SpSt] applies Lang's theorem [Lang] to just this situation and shows that the first cohomology set for the action of σ on the component groups of the $2\cdot E_7(\bar{K})$-stabilizer of any element of M^σ (in our case, these cohomology sets are isomorphic to the component groups) are in bijection with the $2\cdot E_7(\bar{K})$-conjugacy classes of members of M^σ (which in our case give the conjugacy classes of $2B$-pure $2^3{:}7$ subgroups of $2\cdot E_7(\bar{K})$).

2.8. Proposition. *Let K_0 be a field of characteristic not 2 which is either algebraically closed or is a finite field. Take an embedding of E in $G \cong E_8(K_0)$. There is exactly one conjugacy class in E of Frobenius groups of the shape $2^3{:}7$ for which the involutions lie in the G-class 2B.*

Proof. We first prove the result for algebraically closed fields. Later, we reduce the case of a finite field to this case.

Recall [CoGr1, Gr4] that, when the field is algebraically closed, G has two classes of involutions, $2A$ and $2B$, with respective centralizers $2\cdot E_7 A_1$ and $2\cdot D_8$; see Sect. 1.

Proving the required result is equivalent to showing there is one class of pairs (u, X), with respect to G-conjugation, where $u \in 2A$, X is a subgroup of the E_7-factor of $C_G(u)$ and $X \cong \mathrm{Frob}_{56}$. We now classify such pairs.

Let $R := O_2(X)$. Then, by the E_8-conjugacy result [CoGr1, (3.8.i)], $C_G(R) \cong 2^{4\cdot} A_1^8(K_0)\cdot 2^3$ and $N_G(R) \cong 2^{4\cdot} A_1^8(K_0)\cdot 2^3\cdot \mathrm{GL}_3(2)$ and $R \leq Z(C_G(R)^0) \cong 2^4$.

Fix an element $h \in N_G(R)$ of order 7 which centralizes one $\mathrm{SL}(2, K_0)$-factor, say S, and conjugates the other 7 transitively; it exists since there are no outer automorphisms of order 7 of $\mathrm{SL}(2, K_0)$ and 7 is prime to the exponent of $Z(C_G(R))$. All elements of order 7 in $N_G(R) - C_G(R)$ are conjugate to a power of an element of the form hk, where $k \in S$ and $k^7 = 1$. We may assume that $X = \langle R, hk\rangle$, for some $k \in S$ of order 1 or 7. Notice that a such an element of order 7 in $N_G(R) - C_G(R)$ has centralizer in $C_G(R)$ of the form $S_0 \circ T$, where $S_0 \leq S$ is either S or a maximal torus of S and where $T \cong \mathrm{SL}(2, K_0)$ projects isomorphically modulo $Z(C_G(R)^0)$ to each of the seven factors in the nontrivial h-orbit.

We now need to argue that u is in $Z(C_G(R))$. Assume that the statement is false. The action of hk on $C_G(R)$ implies that u has the form $u = u_S u_T$, where $u_S \in S$ and $u_T \in T$. The statement amounts to $u_T \in Z(T)$. We now obtain a contradiction, assuming that $u_T \notin Z(T)$, by showing that such an involution u must be in $2B$. Assuming otherwise, we look in a $\mathrm{PSL}(2, K_0)$-subgroup of $C_{C_G(R)}(h) \cong \mathrm{SL}(2, K_0) \circ \mathrm{SL}(2, K_0)$ which contains u to get a four group V containing u. Letting w be the unique involution in $C_G(hk) \cap Z(C_G(R))$, we then have that $w \in 2A$ and that $D := \langle w, V\rangle$ is a $2A$-pure eights group (it helps to notice that D is in a 2^{1+4} subgroup of $C_{C_G(R)}(h)$). Therefore, $C_G(D) = D \times Y$, where $Y \cong F_4(K_0)$ [CoGr1, (3.9)]. We get $R = [R, h] \leq Y$. This is a contradiction since Y does not contain a $2B$-pure eights group; see [Gr4, (7.3.ii)].

We now have that u is in $Z(C_G(R))$, whence R determines u up to conjugacy in $N_G(R)$. However, the requirements that X be in the $E_7(K_0)$-component of $C_G(u)$ and $hk \in X$ imply that $k = 1$. Thus, up to appropriate conjugacies, R determines u which in turn determines X, and we are done.

Now to reduce the case of a finite field K_0 to that of its algebraic closure $\bar{K}_0$. Let σ be the Frobenius map with fixed point subfield K_0. Fix a matrix δ of order 7 in $\mathrm{GL}_3(2)$. We use the following set in the role of the homogeneous space discussed in [SpSt, (2.7) and (2.8.a)]: the group is $E_7(\bar{K}_0)\langle\sigma\rangle$ and the action is by conjugation on the space of all ordered quadruples (a, b, c, d) in $2E_7(\bar{K}_0)$, where a, b, c generate a $2B$-pure eights group and a, b, c, d generate a Frobenius group of order 56 with d acting by the matrix δ with respect to the basis a, b, c, of $\langle a, b, c\rangle$. Nonemptiness and homogeneity of this space (i.e., transitive action) follow from the fact that $k=1$ (in above notation), [CoGr1, (3.8.i)] and the fact that the isotropy subgroup here is the connected group T (see above notation). We now have verified the hypothesis of [SpSt, (2.8.a)], which then proves the result for K_0. QED

2.9. Proposition. *There is an E-invariant nonsingular alternating bilinear form on its 56-dimensional irreducible (the one which occurs within the E_8 adjoint module).*

Proof. Some extra care is needed for arbitrary fields of characteristic not 2. We begin with a nonzero invariant symmetric bilinear form f on M, the E_8 adjoint module. Such an f exists; for instance see the definition of the Lie algebras associated to a root system as given in [FLM, Kac, Se]. Let z be the involution in $Z(E)$, so that the centralizer of z in E_8 is EA, where A is a central factor and a fundamental $\mathrm{SL}(2, K)$. The restriction of f to the z-eigenspaces M_+ and M_- are nonsingular. A study of the weights shows that $M_- \cong V \otimes W$, where V is our 56-dimensional module and W is the natural 2-dimensional module of A. Since $W \otimes W \cong W_3$, where W_i is irreducible of dimension i, V is embedded in $M_- \otimes W_1$ as the set of invariants of A. Since W_1 carries a nonsingular alternating bilinear form, $M_- \otimes W_1$ and hence V do too. QED

3 The $U_3(8)$ 2-local

We shall be dealing with subgroups of a group $U_3(8){:}12$ as discussed in Sect. 1. Here we give some further details about this group. The outer automorphism group of $U_3(8)$ is $3{:}6 \cong \Sigma_3 \times 3$. The appropriate part of the character table [Atlas, p. 64], for the upward extensions of $U_3(8)$ to $U_3(8){:}6$, is headed by classes $1A$ through $21F$, $3D$ through $12C$, $2B$ through $18I$, $6G$ through $24B$; the remaining outer classes are gotten from these classes by taking inverses. It is a routine matter to deduce the classes, orders of elements and the power maps for $U_3(8){:}12$ from those of $U_3(8){:}6$.

Now, we begin our construction of the relevant Borel. We use the notation introduced in (2.3) and (2.4).

3.1. Notation. Take $U := \langle h_A, n_A \mid A \in \mathscr{H}\rangle$.

3.2. Proposition. *U is isomorphic to a Sylow 2-group of $U_3(8)$.*

Proof. Recall the notations of (2.5). The commutator subgroup of U is generated by all elements of the form $z_{A\cap B} = [n_A, h_B]$, for all $A, B \in \mathscr{H}$. Therefore, as all intersections of subsets in $\mathscr{H}$ are even, U' has order 8 and in fact equals $\{z_A \mid A \in \mathscr{PE}(\Delta)\}$. Every element outside the center has the form $h_A n_B z_C$, for $A, B \in \mathscr{H}$, not both 0, and for $C \in \mathscr{PE}(\tilde{\Delta})$. Now $(h_A n_B z_C)^2 = z_{A\cup B}$, and we note that $A \cup B$ can not be in $\mathscr{H}'$ since it is either in $\mathscr{H}$ or it has weight 6; we conclude that $(h_A n_b z_c)$ has order 4. Thus, since $O_7(F)$ acts transitively on the involutions of U, we have a "Suzuki 2-group" and we refer to the classification [Hig] to get the isomorphism type of U (in the notation of [Hig], type B, with $\theta = 1$). QED

3.3. Notation. Let K have a primitive eighth root of 1; thus, $L_r \cong 2\cdot\Sigma_4$. Now let $B:=\langle U, F, \theta, \phi\rangle$.

3.4. Proposition. *The group B has structure 2^{3+6}:($\mathrm{Frob}_{21} \times 3{:}4$) and it is isomorphic to a Borel subgroup of $U_3(8){:}12$.*

Proof. We observe that F, θ and ϕ all normalize U; this shows that B has the indicated structure. We note that the Borel subgroup of $U_3(8){:}12$ also has this structure. Now, [Land, Theorem 1.1], determines the automorphism group of a Sylow 2-group of $U_3(q)$, $q=2^n$, to be a solvable group of the form $2^{2n^2}{:}M$, where $M\cong(q^2-1){:}2n$ is isomorphic to the normalizer of a Singer cycle in $\mathrm{GL}_{2n}(2)$. Hence $\mathrm{Aut}(U)\cong 2^{18}{:}(63{:}6)$ and a Frattini argument shows that $\mathrm{Aut}(U)$ has a unique conjugacy class of subgroups isomorphic to $\mathrm{Frob}_{21}\times\Sigma_3$. It follows that B must actually be isomorphic to the Borel subgroup of $U_3(8){:}12$. QED

We remark that the commutator subgroup, B', has structure $2^{3+6}{:}21$ and it is isomorphic to a Borel subgroup of $U_3(8)$; moreover $B''\cong U$. Proposition 3.4 provides an explicit embedding of a Borel subgroup of $U_3(8){:}12$ into $2\cdot E_7(K)$. The remaining results in this section establish properties of the automorphisms and representations of B. These results are needed to study overgroups of our Borel subgroups in $\mathrm{GL}_{56}(K)$.

3.5. Lemma. (i) $\mathrm{Aut}(B')\cong 2^{3+6}{:}J$, *where $J\cong 63{:}6$ is isomorphic to the normalizer of a Singer cycle in $\mathrm{GL}_6(2)$.*
(ii) $\mathrm{Aut}(B)\cong 2^{3+6}{:}[\mathrm{Frob}_{21}\times\Sigma_3]\times 2$.

Proof. (i) Let $A:=\mathrm{Aut}(B')$, and let $R:=\{\alpha\in A\mid\alpha$ is trivial on U/U' (and so on U')$\}$, a normal subgroup. Identify $U_3(8)$ with its group of inner automorphisms and let B_1 be the normalizer in $\mathrm{Aut}(U_3(8))$ of U. We identify B_1 with its image in A. Let S be a subgroup of B_1 with structure $63{:}6$, so that $S\cap B'$ is the unique cyclic normal subgroup of order 21 in S. We use a Frattini argument to get $A=B'N$, where $N:=N_A(S\cap B')$. By [Land] (see (3.4)), $N/(R\cap N)$ is contained in a group with structure $63{:}6$, and thus $N=(R\cap N)S$. Now $[N\cap R, S\cap B']\leq R\cap S\cap B'\leq R\cap S=1$. Thus $R\cap N\leq C:=C_R(S\cap B')$. Since the irreducible $S\cap B'$-modules U/U' and U' are nonisomorphic, $[C,U]=1$. Therefore, $C=1$ since we are in $\mathrm{Aut}(B')$. We conclude $A=B'N=B'S=B_1$.
(ii) Let π be the natural map from $\mathrm{Aut}(B)$ to $\mathrm{Aut}(B')\cong 2^{3+6}{:}J$, and let $D=\ker(\pi)$. Then $D\cap\mathrm{Inn}(B)=1$, since $Z(B)=C_B(B')$. Thus $\pi(\mathrm{Inn}(B))\cong\mathrm{Inn}(B)\cong 2^{3+6}{:}[\mathrm{Frob}_{21}\times\Sigma_3]$. It follows that $\pi(\mathrm{Inn}(B))$ is self normalizing in $\mathrm{Aut}(B')$; hence $\pi(\mathrm{Inn}(B))=\pi(\mathrm{Aut}(B))$. We conclude that $\mathrm{Aut}(B)=\mathrm{Inn}(B)\times D$. For any group G, there is a natural inclusion of $C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$ in $\mathrm{Hom}(G,Z(G))$ and the image of this inclusion contains the subset $\{\phi\in\mathrm{Hom}(G,Z(G))\mid\phi^2=0\}$. For B, both objects have cardinality 2 and so D does as well. QED

3.6. Lemma. *The Borel subgroup of $U_3(8)$ is absolutely irreducible on any faithful 56-dimensional module. Such a module is unique up to tensoring by linear characters of order 3. Exactly one of these three representations gives an embedding into the special linear group of degree 56.*

Proof. Let $B'\cong 2^{3+6}{:}21$ be the indicated Borel subgroup. Clifford theory shows that an irreducible representation which is nontrivial on $A:=Z(O_2(B'))$ is induced from an irreducible representation of $B_0:=C'_B(A/A_0)$, for some hyperplane A_0 of A; furthermore, such a representation faithfully represents A/A_0. For any such A_0, $O_2(B')/A_0 \cong 2_+^{1+6}$ and $|B':B_0|=7$. It follows that the degree is at least 56, as required. Furthermore, there is such a representation of degree 56 since the faithful irreducible representation of 2_+^{1+6} extends to any cyclic extension of it by a group of outer automorphisms. The second statement follows from the first since $B'/B'' \cong 21$ and any irreducible character of degree 56 is 0 on 7-singular elements and nonzero on elements of order 3. To prove the third statement, we use the fact that elements of order 7, having trace 0, must have determinant 1 and that all such generate the unique subgroup of index 3 in B'. So, the question of whether the representation lies in the special linear group is answered by looking at the elements of order 3. The third statement now follows since $(3, 56)=1$. QED

3.7. Lemma. *If X is a subgroup of $\mathrm{GL}_{56}(K)$ isomorphic to B, then $N(X)/C(X)$ has trivial center.*

Proof. If false, let $a \in N(X)$ be a matrix which maps to the central direct factor (recall $\mathrm{Aut}(X) \cong 2 \times \mathrm{Inn}(X)$). Then a centralizes X', which is an absolutely irreducible subgroup, so a is a scalar. QED

4 The $Sz(8)$ 2-local

We maintain the notations of (2.3) and (2.4) and assume only that $i \in K$.

4.1. Notation. Define $V := \langle h_A, n_A \mid A \in \mathscr{H}' \rangle$.

Since $[h_A, n_B] = z_{A \cap B}$, V has shape 2^{3+6}. Since there are 3 irreducible F-submodules of V/V', the group $C_L(F) \cong 2 \cdot \Sigma_4$ or $\mathrm{SL}_2(3)$ operates transitively on them; they are generated modulo V' by, respectively, all $h_A, n_A, h_A n_A$, for $A \in \mathscr{H}'$, so the preimage in V of such a submodule is elementary abelian.

We consider the nine $O_7(F)$ submodules of V/V'.

4.2. Notation. Let U be a subgroup between V and V' such that U is stable under $O_7(F)$ but not under F.

The rule $[h_A, n_B] = z_{A \cap B}$ implies that U is nonabelian. Notice that the set of six such U are all isomorphic since $C_L(F)$ acts transitively whenever K contains an eighth root of unity (and we can extend the field to achieve this otherwise). Such a group U is isomorphic to a Sylow 2-group of $Sz(8)$ [Hig].

4.3. Lemma. *The group* $\mathrm{Aut}(U)$ *has structure* $2^9{:}\mathrm{Frob}_{21}$.

Proof. Let $A := \mathrm{Aut}(U)$, and let $R := \{\alpha \in A \mid \alpha \text{ is trivial on } U/U' \text{ (and so on } U')\}$, a normal subgroup. Since $Z(U) = U'$, we have $R \cong \mathrm{Hom}_{\mathbf{Z}}(U/U', Z(U)) \cong \mathrm{Hom}_{\mathbf{Z}}(2^3, 2^3) \cong 2^{3 \otimes 3} \cong 2^9$.

Now, inside the group $Sz(8){:}3$, there is a subgroup isomorphic to U which is normalized by a subgroup isomorphic to Frob_{21}. Therefore, $\mathrm{Aut}(U)$ is at least as large as $2^9{:}\mathrm{Frob}_{21}$. In order to prove that it is no larger we now show that the image of $\mathrm{Aut}(U)$ in $\mathrm{GL}_3(2) \cong \mathrm{Aut}(Z(U))$ is a proper subgroup.

We observe that the action of $\mathrm{GL}_n(2)$ preserves a unique group structure on the set $\mathbb{F}_2^n$ since the stabilizer of two nonzero vectors stabilizes a unique third nonzero vector.

Let $\alpha: A \to \mathrm{Aut}(U/U')$ and $\beta: A \to \mathrm{Aut}(Z(U))$ be the natural homomorphisms. Let $\sigma: U/U' \to Z(U)$ be the squaring map. The map σ is not a group homomorphism (since in the nonabelian group U, $[x, y] = x^2 y^2 (xy)^2$). Therefore the natural group structure on $Z(U)$ is distinct from the linear structure obtained as the σ-image of the natural structure on U/U'. The map σ gives rise to a map $\tilde{\sigma}: \theta \mapsto \sigma^{-1} \theta \sigma$: $\mathrm{Aut}(U/U') \to \mathrm{Aut}(Z(U))$. Moreover, since squaring commutes with conjugation, $\alpha \tilde{\sigma} = \beta$. Hence $\mathrm{Im}(\beta) = (\mathrm{Im}(\alpha))\tilde{\sigma}$ preserves both linear structures on $Z(U)$ and is therefore a proper subgroup of $\mathrm{Aut}(Z(U))$. QED

The lemma shows that there is an essentially unique action of Frob_{21} on U which justifies our choice of the Suzuki parabolic in (4.5).

4.4. Notation. Define H to be a Sylow 3-group of $N_{C_L(F) \times F}(U)$; thus, $H \cong 3$.

4.5. Notation. *The Suzuki Parabolic.* Set $B := U O_7(F) H$.

4.6. Lemma. $B \cong \mathrm{Aut}(B) \cong \mathrm{Aut}(B') \cong 2^{3+3'}:[7:3]$.

Proof. We first prove that $\mathrm{Aut}(B') \cong 2^{3+3'}:[7:3]$. Let $A := \mathrm{Aut}(B')$ and regard B as a subgroup of A; we have a homomorphism α from A to $\mathrm{Aut}(U/U') \cong \mathrm{GL}_3(2)$, whose image is at least $\alpha(B) \cong \mathrm{Frob}_{21}$. Moreover, since α factors through $\mathrm{Aut}(U \cong 2^9:\mathrm{Frob}_{21}$ we must have $\mathrm{Im}(\alpha) \cong \mathrm{Frob}_{21}$.

Let $R := \{a \in A \mid a$ is trivial on U/U' (and therefore on U')$\}$. By the Frattini argument, we have $A = U N_A(C)$, where C is a Sylow 7-subgroup of B'. Note also that $[C, N_R(C)] \leqq C \cap R = 1$, so that $N_R(C) = C_R(C)$. However, $C_R(C)$ induces C-module homomorphisms from U/U' to U', which are irreducible and non-isomorphic. Therefore, $C_R(C) \leqq C_R(U)$, whence $C_R(C) \leqq C_R(B') = 1$. Thus, $N_R(C) = 1$ and so, by α, $N_A(C)$ embeds in a Frobenius group of order 21. We now have the structure of $\mathrm{Aut}(B')$.

The structure of $\mathrm{Aut}(B)$ follows since $C_{\mathrm{Aut}(B)}(B')$ is a normal subgroup of $\mathrm{Aut}(B)$ which meets $\mathrm{Inn}(B)$ trivially, hence is the identity. QED

4.7. Theorem. *Suppose that K_0 is a field of characteristic not 2. If K_0 is algebraically closed: up to conjugacy, there is just one subgroup of $E_0 := 2E_7(K_0)$ isomorphic to $B' \cong 2^{3+3'}:7$; the centralizer of such a subgroup is isomorphic to* Quat_8. *If K_0 is finite, the number of conjugacy classes is* 3; *representing subgroups have centralizers isomorphic to, respectively,* Quat_8 *(twice),* $\mathbb{Z}_4$ *(once).*

Proof. We first deal with the case of an algebraically closed field.

Let X be a subgroup isomorphic to B'. We must show that X is conjugate to B'. We consider E_0 as a subgroup of $E_8(K_0)$.

We claim that the involutions of $Z(X')$ are of E_8-type $2B$. If not, they are all of type $2A$ and so the centralizer of $Z(X')$ in $E_8(K_0)$ has shape $Z(X') \times F_4(\bar{K}_0)$ [CoGr1, Gr4], while this group must also contain X', a contradiction. The claim follows.

The claim implies that $Z(X')$ is in a maximal torus of $E_8(K_0)$ [CoGr1, (3.8.i)]. Therefore, $Z(E_0) Z(X')$ is in a maximal torus T of E_0. Thus, it corresponds to a maximal isotropic subspace of $T_{(2)} := \{x \in T \mid x^2 = 1\}$ [Gr4, (2.16)] whose singular vectors constitute $Z(X')^{\#}$. Since the Weyl group of E_0 induces on $T_{(2)}$ the full group preserving the natural quadratic form [Gr4, (2.16)], we get that $Z(X')$ is conjugate to $Z(U)$, and we now assume them to be equal.

Thus, both X and B' lie in the group $E_0(\Delta)$ defined as in (2.4). Since $E_0(\Delta)^0$ is the centralizer of $Z(U)$, U and B'' are in $E_0(\Delta)^0$. Modulo $Z(U)$, they become elementary abelian and so may be taken as subgroups of Q, whose normalizer in $E_0(\Delta)$ has shape $L\cdot GL_3(2)$; see (2.3) and (2.4).

We claim that X' projects to a fours group in the central quotient of each of the seven components of $E_0(\Delta)$. Its projection is an elementary abelian group of order 2 or 4. This group is invariant under X, which permutes the seven components transitively by conjugation. If the projections all had order 2, X' would be abelian. The claim follows. Therefore, $C_{E_0}(X'/Z(E_0(\Delta)^0))$ is Q, which means that $X \leqq N(X') \leqq N(Q)$, whose shape was described above. If $x \in X$ has order 7, we find that $C_Q(x) \cong \mathrm{Quat}_8$ and equals $C_E(X)$ since $C(X') \leqq Q$. Note that, by Sylow's theorem for the prime 7, $N(X')$ induces Alt_4 on $C_Q(X)$.

There is one $E_0(\Delta)$ conjugacy class of groups of order 7 in $E_0(\Delta)$ which does not lie in $E_0(\Delta)^0$. By Sylow's theorem, we may assume that X and B' contain the same subgroup Y or order 7; thus $X \cap B'$ contains a Frobenius group of order 56. We get the rest of O_2 of X and B' by choosing a Y-submodule of Q/Q'. In fact, Q/Q' has the required 3-dimensional irreducible with multiplicity 2 and so there are exactly three submodules of the required form. All such form an orbit under a group of order 3 in $C_L(Y)$. We conclude that X and B' are conjugate.

In the case of finite fields, we shall finish in the same style as at the end of the proof of (2.8).

In the group B' from (4.5), there are generators (a, b, c, d), where $O_2(B') = \langle a, b, c \rangle, d$ is an element of order 7; there are words $w_i(x, y, z)$ in the free group on symbols x, y, z such that

$$(*) \qquad a^d = w_1(a, b, c), \quad b^d = w_2(a, b, c \quad \text{and} \quad c^d = w_3(a, b, c).$$

The homogeneous space we take for our application of Lang's theorem is that of all quadruples (a, b, c, d) where the group, say X, generated by a, b, c, d is isomorphic to B' and where (a, b, c, d) satisfy $(*)$.

The stabilizer of such a quadruple (= centralizer in E_0) is a quaternion group of order 8, as proven above. Then [SpSt, (2.8.a)] gives 5 conjugacy classes (= number of conjugacy classes in Quat_8) of such quadruples. The stabilizers are the centralizers in Quat_8 for the various conjugacy classes, i.e., Quat_8 twice and $\mathbb{Z}_4$ three times. Above remarks on the action of $N(B')$ on $C(B')$ imply that the latter three classes of quadruples correspond to a single class of subgroups isomorphic to B'. QED

4.8. Remark. Suppose K_0 is finite. There is just one $2E_7(K_0)\cdot 2$-conjugacy class of subgroups of $2E_7(K_0)$ isomorphic to $\mathrm{Quat}_8 \times 2^{3+3'}{:}7$; this follows from an argument like (4.7) applied to the adjoint form $E_7(\bar{K}_0)$ and noticing that the two cases in (4.7) of Quat_8-centralizer for $2^{3+3'}{:}7$ collapse to one case here; the fixed point subgroup of the Frobenius endomorphism is $E_7(K_0)\cdot 2$, and this slight enlargement explains fusion of the above two $2E_7(K_0)$-classes in $2E_7(K_0)\cdot 2$.

5 The Rudvalis parabolic

In Sect. 5, we require only that $\mathrm{char}(K) \neq 2$ but do need to use $\bar{K}$, the algebraic closure. We give a description of how the parabolic $2\cdot 2^{3+8}GL_3(2)$ of $2\cdot Ru$

embeds in E. In Sect. 7, where $K=\mathbb{F}_5$, we need only a subgroup of this group of the form $2^{4+8}\cdot[7:3]$ to carry out our construction of a Ru-subgroup of E. To locate a useful subgroup of E, it seems easier to work initially with subgroups of shape $2\cdot 2^{3+8}\mathrm{GL}_3(2)$. The right subparabolic $2^{4+8}\cdot[7:3]$ is explicitly defined here (5.7) but we get a correct $2\cdot 2^{3+8}\mathrm{GL}_3(2)$ subgroup of E only by use of Sect. 7.

Aspects of this 2-local $2\cdot 2^{3+8}\mathrm{GL}_3(2)$ and its image in Ru have been discussed in, e.g., [D, GMS, Wi]. In [Gr2], a code-theoretic description of the group $2^{3+8}\mathrm{GL}_3(2)$ is given. We give a setting for the double cover $2\cdot 2^{3+8}\mathrm{GL}_3(2)$ which does involve codes, but in a different role. We mention that the group $2\cdot 2^{3+8}\mathrm{GL}_3(2)$ of $2\cdot Ru$ contains a subgroup isomorphic to $\mathrm{GL}_3(2)$; see, e.g. [Wi]. See also [Ru1, Ru2, I, II].

5.1. Notation. Let $E:=2E_7(K)$ and $\bar{E}:=2E_7(\bar{K})$. We maintain the notation of (2.2) and (2.3). In addition, we let $Y\cong 2^4\cdot\mathrm{GL}_3(2)\leqq\bar{E}(\Delta)$ satisfy $Y\cap Z(\bar{E}(\Delta))=O_2(Y)$, $Y>F$ (see (2.4)) and $Y/O_2(Y)$ acts on the set of factors by permuting the coordinates. We may and do assume that $Y\leqq N_{\bar{E}(\Delta)}(Q)$. Let Y_1 be a group between $O_2(Y)$ and Y which satisfies $Y_1/O_2(Y)\cong\mathrm{Frob}_{21}$ and is contained in E.

We see that such Y and Y_1 exist as follows. We note that the algebraic automorphism group (i.e., no field automorphisms) of a direct product of n copies of $\mathrm{PSL}_2(\bar{K})$ is complemented over the inner-diagonal automorphisms by a copy of the symmetric group acting via coordinate permutations; furthermore, all such complements are conjugate. We then take $n=7$, and let Y correspond to a copy of our $\mathrm{GL}_3(2)$ in the complementing group of direct factor permutations. If we attempt to do this over K, we must consider outer diagonal automorphisms which appear in the quotient of $E(\Delta)$ modulo the subgroup E_0 of type A_1^7 generated by root groups. The quotient $E(\Delta)/E_0$ has a normal subgroup $C(Z(E(\Delta)))/E_0$ of exponent 2 and quotient $E(\Delta)/C(Z(E(\Delta)))\cong\mathrm{GL}(3,2)$. There is a splitting question here. However, there is splitting if we restrict the group on top to a Frob_{21} subgroup. This gives existence of Y_1 in $E(\Delta)$. Upon extending the field to $\bar{K}$, we see that Y is conjugate in $\bar{E}(\Delta)$ to a subgroup containing Y constructed above.

5.2. Lemma. (i) $C_{\bar{E}(\Delta)}(Y)\cong\mathrm{SL}_2(K)$.
(ii) $Y'\cong 2^3\cdot\mathrm{GL}(3,2)$, *so that* $Y\cong 2\times 2^2\cdot\mathrm{GL}(3,2)$.

Proof. (i) Since the action of Y is transitive on the seven components of $\bar{E}(\Delta)$, $C_{\bar{E}(\Delta)}(Y)$ is isomorphic to one of $\mathrm{SL}(2,K)$ or $2\times\mathrm{PSL}(2,K)$. In case of the latter, we have seven elements h_r of order four, for $r\in\Delta$ such that $h_\Delta=1$. But then we deduce $z_\Delta=1$, a contradiction since z_Δ generates $Z(\bar{E})\cong 2$.
(ii) In $C_{\bar{E}(\Delta)}(Y)$, take a subgroup $S\cong\mathrm{Quat}_8$. From [Gr4, (9.8)], we know that $C_{\bar{E}}(S)\cong 2\times X$, where $X\cong\mathrm{PSO}(8,K)$ or $F_4(K)$. Since $Y\leqq C_{\bar{E}}(S)$ and $Y\cap S=Z(S)=Z(C_{\bar{E}}(S))$, (ii) follows. QED

Thus, Q/Q' is isomorphic to the degree 7 permutation module for $\mathbb{F}_4[Y/O_2(Y)]$; note that we may use the action of $C_L(Y)\cong 2\cdot\Sigma_4$ on Q/Q' to get a copy of $\mathbb{F}_4^\times$ extended by field automorphisms in the commuting algebra. The group Y, therefore, permutes the A_1-factors of $E(\Delta)$ in the natural way. However, we need a different group of shape $2^3\cdot\mathrm{GL}_3(2)$ and this requires a result in "nonabelian cohomology of groups". Recall the interpretation of the first cohomology group as conjugacy classes of complements in a split extension and recall the Eckmann-Shapiro Lemma [Gruen, p. 91].

5.3. Lemma. *Let* $X := \Sigma_3$ *wr* $GL_3(2)$ *with given base group* $J = O_{2,3}(X) \cong \Sigma_3^7$ *and complement* $W \cong GL(3,2)$ *acting by a transitive permutation representation of degree 7. Let* $T \cong \Sigma_3$ *be one of the seven indecomposable direct factors of J and let* $S \cong \Sigma_4$ *be the subgroup of W which centralizes T.*

(i) *There are three conjugacy classes of subgroups of X isomorphic to* $GL_3(2)$.

(ii) *These conjugacy classes are distinguished in the following manner. If* W_0 *is a subgroup of X isomorphic to* $GL_3(2)$, *let* S_0 *be the subgroup of* W_0 *which is congruent to S modulo J. The projection of* S_0 *to the first factor of* $T \times S$ *is a group of order* 1, 2 *or* 6, *and this order determines the conjugacy class of* W_0. *Correspondingly, we call these subgroups isomorphic to* $GL_3(2)$, *untwisted, 2-twisted or* Σ_3*-twisted* $GL_3(2)$.

(iii) *Equivalently, the conjugacy class is determined by* $C_X(W_0) = C_J(W_0)$, *which is isomorphic to a subgroup of respective order* 6, 2 *or* 1 *of* Σ_3.

Proof. (i) We can use cohomology of groups for the action of W on J by piecing together information from the action on the abelian groups $O_3(J)$ and $J/O_3(J)$. For action on $J/O_3(J)$, the Eckmann-Shapiro Lemma gives two conjugacy classes of complements to $J/O_3(J)$ in $X/O_3(J)$ since $H^1(X/J, J/O_3(J)) \cong H^1(\Sigma_4, \mathbb{Z}_2) \cong \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$. Now consider the preimage of each over $O_3(J)$ to get an extension of $GL(3,2)$ by $O_3(J)$.

For $O_3(J)\,W$, the Eckmann-Shapiro Lemma gives one conjugacy class of subgroups isomorphic to W since $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_3) = 0$. Let $W_1 \cong GL_3(2)$ be a subgroup of X such that $O_3(J)\,W$ and $O_3(J)\,W_1$ are not conjugate in X/J. We apply the Eckmann-Shapiro Lemma to the action of W_1 on $O_3(J)$ noting that the subgroup S_1 of W_1 defined by $JS = JS_1$ inverts the direct factor $O_3(T)$ under conjugation whereas the action of S centralizes it. It follows from the Eckmann-Shapiro lemma that $H^1(W_1, O_3(J)) \cong H^1(S_1, O_3(T)) \cong 3$. Let W_2 and W_3 be the two additional complements to $O_3(J)$ in $O_3(J)\,W_1$ obtained by this procedure and let S_i, $i = 2, 3$, be their respective subgroups congruent to S_1 (or S) modulo J. We finish by proving that W_2 and W_3 are conjugate in X, even by an involution $u \in C_T(S_1) \cong 2$. We see this by noticing that $S_1 \cap S_i \cong \mathrm{Dih}_8$ and that we get S_i from S_1 by replacing an element s of order 3 in S_1 by $m^{i-1}s$ for $i = 2, 3$, where m generates $O_3(T)$, i.e., $S_i = \langle S_1 \cap S_i, ms^{i-1} \rangle$; consequently, $(ms)^u = m^{-1}s$.

(iii) The correspondence of these classes of complements with their indicated fixed points on J is an exercise. We hint that, if Z is a transversal to S_i in W_i, the map $x \mapsto \prod x^z$, $z \in Z$, gives an isomorphism of $C_T(S_i)$ with $C_J(W_i)$. QED

5.4. Notation. We maintain the notations of Sect. 2 and (5.1). We take a group W_0 satisfying $W_0 \cap Y = O_2(T) = Q' \cong 2^4$ and $TY = TW_0$ (see (5.1)), with W_0 corresponding, in the notation of (5.2.ii), to a Σ_3-twisted $GL_3(2)$-subgroup of $TY/O_2(T)$. We also arrange for $Y_1 \leqq W_0$ and we let W_1 denote a subgroup of Y_1 isomorphic to Frob_{21}. For each $r \in \Delta$, let T_r be a 3:4 subgroup of L_i normalized by a Σ_4-subgroup of W_0 and let $T := T_\Delta$.

5.5. Lemma. Q/Q' *has constituents of isomorphism type* 3, 3′, 8 *for the action of* $W_0/O_2(W_0) \cong GL_3(2)$ (*the* 8 *designates the Steinberg module*).

Proof. This just amounts to establishing the Brauer character. The only odd orders here are 1, 3 and 7. Since the module here is induced from a module for a Σ_4-subgroup, the trace for an element of order 7 is 0. If x is an element of order 3 in W_0, it permutes the seven groups Q_r with a single fixed point. If Q_r is the unique such group normalized by x, x acts by a nontrivial outer automorphism, due to the Σ_3-twisting. It follows, if ω denotes a primitive cube root of unity in an algebraic closure of $\mathbb{F}_4$, that on Q/Q', the multiplicities of 1, ω and ω^{-1} as eigenvalues of x are 4,5 and 5, respectively. QED

5.6. Lemma. *Let Q_3 and $Q_{3'}$ be subgroups of Q containing Q' whose images in Q/Q' are W_1-submodules which cover the W_0-composition factors of type* 3 *and* 3' *as indicated above and such that $Q_3 Q_{3'}$ is invariant under W_0. Then,*
(i) *If $Q_{3'}$ is invariant under W_0, it is elementary abelian;*
(ii) $Q_3 \cong 4^3 \times 2$.

Proof. (i) Suppose false. If $Q_{3'}/Z(E)$, were nonabelian, the action of W_1 implies that its quotient modulo $Z(E)$ would be isomorphic to a Sylow 2-group of $Sz(8)$. Since the group $Q_{3'}/Z(E)$ has $GL_3(2)$ involved in its outer automorphism group, it is abelian (we have used this argument in the proof of (4.3)). Since it has two different nontrivial $GL_3(2)$ composition factors, it is elementary abelian. Finally, if $Q_{3'}$ were nonabelian, it would be extraspecial, due to the action of W_0. But this is impossible since every coset of $Z(E)$ in Q' consists of involutions from different E-conjugacy classes.
(ii) The set of composition factors for the action of W_1 implies that the group is abelian. Suppose that it is elementary abelain. From Table 1 of [Gr4], we see that it is toral. However, the composition factors for the action of W_1 make it impossible to support the natural quadratic form inherited from a torus [Gr4, (2.15)], contradiction. QED

5.7. Notation. Let Q_0 be the subgroup between Q and Q' which corresponds to the Steinberg module, as in (5.5). Let $P_1 := Q_0 W_1 \cong 2^{4+8}{:}\mathrm{Frob}_{21}$. We call P_1 our *standard Rudvalis subparabolic.*

The results of Sect. 7 imply that there is a group P between P_1 and QW_0 with P isomorphic to the Rudvalis parabolic of shape $2^{4+8}{:}GL(3,2)$. Naming the exact group P at this point seems hard.

5.8. Notation. $Q_\infty := Q_3 Q_{3'}$.

5.9. Corollary. $Q_\infty \cong 2^{1+3+3+3'} \cong [2 \times 4^3] \cdot 2^{3'}$ *(with compostion factors as indicated for $W_0/O_2(W_0) \cong GL_3(2)$). Then $Q_\infty = C_Q(Q_\infty)$. With respect to this operator group, we have that $Q_\infty/Z(Q_\infty)$ is an indecomposable module with ascending factors* 3, 3' *and Q_∞ has an invariant subgroup of the shape 2×4^3. Also, $Q_3/Z(E) = Z(Q_\infty/Z(E))$.*

Proof. We give only a sketch. If there were a $W_0/O_2(W_0)$-invariant subgroup, U, which is elementary abelian of rank 6, we can use the results of [Gr4, (1.8), Table 1], to get a contradiction as follows. Such a group must be toral, due to the action of W_0, whence $\langle U, Z(E) \rangle$ is the set of elements of square 1 in a maximal torus, T. Its centralizer is $\langle T, u \rangle$, where u^2 is the involution in $Z(T)$ and u inverts T. Since Q_0 is in the centralizer and both are stable under W_0, we get that $Q_0 = [Q_0, W_0] \leqq T$, which is a contradiction since Q_0 is nonabelian. So, no such U exists.

In case there is a nonabelian section of shape $2^{3+3'}$, we get a contradiction as in the proof of (3.5.i).

When we consider the commutation map from $Q_\infty \times Q_0$ to $Z(Q_\infty) = Z(Q_0)$, we deduce from $3 \otimes 3' \cong 8 \oplus 1$ that $C_{Q_\infty}(Q_0)$ equals Q_∞ or a proper subgroup of it with factors 1, 3, 3', a possiblity which has just been eliminated. This proves that $Q_\infty \leqq C(Q_0)$.

This tensor decomposition also shows that $Q_3/Z(E)$ is central in Q_∞. If the center were larger than $Q_3/Z(E)$, Q_∞ would be abelian and so $\Omega_1(Q_\infty/Z(E))$ would be a characteristic subgroup of order 64 which covers $Q_{3'}/Q'$; thus, $\Omega_1(Q_\infty)=Q_{3'}$. Let R be a W_1-invariant subgroup of Q_0 such that R/Q' represents the unique $3'$ factor within Q_0/Q'. Then, since $Q_{3'}/Z(E)$ is elementary abelian and centralizes $R/Z(E)$, the nine $O_7(W_1)$-invariant submodules of order 64 in $Q_{3'}R/Z(E)$ consist of eight groups isomorphic to $R/Z(E)$ and one elementary abelian one. This is incompatible with remarks after (4.1) and (4.2). We conclude that $Q_3/Z(E)=Z(Q_\infty/Z(E))$. QED

5.10. Corollary. *P_1 contains a subgroup of the form $B\times Q_B$, where B is isomorphic to a Borel of $Sz(8)$ and $Q_B\cong \mathrm{Quat}_8$; consequently, $|O_2(P_1)'|=16$.*

Proof. This follows from the fact that we may construct a $Sz(8)$ Borel in P_1 by performing the procedure of Sect. 4 within P_1. Letting X be a Sylow 7-subgroup of B, we find that $C_{P_1}(X)\leqq C_{P_1}(B)$ and so, by (4.7), $C_{P_1}(X)\cong \mathrm{Quat}_8$. QED

5.11. Notation. V is a 56-dimensional irreducible module for E.

5.12. Lemma. (i) *If $U:=O_2(B)$ (as in (5.10)), and Z is any hyperplane of $Z(U)$, then $U/Z\cong 2^{1+2}\circ 4$.*

(ii) *The restriction of V to QW_1 is irreducible and is induced from the subgroup $C_0:=C_{QW_1}(A/A_0)$, where $A:=Z(Q_0)$ and A_0 is a hyperplane of A which does not contain $Z(E)\cong 2$ and which is not normal in QW_1. We have $|QW_1:C_0|=7$.*

(iii) *The restriction of V to P_1 is the direct sum of two distinct, dual degree 28 irreducibles, distinguished by the traces $4i$ and $-4i$ of an element of order 4 in $O_2(B)$.*

(iv) *There are exactly 12 irreducible representations of degree 28 of P_1 with the property that the kernel intersects A trivially. Of these, only four have determinant 1 and these come in two equivalence classes of size two, distinguished by the traces of an element of order 4 in $O_2(B)$. There is an automorphism of order 2 of P_1 which acts transitively on each class.*

(v) $\mathrm{Out}(P_1)\cong 2$.

Proof. (i) This is well-known and easily proved, e.g. [AGo] or [Gr1].

(ii) Consider an irreducible QW_1-constituent of V. It is induced from an irreducible representation of C_0, say V_0. Since $|QW_1:C_0|=7$, $\dim V_0\leqq 8$. The subgroup $B\times Q_B$ of (5.10) satisfies the condition that $[B'\times Q_B]/A_0\cong 2_+^{1+4}\circ 4$. This implies that $\dim V_0$ is divisible by 4. We have that $B\times Q_B$ centralizes Q_∞, which maps to a subgroup of the commuting algebra for the action of $B'\times Q_B$ on V_0. Since Q_∞ contains an isomorphic copy of B'(in $Q_{3'}$) which does not contain A_0, the image of Q_∞ in $\mathrm{GL}(V_0)$ is nonabelian, by (i). Therefore, $\dim V_0$ is greater than 4 and so is 8.

(iii) An irreducible constituent V_0 of P_1 occurring in the restriction V_{P_1} is induced from $P_0:=C_{P_1}(A/A_0)$, where $A:=Z(O_2(P))$ and A_0 is a hyperplane of A which does not contain $Z(P_1)\cong 2$ and which is not normal in P_1. We have $|P_1:P_0|=7$. The subgroup $B\times Q_B$ of (5.10) satisfies the condition that $[B\times Q_B]/A_0\cong 2_+^{1+4}$. Therefore, the degree of a representation of P_0/A_0 faithful on A/A_0 is a multiple of 4. Thus, $\dim V_0$ is divisible by 28.

We may assume that $P_0=C_0$, from (ii). Thus, we see that P_0/A_0 has such a representation of degree 4.

A degree 28 representation of P_1 is irreducible and is induced from an irreducible degree 4 representation of P_0, say λ. Let $M_0:=\ker\lambda$. Then any representation of $P_0/M_0 \cong [2^{1+4} \circ 4]:3$ whose restriction to $Q_0/M_0 \cong [2^{1+4} \circ 4]$ is identfied with a multiple of $\lambda | Q_0$ has the form $\lambda \otimes \alpha$ where α is the inflation of a representation of P_0/Q_0 to P_0/M_0; see [CuRe, (51.7)]. Clifford theory implies that distinct irreducible representations of $P_0/Q_0 \cong 3$ give 3 distinct irreducible representations of P_1 and all of these have degree 28.

Let $h \in P_0$ have order 3 and let $T := C_{P_0}(h)$, a group of order 16. We have that $[O_2(B) \times Q_B, h] \cong 2_+^{1+4}$. Since $[T, [O_2(B) \times Q_B, h]] = 1$, it follows that $T' \leqq A_0$ and so $T/T \cap A_0 \cong 2 \times 4$. It follows that any such M_0 is a subgroup of Q_0 and that, given A_0 as above, there are exactly two groups M_0 which intersect A in A_0.

If f is an element of order 4 in B corresponding to the central factor in (i), f has trace $4i$ or $-4i$ on V_0 and trace 0 on the other 6 transforms of V_0 by P_1 (one can see this by looking at V_B). Since V has an E-invariant alternating bilinear form (2.9), V_{P_1} has a self dual character. Above remarks about an element of order 4 imply that V_{P_1} is a sum of distinct, dual degree 28 irreducibles.

(iv) The count follows from the proof of (iii). In more detail, we observe that such a representation is induced from a subgroup of index 7 in P_1; such a subgroup is conjugate to C_0. If X is the kernel of such a character, we note that $C_0/X \cap C_0 \cong [2^{1+4} \circ 4]:3$, $X \geqq [Z(Q), C_0]$ and that $X \ngeq [Z(Q), QW_1]$. Thus, there are two choices for X, two for the character on $Z(C_0/X \cap C_0) \cong 4$ and three choices for linear character of C_0.

Now to show existence of $\beta \in \mathrm{Aut}(P_1)$ with the right properties. As a module for W_1, $Q_0/\Phi(Q_0)$, which is the Steinberg module for $Y/O_2(Y) \cong \mathrm{GL}(3,2)$, breaks up as $3 \oplus 3' \oplus 2$, a sum of three irreducibles. Let R and R^* be the preimage in Q_0 of 3 and $3' \oplus 2$. Then $R \cong 4^3 \times 2$. Define β as the automorphism of Q_0 which inverts R and is trivial on $R^* W_1$; this map commutes with the action of W_1.

We need to prove that β interchanges the two groups M_0 containing A_0 which are associated to representations of P_0 as in (iii). Let $x \in M_0 - A$ satisfy $x^2 \in A_0$. Then, $x = pq$, where $p, q \in T$, $p \in R$, $q \in R^*$. We claim that $|p| = 4$. If false, $p \in Z(Q_0)$ and so $x^2 = q^2$, a generator for $Z(E)$, which is not in A_0. The claim then implies that $x^\beta = p^3 q$. We are done once we show that $p^2 \notin A_0$; but $p^2 \in [R, W_1]\, Z(E)$ implies that $p^2 \in [A, W_1] \cap T$ and $[A, h] \leqq A_0$, so if p^2 were in A_0, we would conclude that $A_0 = [A, W_1]$, a normal subgroup of P_1, a contradiction.

We now have that β interchanges the two groups M_0 over A_0. Observe that β fixes the linear character α and any linear character of $T \cap R^*/T \cap A_0 \cong 4$. Thus, the action of β on the set of twelve faithful irreducible representations of degree 28 is as claimed.

(v) We show that an automorphism is inner or in the outer automorphism class of β. Let $\alpha \in \mathrm{Aut}(P_1)$. Since the irreducible 3 for W_1 which occurs in $Z(Q_0)$ is not stable under $\mathrm{Aut}(W_1) \cong \mathrm{Frob}_{42}$, α is inner on P_1/Q_0, so may be assumed to be trivial. By coprimeness, we may assume that α centralizes W_1. Since α leaves R and R^* invariant (see (iii)), the facts that the W_1-chief factors in R are $1, 3, 3$ and those in R^* are $1, 3, 3', 2$, it follows that $\alpha = 1$ or β. QED

5.13. Proposition. *P_1 is embedded in the Rudvalis group.*

Proof. Fortunately, this is easy to prove with available results. We take the group $P := Q_0 W_0$ of (5.7) and observe that the result [Gr2, (7.2)] may be modified to prove that the isomorphism type of the group P is determined by the cohomology class in $H^2(P/Q_0, Q')$ associated to any subgroup of P containing Q' which complements Q_0 modulo Q' (all such are conjugate since Q_0/Q' is the Steinberg module). In particular, if we drop down from P to its subgroup P_1, which does split over Q_0, we may embed P_1 in the version of P which gives the 0-class of $H^2(P/Q_0, Q')$ and then conclude that the isomorphism type of P_1 is determined. Since $2 \cdot Ru$ has a subgroup of shape $2^{4+8} \cdot \mathrm{GL}(3,2)$ with the right composition factors and complete reducibility on the 2^4 factor, it follows that $2 \cdot Ru$ has a subgroup isomorphic to P_1. QED

6 The $Sz(8)$ case

In (4.7), we showed that, up to conjugacy, there is only one way to embed the Borel subgroup B of $Sz(8)$ into $E = 2 \cdot E_7(\mathbb{C})$. We show below that this embedding of B cannot be extended to one for $Sz(8)$; this result also rules out embeddings in all characteristics which do not divide $|Sz(8)| = 2^6 \cdot 5 \cdot 7 \cdot 13$. By definition, $Sz(8) \cong {}^2B_2(8)$, so $Sz(8)$ is embedded in $E_7(K)$ when char $K = 2$. In characteristics 7 and 13 we use a modified version of the characteristic 0 argument to eliminate any embeddings. We shall conclude this section by observing that there is an embedding in characteristic 5, via a chain of the form $Sz(8) < D_4(5) < 2 \cdot E_7(5)$, where the first two groups have E-centralizer isomorphic to Quat_8.

6.1. Notation. Call two distinct elements of E *negatives* if they are congruent modulo $Z(E)$.

We note that any embedding of $Sz(8)$ into $E/Z(E)$ must lift to an embedding of $Sz(8)$ into E (since involutions of E are not conjugate to their negatives [CoGr1], whereas in the double cover of $Sz(8)$ non-central involutions congruent modulo the center are conjugate [AGo]). In order to establish a contradiction, assume for the moment that $E > S > B \cong 2^{3+3'}:7$ and $S \cong Sz(8)$. The group S has two 14-dimensional representations (14a and 14b, say) and these restrict to distinct irreducibles of B.

6.2. Lemma. (i) *The 56-dimensional representation of E must restrict to a direct sum* $14a + 14a + 14b + 14b$ *of representations of S.*

(ii) $C_E(B) = C_E(S)$.

Proof. (i) The 56-dimensional representation of E is self dual and has trace ± 8 at involutions of E. Any restriction to S must contain $14a$ and $14b$ with equal multiplicity (since these representations are mutually dual). Using the character table from [Atlas, p. 28], we see that the only feasible restriction to S is $14a + 14a + 14b + 14b$.

(ii) From the 56-dimensional representation of E we obtain $G = \mathrm{GL}_{56}(\mathbb{C}) > E > S > B$. The decomposition of (i) gives: $C_G(S) \cong \mathrm{GL}_2(\mathbb{C})$. Moreover, we recall that $14a$ and $14b$ restrict to distinct 14-dimensional irreducible representations of B and so $C_G(B)$ is also isomorphic to $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$. Thus $C_G(S) = C_G(B)$ and intersection with E we obtain $C_E(B) = C_E(S)$. QED

In (4.7), we showed that any subgroup B of E must centralize a quaternion subgroup Q of order 8 in E, where $C_E(Q) \cong 2 \times \mathrm{PSO}_8(\mathbb{C})$. Combining this with the above lemma we see that S also centralizes Q and thus $S < O_8(\mathbb{C})$. Since neither S nor its double cover possesses a faithful 8-dimensional complex representation this is clearly impossible. This contradiction shows that our supposed embedding $S < E$ is also impossible and thus $Sz(8)$ cannot be a subgroup of $E_7(\mathbb{C})$.

Using the Lifting Lemma of [Gr4, Appendix B], we can now conclude that there is no embedding of $Sz(8)$ into a Chevalley group $E_7(K)$ whenever char (K) is other than 2, 5, 7 or 13 (the divisors of $|Sz(8)|$).

The proof of (6.2) can be carried out almost without change to show that there is no embedding in characteristics 7 and 13; the only minor problem is to show that the sum $14a + 14a + 14b + 14b$ of $Sz(8)$-modules is still direct in these characteristics. In characteristic 7, the modules $14a$ and $14b$ are projectives and therefore the sum must be direct.

In characteristic 13, we can use the Brauer tree of the principal block of $Sz(8)$ (given below) to show that the only way to combine irreducible 14-dimensional modules is as a direct sum. Here, as in our other calculations of Brauer trees we refer the reader to the first 4 chapters of [HL] for a good overview of available techniques. The real stem (see [HL, Sect. 1.1]) of the tree must be:

35——64——1.

Theorem 2.1.20(iii) of [HL] (which states that complex conjugation gives a graph automorphism of the Brauer tree) now determines the planar embedded Brauer tree as:

```
       14
       |
35——64——1.
       |
       14
```

We now apply Lemma 2.1.22 of [HL] to deduce that the only possible extensions of 14-dimensional modules (in characteristic 13) are split (since the two edges linking the 64-dimensional node to 14-dimensional nodes are not consecutive in the cyclically ordered arrangement of edges through the 64-dimensional node).

In characteristic 5, the following lemma (known to Thompson around 1970; see [Atlas, p. 28]) shows that we actually do get an embedding $Sz(8) < D_4(5) < E_7(5)$ of the kind described earlier.

6.3. Lemma. (i) *The irreducible 5-modular representations of* $Sz(8)$ *have degrees* 1, 14, 14, 63, 35, 35, 35, 65, 65 *and* 65.

(ii) *The double cover of* $Sz(8)$ *has a faithful self-dual irreducible 5-modular representation of degree* 8.

(iii) $Sz(8)$ *is embeddable in* $D_4(5)$.

Proof. (i) The ordinary characters of degrees 1, 14 (two characters), 64 and 91 lie in the principal block. The characters of degree 14 restrict to irreducible ordinary characters of the Borel subgroup whose order is prime to 5; thus these ordinary characters are end nodes of the Brauer tree. Therefore the Brauer tree must be:

```
                ╱14
1——64——91
                ╲14.
```

This give rise to the modular irreducible characters of degrees 1, 14, 14 and 63. The other modular irreducibles all arise from defect 0 ordinary representations.

(ii) For the Brauer tree on the faithful irreducible characters of $2\cdot Sz(8)$, congruences of ordinary degrees modulo 5 show that the characters of degrees 64 and 104 can be joined only to characters of degree 56. Now, since $64>56$ and $104>56$, we deduce that 64 and 104 can not be end nodes of the tree, and thus all of the end nodes have degree 56. Hence, the Brauer tree for the faithful characters of $2\cdot Sz(8)$ must be:

56——64——56——104——56.

This gives modular irreducibles of degrees 56, 8, 48 and 56. Since there is just one 8-dimensional irreducible it must be self dual.

(iii) The skew square of the 8-dimensional 5-modular representation constructed in (ii) must break up into two irreducible 14-dimensional constituents (use the character degrees of (i) to see this). The invariant bilinear from on the 8-dimensional module is therefore not symplectic so it must be orthogonal. QED

We note that $2\cdot E_7(5)$ has a subgroup $\mathrm{Quat}_8\times D_4(5)$ [Gr4, (9.5)] and thus:

6.4. Corollary. *The double cover of* $E_7(5)$ *has a subgroup* $\mathrm{Quat}_8\times Sz(8)$.

In the next section this corollary will be the starting point of our proof that $\mathrm{Quat}_8\times Sz(8)<2\cdot Ru<2\cdot E_7(5)$. Since Ru has a subgroup $Sz(8){:}3$ we will be able to extend $Sz(8)$ to $Sz(8){:}3$ inside $E_7(5)$. The embedding of $2\cdot Ru$ will also show that, in characteristic 5, the 56-dimensional representation of $2\cdot E_7(5)$, restricts to a $Sz(8)$-module with just two (isomorphic) indecomposable summands, but irreducible constituents $14a+14a+14b+14b$.

7 The *Ru* case

7.1. Notation. In this section, $E\cong 2\cdot E_7(5)$ is a fixed double cover of $E_7(5)$ and $P\cong 2^{4+8}{:}\mathrm{Frob}_{21}$ denotes a subgroup isomorphic to the standard Rudvalis subparabolic (obtained by performing the embedding of Sect. 5 over a field K of characteristic 5). The group P contains a subgroup $Q\times B$ with $Q\cong\mathrm{Quat}_8$ and $B\cong 2^{3+3'}{:}7$ where $Z(O_2(B))$ is $2B$-pure in E (by (5.10)). Let V denote the 56-dimensional E-module.

By (4.8), $Q\times B$ is unique up to conjugacy in $E\cdot 2$ and thus by (6.4) $Q\times B<Q\times S<E$ with $S\cong Sz(8)$.

7.2. Notation. $R:=\langle P,S\rangle$.

We shall show that R is a proper subgroup of E and is isomorphic to $2\cdot Ru$. We begin with the following lemma which analyzes the action of R on V.

7.3. Lemma. (i) *The restriction* V_B *is a direct sum* $14a\oplus 14a\oplus 14b\oplus 14b$ *of irreducible B-modules* ($14a$ *and* $14b$ *are the 5-modular reductions of the ordinary representations of* (6.2)).

(ii) *The module* $V_{Q \times B}$ *is the direct sum* $2 \otimes 14a \otimes 2 \otimes 14b$ *of two non-isomorphic 28-dimensional irreducible submodules (the ordered direct factors* Q *and* B *act on the respective tensor factors).*

(iii) *Every* $Q \times B$*-submodule of* V *is also a* P*-submodule.*

(iv) *The restriction* $V_{Q \times S}$ *has two irreducible 28-dimensional constituents* $2 \otimes 14a$ *and* $2 \otimes 14b$.

(v) *The group* R *stabilizes a 28-dimensional subspace* $W \leqq V$.

(vi) *The center of* R *is a 2-group.*

Proof. (i) Since $5 \nmid |B|$, the analysis following (6.1) can be reduced modulo 5 to give the decomposition of V_B.

(ii) Follows from (i) and the fact the center of Q acts as -1 on V.

(iii) According to (5.12.iii), V_P is a direct sum of two non-isomorphic irreducible submodules. Thus, every $Q \times B$-submodule of V is a P-submodule.

(iv) From (ii) we deduce that: the character of $V_{Q \times S}$ is of the form $2 \otimes \psi$ where ψ is a (reducible) 5-modular character of S that restricts to $14a \oplus 14b$ on B. Now the list of 5-modular characters of S (see (6.3.i)) shows that $\psi = 14a + 14b$.

(v) The group $Q \times S$ does fix at least one 28-dimensional subspace $W \leqq V$ (by (iv)). (Later we shall show that actually there is just one choice for W, but at the moment any 28-dimensional submodule will meet our needs.) The subgroup $Q \times B$ also fixes W, and hence so does P (by (iii)).

(vi) The centralizer in $GL_{56}(5)$ of $Q \times B$ is a direct product of two copies of the multiplicative group of $\mathbb{F}_5$ (from (ii), absolute irreducibility of 14 and Schur's Lemma). Thus $Z(R)$ is contained in a group of order 16. QED

Lemma 7.3 shows that R is a proper subgroup of E and its leads to a pair of homomorphic images of R (R_W given by the action of R on W and R^W given by the action of R on V/W). The flavor of the next result is that a finite group which looks like the Rudvalis group 2-locally and which has a degree 28 projective representation is uniquely determined up to isomorphism; here, we are only interested in identifying R, but remark that a proper uniqueness result for Ru in our setup seems reasonable.

7.4. Lemma. (i) $R_W \cong 2 \cdot Ru$.

(ii) $R^W \cong 2 \cdot Ru$.

Proof. (i) Let π be the homomorphism from R to R_W. Let $G = GL(W) \cong GL_{28}(5)$. Now let $R^* \cong 2 \cdot Ru$ satisfy $G > R^* > \pi(P_1)$ (there is such an R^* since (5.12.iv) and (5.13) show that any irreducible embedding of P_1 into $SL_{28}(5)$ extends to an embedding of $2 \cdot Ru$). Now, $R^* > \pi(P_1) > \pi(Q \times B) > \pi(Q) \cong Q$, ($\pi(Q) \cong Q$ since $Z(Q)$ is non-trivial on W). Let $S^* = C_{R^*}(\pi(Q)) \cong 2 \times Sz(8)$, we note that $\pi(B) < S^*$. The argument of (7.3.iv) shows that $\pi(S)$ and S^* are conjugate in G.

Now select $g \in G$ with $S^{*g} = \pi(S)$. Since there is only one class of subgroup isomorphic to B in S, there is an element $s \in \pi(S)$ with $\pi(B)^{gs} = \pi(B)$. Moreover, since $\mathrm{Aut}(\pi(B)) \cong 2^{3+3'}:7:3 < \pi(P_1)$, there is an element $p \in \pi(P_1)$ such that conjugation by p has the same effect on $\pi(B)$ as conjugation by gs. Thus $p^{-1} gs \in C_G(\pi(B))$. But, by Schur's Lemma $C_G(\pi(B)) \cong GL_2(5) \cong C_G(\pi(S))$ (since the underlying 28-dimensional G-module gives the same direct sum of a pair of isomorphic irreducible submodules when restricted either to B or to S). Thus $C_G(\pi(B))=C_G(\pi(S))$ and hence $p^{-1}gs\in C_G(\pi(S))$. We conclude that

$$R_W=\langle\pi(P_1),\pi(S)\rangle=\langle\pi(P_1),\pi(S)^{s^{-1}g^{-1}p}\rangle=\langle\pi(P_1),\pi(S)^{g^{-1}p}\rangle$$
$$\cong\langle\pi(P_1),\pi(S)^{g^{-1}}\rangle=\langle\pi(P_1),S^*\rangle=R^*.$$

(ii) A similar arguments shows that $R^W\cong 2\cdot Ru$. QED

7.5. Lemma. *Let K_W and K^W be the kernels of the maps from R to R_W and R^W, respectively.*
(i) $|K_W:K_W\cap K^W|\leq 2$.
(ii) *$K_W\cap K^W$ is an elementary abelian 5-group.*
(iii) $K_W=K_W\cap K^W$.
(iv) $K_W=K^W=1$. *Thus $R\cong 2\cdot Ru$.*

Proof. (i) The quotient $K_W/(K_W\cap K^W)$ is isomorphic to a normal subgroup of $2\cdot Ru$. But Ru itself cannot be a composition factor of K_W, otherwise both $|R/K_W|$ and $|K_W|$ would be divisible by $|Ru|=2^{14}3^3 5^3 7\cdot 13\cdot 29$ whence 29^2 would divide $|R|$ and $|E|$, a contradiction.

(ii) Let k and l be any elements of $K_W\cap K^W$, then $k-1$ and $l-1$ both map V to W and W to $\{0\}$. Thus $(k-1)(l-1)=0=(l-1)(k-1)$; it follows that k and l commute. Also $(k-1)^2=0$ implies $(k-1)^5=0$ and $k^5=1$.

(iii) Otherwise, by (i) and (ii), $|K_W|=2|K_W\cap K^W|$ is twice a power of 5. In this case we can choose an involution $x\in K_W-K^W$. The definitions of K_W and K^W now show that x acts as $+1$ on W and -1 on V/W; and so x has trace 0. But here are no involutions of E with trace 0.

(iv) We now write $K:=K_W=K_W\cap K^W$, an elementary abelian 5-group, which we can regard as a module for $R_W\cong 2\cdot Ru$. Recall that the central involution of E acts as -1 on W and therefore maps to the central involution of R_W. But the central involution of E certainly acts trivially on K, and therefore K must actually be a Ru-module. We deduce, by restricting to a subgroup $2^6:U_3(3)$, that all non-trivial irreducible $\mathbb{F}_5\,Ru$-modules have degree at least 63. In view of the fact that 63 is the exact power of 5 dividing $|E|$, which has nonabelian Sylow 5-groups, we are forced to conclude that all composition factors of the Ru-module K are trivial. In particular, if $K>1$, $Z(R)$ must contain an element of order 5, contradicting (7.3.vi). We must now concede that $K=1$ and $R\cong 2\cdot Ru$. QED

We could use the E-invariant symplectic form on V given by (2.9) to obtain a duality between the R-modules W and V/W; this gives a geometric proof of (7.5.iii). We now use Lemma 7.5 to obtain the main result of this section:

7.6. Theorem. *Let F be a field of characteristic 5. The Chevalley group $E_7(F)$ contains the sporadic simple group Ru but not $2\cdot Ru$. Also, $2\cdot E_7(F)$ contains $2\cdot Ru$ but not Ru.*

Proof. The existence of an embedding $2\cdot Ru<2\cdot E_7(F)$ follows from (7.2) and (7.5). We immediately obtain the embedding $Ru<E_7(F)$ by factoring out the common center.

Since Ru has only trivial 5-modular representations of degree less than 63 (see the proof of (7.5.iv)), there is no embedding of Ru into $2\cdot E_7(F)$. It is also impossible to embed $2 \cdot Ru$ into $E_7(F)$ since this would place Ru in a smaller algebraic group in characteristic 5 (an involution centralizer). This possibility is easily eliminated, see [KlWi]. QED

According to [KMR], V_R is a uniserial module. We claim that $V_{Q \times S}$ must also be uniserial. Otherwise $V_{Q \times S}$ would have two 28-dimensional invariant submodules, W and W' say. These would also be invariant under the subgroup $Q \times B$; and hence (by (7.3.iii)) under P. Thus W and W' would be distinct R-submodules of V contradicting the uniseriality of V_R. The claim implies that V_S has only two direct summands (but four irreducible constituents) as stated at the end of Sect. 6.

In our proof of Theorem 7.6 we have not used Aschbacher's embedding ${}^2F_4(2) < E_6(5)$ which was the starting point for one of the constructions of $Ru < E_7(5)$ given in [KMR]. It is possible to use the classification of vector stabilizers in the action of E on V (see [LiSa]), together with (7.6) to show that ${}^2F_4(2)'$ is a subgroup of at least one of the two groups $E_6(5)$ and ${}^2E_6(5)$. Although we shall omit the details, we did check this weak form of Aschbacher's result by employing a computer calculation to show that a subgroup ${}^2F_4(2)'$ of $2Ru$ does fix a vector in V.

8 The $U_3(8)$ case

In this section we shall describe our computer construction of $U_3(8){:}12 < E_7(25)$. Since $5 \nmid |U_3(8)|$, the Lifting Lemma of [Gr4, Appendix 2], proves that $U_3(8) < E_7(\mathbb{C})$. After describing the computations which we performed, we finish by providing (in (8.6)) an explicit matrix representation of the Lie algebra of type E_7 and (in (8.7)) generating matrices for $U_3(8)$. The information included in (8.6) and (8.7) makes it straightforward for the reader to check (for example, by our algorithm (8.4)) that $U_3(8)$ is a subgroup of $E_7(5)$.

For computational purposes we shall work within fixed 56-dimensional 5-modular representations of various groups and Lie algebras. Let G be $\mathrm{GL}_{56}(25)$ and let V be the natural 56-dimensional G-module. We let $\mathscr{E}$ be a particular linear space of 56×56 matrices representing the Lie algebra of type E_7 over a field of 25 elements. We then let E be the commutator subgroup of the subgroup $\tilde{E}$ of G which preserves $\mathscr{E}$ (under conjugation). Generators of $\mathscr{E}$ are given in (8.6) and generators of E can then be obtained by exponentiating the actions of root elements, which have square 0 on this space.

8.1. Lemma. $\tilde{E} \cong 24 \cdot E_7(25)$ *and* $\tilde{E}' = E \cong 2 \cdot E_7(25)$.

Proof. The automorphism group of $\mathscr{E}$ is just $E_7(25)$ [St]. It is clear from weight theory that the degree 56 matrix group $\tilde{E}'$ is the universal central extension of E_7. Only the 24 scalar matrices can act trivially on $\mathscr{E}$. QED

The construction of (3.3) gives a subgroup $B \cong 2^{3+6}{:}(\mathrm{Frob}_{21} \times 2 \cdot \Sigma_3)$ of E.

8.2. Lemma. *There is a unique extension of B to a group $U \cong U_3(8){:}12$ inside G.*

Proof. The restriction, V_B also arises as the restriction of a 56-dimensional representation of $U_3(8){:}12$ (from (3.6)). Therefore, we can find at least one extension, $B \leqq U \leqq G$, with $U \cong U_3(8){:}12$. Moreover, up to equivalence, there is just one 56-dimensional representation of U which restricts to the B-module V_B; thus if $B \leqq U^* \leqq G$, with $U^* \cong U$ then $U^* = U^g$ for some $g \in G$. Now, B^g and B are normalizers of Sylow groups of U^* and so we can find $u \in U^*$ with $B^{gu} = B$.

Now, gu must induce an inner automorphism on B (by (3.7)). We may therefore assume u chosen so that gu is a scalar. Then $U^* = U^{*u} = U^{gu} = U$. QED

In order to locate the extension of B to U, we examine a subgroup $D \cong 2 \cdot \Sigma_3 \times 2^3{:}\mathrm{Frob}_{21} \cong 3{:}4 \times 2_3{:}\mathrm{Frob}_{21}$ inside B. From the subgroup structure of $U_3(8)$, we know that D can be enlarged to a group $S \cong 2 \cdot \Sigma_3 \times L_2(8){:}3$ inside the group U.

8.3. Lemma. *Let $\bar{G}$ be the algebraic group* $\mathrm{GL}_{56}(\bar{\mathbb{F}}_5)$ *containing G.*
(i) *The restriction V_D has five (non-isomorphic) absolutely irreducible constituents. Hence the algebraic group $C_{\bar{G}}(D)$ has dimension* 5.
(ii) *The restriction V_S has three (non-isomorphic) absolutely irreducible constituents. Hence the algebraic group $C_{\bar{G}}(S)$ has dimension* 3.

Proof. We use tensor product notation to denote a representation of a direct product with the direct factors acting on the respective tensor factors.
(i) The group $2^3{:}\mathrm{Frob}_{21}$ has exactly three faithful representations, all three have degree 7. The representations are distinguished by the character values on an element of order 3. We shall call these representations $7a$, $7b$ and $7c$, where the labels are chosen to that an element of order 3 has trace 1 on $7a$ and has a pair of conjugate cube roots of 1 as its traces on $7b$ and $7c$. We restrict a symplectic 56-dimensional character of $U_3(8){:}12$ to get a character $1 \otimes 7b \oplus 1' \otimes 7a \oplus 2 \otimes 7a \oplus 2 \otimes 7b \oplus 2 \otimes 7c$ of D. Here, we use symbols 1, 1′ and 2 to denote the three representation of $2 \cdot \Sigma_3$, of the indicated degrees, in which the central involution acts as -1.
(ii) Similarly, we can restrict the 56-dimensional representation of $U_3(8){:}12$, to obtain the representation $1 \otimes 7a \oplus 1' \otimes 7a \oplus 2 \otimes 21$ of S. (Here 21 represents the unique 21 dimensional irreducible representation of $L_2(8){:}3$, the representation is obtained as a sum of three algebraically conjugate irreducible representations of $L_2(8)$.) QED

The lemma implies that within G, there is a "2-parameter family" of extensions from D to G-conjugates of S; see [PaWi, Ry] for analysis of similar situations. For all extensions but one, a computer check shows that the group generated by B and the relevant conjugate of S has an element of order not found in $U_3(8){:}12$. The computer algorithm (8.4) (explained below) then proves that the group generated by B and the remaining conjugate of S does remain inside E. This strategy is very similar to our earlier construction of Ru; however, in the work on Ru we were lucky enough to obtain a single possible extension of the parabolic subgroup inside the general linear group. The tighter control in the Rudvalis situation allowed us to be independent of computer calculations.

8.4. Algorithm. *To test whether an element $g \in G$ belongs to $\tilde{E}$.*
(i) *Select a basis $b_1, b_2, \ldots, b_{133}$ of $\mathscr{E}$.*
(ii) *Select a pair of algebra generators a_1 and a_2 for $\mathscr{E}$.*
(iii) *Calculate $a_i^g = g^{-1} a_i g$ for $i = 1, 2$.*
(iv) *Then $g \in \tilde{E}$* iff *a_1^g and a_2^g are linear combinations of $b_1, b_2, \ldots, b_{133}$.*

To justify the algorithm, observe that, if conjugation by g maps generators of $\mathscr{E}$ into $\mathscr{E}$, it must preserve $\mathscr{E}$; whence $g \in E$. The main calculation required in 8.4 is the Gaussian elimination implicit in (iv). We used the following standard sampling technique in order to avoid working in the full 56×56 dimensional underlying matrix space.

8.5. Implementation of (8.4.iv) (i) *Select a random 3×56 matrix, M say.*
(ii) *Calculate the 3×56 matrices, $A_i = M a_i^g$, and $B_j = M b_j$.*

(iii) *Apply Gaussian elimination (in a 168-dimensional space) to obtain scalars* λ_{ij} *with* $A_i = \sum \lambda_{ij} B_j$. *(If there are no such scalars then* $g \notin \tilde{E}$. *In the unlikely event that more than one combination is found we go back to* (8.5.i) *and make a more random choice for M.)*
(iv) *Test whether the corresponding linear combinations:* $\sum \lambda_{ij} b_j$ *give* a_1^g *and* a_2^g. *The element* $g \in \tilde{E}$ *if and only if these linear combinations do give* a_1^g *and* a_2^g.

In our opinion, (8.4) gives an efficient algorithm for checking whether a particular degree 56 matrix lies in the subgroup of type E_7. Our impression is that it is more convenient than, say, an algorithm which checks invariance of the 4-linear form.

We now give an explicit matrix representation of generators of $\mathscr{E}$. With respect to an appropriate Cartan decomposition, the generators that we give are members of the seven fundamental root spaces. Their transposes are seven elements from the corresponding negative root spaces. We obtained these generators by applying the MeatAxe [Pa] to a 248-dimensional (E_8-adjoint) representation of E_7. However, once they are known, we can check that they do indeed represent a Lie algebra E_7 by simply verifying Serre's presentation [Hum, p. 99]; this check is an easy computation of commutators of integral matrices which can be done either by hand or computer. Our matrices actually give generators of the Lie algebra $E_7(\mathbb{Z})$ (by interpreting their entries as integers) as well as generators of any classical Lie algebra of type E_7 (by viewing the entries as elements of an appropriate field).

We number the nodes of the Dynkin diagram of E_7 as follows:

```
              2
              |
7 —— 1 —— 3 —— 4 —— 5 —— 6.
```

The generator of $\mathscr{E}$ which we call e_i in (8.6) belongs to the fundamental root space corresponding to node i of the diagram; and the generator which we call f_i belongs to the corresponding negative root space. We shall write our generators as sums of elementary matrices; $\varepsilon_{i,j}$ denotes the elementary matrix which consists of a single non-zero entry, namely a 1 in the i, j position.

8.6. Generators of $\mathscr{E}$. *The following seven* 56×56 *matrices,* $e_1, e_2, \ldots, e_7$, *together with their transposes,* $f_1 = e_1^t, f_2 = e_2^t, \ldots, f_7 = e_7^t$, *generate* $\mathscr{E}$.

$\varepsilon_{6,5} + \varepsilon_{8,7} + \varepsilon_{16,15} + \varepsilon_{18,17} + \varepsilon_{22,21} + \varepsilon_{28,27} + \varepsilon_{30,29} + \varepsilon_{36,35} + \varepsilon_{40,39} + \varepsilon_{42,41} + \varepsilon_{50,49} + \varepsilon_{52,51}.$
$\varepsilon_{7,5} + \varepsilon_{8,6} + \varepsilon_{14,13} + \varepsilon_{20,19} + \varepsilon_{24,23} + \varepsilon_{26,25} + \varepsilon_{32,31} + \varepsilon_{34,33} + \varepsilon_{38,37} + \varepsilon_{44,43} + \varepsilon_{51,49} + \varepsilon_{52,50}.$
$\varepsilon_{5,4} + \varepsilon_{9,8} + \varepsilon_{15,14} + \varepsilon_{19,18} + \varepsilon_{23,22} + \varepsilon_{27,26} + \varepsilon_{31,30} + \varepsilon_{35,34} + \varepsilon_{39,38} + \varepsilon_{43,42} + \varepsilon_{49,48} + \varepsilon_{53,52}.$
$\varepsilon_{4,3} + \varepsilon_{10,9} + \varepsilon_{17,15} + \varepsilon_{18,16} + \varepsilon_{25,23} + \varepsilon_{26,24} + \varepsilon_{33,31} + \varepsilon_{34,32} + \varepsilon_{41,39} + \varepsilon_{42,40} + \varepsilon_{48,47} + \varepsilon_{54,53}.$
$\varepsilon_{3,2} + \varepsilon_{11,10} + \varepsilon_{21,17} + \varepsilon_{22,18} + \varepsilon_{23,19} + \varepsilon_{24,20} + \varepsilon_{37,33} + \varepsilon_{38,34} + \varepsilon_{39,35} + \varepsilon_{40,36} + \varepsilon_{47,46} + \varepsilon_{55,54}.$
$\varepsilon_{2,1} + \varepsilon_{12,11} + \varepsilon_{29,21} + \varepsilon_{30,22} + \varepsilon_{31,23} + \varepsilon_{32,24} + \varepsilon_{33,25} + \varepsilon_{34,26} + \varepsilon_{35,27} + \varepsilon_{36,28} + \varepsilon_{46,45} + \varepsilon_{56,55}.$
$\varepsilon_{13,6} + \varepsilon_{14,8} + \varepsilon_{15,9} + \varepsilon_{17,10} + \varepsilon_{21,11} + \varepsilon_{29,12} + \varepsilon_{45,28} + \varepsilon_{46,36} + \varepsilon_{47,40} + \varepsilon_{48,42} + \varepsilon_{49,43} + \varepsilon_{51,44}.$

Our computations established:

8.7. Theorem. *The following four matrices generate* $U_3(8)$ *and lie in the subgroup* $2 \cdot E_7(5)$ *of E.*

The first three are the following diagonal matrices (they arose in our construction as generators of the form h_A in (3.1).

$l = \mathrm{diag}(33434422121121331134334234223342342242112213113133442422)$
$m = \mathrm{diag}(23343413133243133431343143132124124212422124322121424223)$
$n = \mathrm{diag}(42334331332143434343121212123131313124242424132212242234)$

The fourth is the following matrix, d say (each entry is a single integer modulo 5).

11232422302033030000202401402324014022340140112004000000
12334202220213222024000101043202224300101010014401000000
23312433302010223232320221340000040104014040144010000000
33134400000022302230033430113334001121001100334311000000
24241202320320020201032021430034200421430410040412000000
42442320032220202034303000410021213020414130404420000000
40100432120044340020321020033300223300020001020230024044
44104020402202403013022020301230200002030004202203241400
14101014430300232303002230023210000030003231000023142010
040004120332013020342021300003443330022300030000030404143
40400402023313433033344003202030002040032020000003042441
04001402303013341240200333220220403030020201000030403404
31122220033312000304330423002200244404443300400111000000
11040042033344023130203200333130000000220000033200210444
04410434204300042323222003200103200320003002300200111004
14404040323402430300430303303020230220000303030200104440
04140403203103200220310020021422030020000034220033430104
00444000330211332302102020204303402233000021002000410012
04110121043403202000023330300330223210300004002020014402
01304101141010100403010000320014021001222300243300232003
401001300032122431001420300303233200023022201000003011430
00411022024010231023430320023300300012200004330002204014
40014112214003200230201022000323000332033001002002100124
14310000410114010010010404330012002303403202403023320030
00414022300340002230322013031040330340330300003000441034
44404000001120110000004413030004130343022030331120223003
10241101004401410413000300000000032302310142204322002033
01142410400401004002140313030001003000113033022343003033
41010031332043031400030010003032203320030020200302434410
140100322402411043303330000000000300023303330320020411201
40011403143203022033202340003032120000002303320000010141
13011400010000104104101204014041032203320003200332033300
040004422030440220420330033002310403003003021202200004311
43100440010010011042400013300043010202200220011324222002
11040210004110000411000200231002103142013302043032023300
02441010000010140440001313301002021100121233220002200302
40044400324300222310013042002200002030200304022230110210
30144004040010000101444303200103122000020330131042303220
24001100010014000012140410310103020140332343203200023200
10402441001414000002001012111002001202300330430220202220
04140100104020100002401302030140103100203120133040303222
40041400400420010003400210100110023213331000310342302230
14104200114000001400000003434100420303432043232034002032
00000041400400321410414400410021404110100011141001323143
40420104000010104002010403204112400201241323020220330030
44121040000001014004010003020440014243033132233030200303
31110104000001000443004311420000413041303023033203220030
01021144000044440003000001331003430040223300202411020222
00444310410140001040000202240403023014020430230103020332
14043001101040001000144300234002042202004243003133200203
00000003410103441104034114001100040441040103144004334121
00000031101004402441400414002441044040401002104440311212
00000014313200410014410041411401141001140443000000413432
00000001041101014010104000001341201134441401010414124322
00000010411001010400243120114010400044044114101410213224
00000010024101101331011011110440440400004043010441122241

Algorithm 8.4 can be used to check that these matrices do belong to $\tilde{E}$; and since the entries of our matrices lie in $\mathbb{F}_5$, they must belong to a subgroup $4\cdot E_7(5)$ in $\tilde{E}$. Moreover, once the above matrices have been constructed we can verify fairly simply that they generate $U_3(8)$. For example, we can check that the matrices given above satisfy the presentation (8.8) of $U_3(8)$ (which can in turn be verified via a coset enumeration, for example using the computer package Cayley). For convenience we introduce three new auxiliary generators

$$x=(d\,l^2\,dm^2\,n^2\,dm^2)^5, \quad w=d\,l\,d\,n\,x, \quad \text{and} \quad y=n^3(w^3\,l^2\,w\,l^2\,w^3)^2$$

into the presentation. The generators l, m, n, x together with y generate a subgroup $2^{3+6}{:}21$ and the generators l^2, m^2, n^2, x, y together with d generate a subgroup $3\times L_2(8)$.

8.8. Lemma. *The following relations present $U_3(8)$ and are satisfied by the matrices of* (8.7).

$$l^4=m^4=n^4=d^2=[l,m]=[l,n]=[m,n]=1, \quad x=(d\,l^2\,dm^2\,n^2\,dm^2)^5,$$
$$l^x=m, \quad m^x=n, \quad n^x=lm^3\,n^2, \quad w=d\,l\,d\,n\,x, \quad y=n^3(w^3\,l^2\,w\,l^2\,w^3)^2,$$
$$x^7=y^3=[y,l^2]=[y,m^2]=[y,n^2]=1,$$
$$[y,d]=[y,x]=(l\,y)^3=(m\,y)^3=(n\,y)^3=(x^2\,dm^3\,n^2\,m\,l^2)^2=(d\,y\,n^3)^3=(dm^2\,y\,l)^4=1.$$

The larger field, of 25 elements, is needed only to accommodate elements of the full group $U_3(8){:}12$. Rather than present another matrix to exhibit an outer element of $U_3(8){:}12$ we note that the following theorem, together with (3.4), guarantees that our copy of $U_3(8)$ extends to a copy of $U_3(8){:}12$ in E. Moreover, an application of Parker's Standard Basis algorithm can be used to conjugate the Borel subgroup constructed in (3.4) to an explicit matrix representation of the group which we call B_E in (8.9).

8.9. Lemma. *Suppose that B_E and U are subgroups of $E\cong 2\cdot E_7(5^2)$ with $B_E\cong 2^{3+6}{:}(\mathrm{Frob}_{21}\times 2\cdot\Sigma_3)$, $U\cong U_3(8)$, and $B_E\cap U=B'_E\cong 2^{3+6}{:}21$. Then $\langle B_E, U\rangle\cong U_3(8){:}12$.*

Proof. Let $G\cong \mathrm{GL}_{56}(25)$; and as usual we may assume that $G>E$. Select $G>U_2>U_1>U$, with $U_2\cong U_3(8){:}(3\times 2\cdot\Sigma_3)$, and $U_1\cong U_3(8){:}12$; there are several extensions of this form corresponding to the different extensions of a 56-dimensional representation of U to a representation of U_1. By making an appropriate choice of U_1 (and U_2), we may assume that U_1 contains $B=B_E^g$ for some $g\in G$. Now, U_1 has a single conjugacy class of subgroups isomorphic to B'_E, so we may further assume that g has been adjusted so that $B'=B'_E$. Note that such a choice of g must normalize B'_E. Now, U_2 has a subgroup, B_2 say (with structure $2^{3+6}{:}(7\times 9{:}4){:}3$) which normalizes B'_E. Moreover, any automorphism of B'_E arises as a conjugation by an element of B_2 (see (3.5.i)). Thus we may select $u\in U_2$ with $gu\in C_G(B'_E)$. However, from (3.6), we know that $C_G(B'_E)$ just consists of scalar matrices, and so $gu\in C_G(U)$. Hence, $g\in N_G(U)$. We now conclude that $U_3(8){:}12\cong\langle B, U\rangle=\langle B_E^g, U^g\rangle\cong\langle B_E, U\rangle$. QED

Finally, we mention that, if M is a complex irreducible 133-dimensional module for $U\cong U_3(8){:}6$, then M is self-dual and the space of invariants for

$\mathrm{Hom}_U(\Lambda^3 M, \mathbb{C})$ is 1-dimensional. Our embedding of U in $E_7(\mathbb{C})$ is essentially unique and proves that the essentially unique skew-commutative algebra structure on M preserved by U is the Lie algebra of type E_7. It seems difficult to prove this by working directly with spaces of invariants.

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