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L^2-Topological invariants of 3-manifolds

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Summary. We give results on the L^2-Betti numbers and Novikov-Shubin invariants of compact manifolds, especially 3-manifolds. We first study the Betti numbers and Novikov-Shubin invariants of a chain complex of Hilbert modules over a finite von Neumann algebra. We establish inequalities among the Novikov-Shubin invariants of the terms in a short exact sequence of chain complexes. Our algebraic results, along with some analytic results on geometric 3-manifolds, are used to compute the L^2-Betti numbers of compact 3-manifolds which satisfy a weak form of the geometrization conjecture, and to compute or estimate their Novikov-Shubin invariants.

0. Introduction

The L^2-Betti numbers of a smooth closed manifold M, introduced by Atiyah [1], are invariants of M which are defined in terms of the universal cover $\widetilde{M}$. Roughly speaking, if M is Riemannian then the p-th L^2-Betti number $b_p(M)$ measures the size of the space of harmonic L^2 p-forms on $\widetilde{M}$, relative to the action of the fundamental group π on $\widetilde{M}$. We give the precise definition later. The L^2-Betti numbers are homotopy invariants of M (Dodziuk [12]), and can be extended to become Γ-homotopy invariants of topological spaces upon which a countable group Γ acts (Cheeger and Gromov [10]).

By means of a Laplace transform, there is an interpretation of the L^2-Betti numbers in terms of the large-time asymptotics of heat flow on $\widetilde{M}$. Let $e^{-t\widetilde{\Delta}_p}(x,y)$ be the Schwartz kernel of the heat operator acting on L^2 p-forms on $\widetilde{M}$. The von Neumann trace of the heat operator is given by

$$\operatorname{tr}_{N(\pi)}\left(e^{-t\widetilde{\Delta}_p}\right) = \int_{\mathcal{F}} \operatorname{tr}\left(e^{-t\widetilde{\Delta}_p}(x,x)\right) d\operatorname{vol}(x),$$

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where $\mathcal{F}$ is a fundamental domain for the π-action on $\widetilde{M}$ and the trace on the right-hand-side is the ordinary trace on $\mathrm{End}(\wedge^p(T_x^*\widetilde{M}))$. The L^2-Betti numbers of M can be expressed by

$$b_p(M) = \lim_{t\to\infty} \mathrm{tr}_{N(\pi)}\left(e^{-t\widetilde{\Delta_p}}\right).$$

In many examples one finds that $\mathrm{tr}_{N(\pi)}(e^{-t\widetilde{\Delta_p}}) - b_p(M)$ approaches zero with an exponential or power decay as $t \to \infty$. Novikov and Shubin [37] introduced invariants which quantify this phenomenon. If there is an exponential decay, put $\widetilde{\alpha}_p(M) = \infty^+$. Otherwise, put

$$\widetilde{\alpha}_p(M) = \sup\left\{\beta_p : \mathrm{tr}_{N(\pi)}\left(e^{-t\widetilde{\Delta_p}}\right) - b_p(M) \text{ is } O(t^{-\beta_p/2}) \text{ as } t\to\infty\right\} \in [0,\infty].$$

Roughly speaking, $\widetilde{\alpha}_p(M)$ measures the thickness of the spectrum of $\widetilde{\Delta_p}$ near 0; the larger $\widetilde{\alpha}_p(M)$, the thinner the spectrum near 0. Novikov and Shubin stated that these invariants are independent of the choice of Riemannian metric on M, and hence are smooth invariants of M. The first author showed that they are defined for all topological manifolds and depend only on the homeomorphism type of M, and computed them in certain cases [24]. The Novikov-Shubin invariants are homotopy invariants (see Gromov and Shubin [18] and Theorems 2.6 and 5.7 of the present paper.) A combinatorial Novikov-Shubin invariant was defined by Efremov in [14] and shown to be the same as the analytically defined invariant, again under the assumption that M is closed.

In this paper we give some results on the L^2-Betti numbers and Novikov-Shubin invariants of compact manifolds (possibly with boundary), especially 3-manifolds. Our interest in these invariants comes from our work on related L^2-invariants, the L^2-Reidemeister and analytic torsions [6, 24, 29, 31, 32]. In particular, one wishes to know that the Novikov-Shubin invariants of a manifold are all positive, in order for the L^2-torsions to be defined. We make some remarks on the L^2-torsions in Section 7.

We define an invariant $\alpha_p(M)$ in terms of the boundary operator acting on p-chains on $\widetilde{M}$ (compare [18, 19]). The relationship with $\widetilde{\alpha_p}(M)$ is that $\widetilde{\alpha_p}(M) = \min(\alpha_p(M), \alpha_{p+1}(M))$, where the left-hand-side is defined using p-forms on $\widetilde{M}$ which satisfy absolute boundary conditions if M has boundary. Let us say that a prime 3-manifold is *exceptional* if it is closed and no finite cover of it is homotopy equivalent to a Haken, Seifert or hyperbolic 3-manifold. No exceptional prime 3-manifolds are known, and standard conjectures (Thurston geometrization conjecture, Waldhausen conjecture) imply that there are none. The main results of this paper are given in the following theorem:

Theorem 0.1. *Let M be the connected sum $M_1\#\ldots\#M_r$ of (compact connected orientable) nonexceptional prime 3-manifolds M_j. Assume that $\pi_1(M)$ is infinite. Then*

1. a. *The L^2-Betti numbers of M are given by:*

$$b_0(M) = 0$$

$$b_1(M) = (r-1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|} - \chi(M) + |\{C \in \pi_0(\partial M)\, s.t.\, C \cong S^2\}|$$

$$b_2(M) = (r-1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|} + |\{C \in \pi_0(\partial M)\, s.t.\, C \cong S^2\}|$$

$$b_3(M) = 0.$$

b. *Equivalently, if $\chi(\pi_1(M))$ denotes the rational-valued group Euler characteristic then $b_1(M) = -\chi(\pi_1(M))$ and $b_2(M) = \chi(M) - \chi(\pi_1(M))$.*

c. *In particular, M has trivial L^2-cohomology iff M is homotopy equivalent to $RP^3 \sharp RP^3$ or a prime 3-manifold with infinite fundamental group whose boundary is empty or a union of tori.*

2. *Let the Poincaré associate $P(M)$ be the connected sum of the M_j's which are not 3-disks or homotopy 3-spheres. Then $\alpha_p(P(M)) = \alpha_p(M)$ for $p \leqq 2$. We have $\alpha_1(M) = \infty^+$ except for the following cases:*

(a) *$\alpha_1(M) = 1$ if $P(M)$ is $S^1 \times D^2$, $S^1 \times S^2$ or homotopy equivalent to $RP^3 \sharp RP^3$.*

(b) *$\alpha_1(M) = 2$ if $P(M)$ is $T^2 \times I$ or a twisted I-bundle over the Klein bottle K.*

(c) *$\alpha_1(M) = 3$ if $P(M)$ is a closed R^3-manifold.*

(d) *$\alpha_1(M) = 4$ if $P(M)$ is a closed Nil-manifold.*

(e) *$\alpha_1(M) = \infty$ if $P(M)$ is a closed Sol-manifold.*

3. *$\alpha_2(M) > 0$.*

4. *If M is a closed hyperbolic 3-manifold then $\alpha_2(M) = 1$. If M is a closed Seifert 3-manifold then $\alpha_2(M)$ is given in terms of the Euler class e of the bundle and the Euler characteristic χ of the base orbifold by:*

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	∞^+	3	1
$e \neq 0$	∞^+	2	1

If M is a Seifert 3-manifold with boundary then $\alpha_2(M)$ is ∞^+ if $M = S^1 \times D^2$, 2 if M is $T^2 \times I$ or a twisted I-bundle over K, and 1 otherwise. If M is a closed Sol-manifold then $\alpha_2(M) \geqq 1$.

5. *If ∂M contains an incompressible torus then $\alpha_2(M) \leqq 2$. If one of the M_j's is closed with infinite fundamental group and does not admit an R^3, $S^2 \times R$ or Sol-structure, then $\alpha_2(M) \leqq 2$.*

6. *If M is closed then $\alpha_3(M) = \alpha_1(M)$. If M is not closed then $\alpha_3(M) = \infty^+$.* □

Let us briefly indicate how we prove that $\alpha_2(M)$ is positive. The important case is when M is an irreducible Haken 3-manifold with infinite fundamental group whose boundary is empty or consists of incompressible tori; the general case follows by further arguments. The Jaco-Shalen-Johannson splitting of M, together with the work of Thurston, gives a family of embedded incompressible tori which cut the manifold into pieces that are either Seifert manifolds or whose interiors admit complete finite-volume hyperbolic metrics. The α_2-invariants of the Seifert pieces can be computed explicitly. By analytic means we derive a lower bound for the α_2-invariants of the (compact) hyperbolic pieces. We then face the problem of understanding what happens to the Novikov-Shubin invariants when one glues along incompressible tori. This is done algebraically by means of inequalities among the Novikov-Shubin invariants of the terms in a short exact sequence.

A description of the contents of the paper is as follows. The natural algebraic setting for our work is that of Hilbert $\mathscr{A}$-modules, where $\mathscr{A}$ is a finite von Neumann algebra. In Section 1 we define the Betti numbers and Novikov-Shubin invariants of a (left-Fredholm) morphism of Hilbert $\mathscr{A}$-modules, and derive some useful inequalities on the Novikov-Shubin invariants. In Section 2 we define the Betti numbers and Novikov-Shubin invariants of a Fredholm Hilbert $\mathscr{A}$-chain complex. If one has a short exact sequence of Fredholm Hilbert $\mathscr{A}$-chain complexes then there is an induced long weakly exact homology sequence, with which one can relate the Betti numbers of the chain complexes (Cheeger and Gromov [9]). We show that in Theorem 2.3 that the Novikov-Shubin invariants of the chain complexes are related by certain inequalities.

In Section 3 we specialize to the case of manifolds, in which $\mathscr{A}$ is the group von Neumann algebra $N(\pi)$ of the fundamental group π. Proposition 3.2 gives the relations on the L^2-Betti numbers and Novikov-Shubin invariants due to Poincaré duality, and Proposition 3.7 computes the L^2-Betti numbers and Novikov-Shubin invariants of connected sums. In Theorem 3.8 we show that if M admits a homotopically nontrivial S^1-action then the L^2-Betti numbers vanish and the Novikov-Shubin invariants are bounded below by 1. In Corollary 3.4 we show that the Novikov-Shubin invariants of closed manifolds of dimension less than or equal to 4 depend only on the fundamental group. In Section 4 we compute the L^2-Betti numbers and Novikov-Shubin invariants of Seifert 3-manifolds (Theorems 4.1 and 4.4).

Section 5 first extends the results of [12, 14] on the equality of combinatorial and analytic L^2-topological invariants from the case of closed manifolds to that of manifolds with boundary. We then consider the Novikov-Shubin invariants of a compact 3-manifold M whose interior admits a complete finite-volume hyperbolic structure. If M is closed, the Novikov-Shubin invariants were computed in [24]. If M is not closed then we use a Mayer-Vietoris construction in the analytic setting, along with Theorem 2.3, to derive needed inequalities on the Novikov-Shubin invariants of the compact manifold, defined with absolute boundary conditions.

Theorem 0.1 is proven in Section 6. Section 7 has some remarks and gives some conjectures that are supported by the results of this paper. To understand the statements of Sections 3-7, it suffices to understand Definitions 1.3, 1.8 and 2.1.

1. L^2-Betti numbers and Novikov-Shubin invariants of morphisms of Hilbert $\mathcal{A}$-modules

In this section we introduce the L^2-Betti numbers and Novikov-Shubin invariants of morphisms of Hilbert $\mathcal{A}$-modules over a finite von Neumann algebra $\mathcal{A}$. We study their behaviour under composition. For background material on finite von Neumann algebras and their Hilbert modules, we refer to [1, 8, 11, 31].

Let $\mathcal{A}$ be a von Neumann algebra with finite faithful normal trace $\mathrm{tr}_{\mathcal{A}}$. Let $l^2(\mathcal{A})$ denote the Hilbert completion of $\mathcal{A}$ with respect to the inner product given by $\mathrm{tr}_{\mathcal{A}}(a^*b)$ for $a, b \in \mathcal{A}$. A *Hilbert $\mathcal{A}$-module* is a Hilbert space V with a continuous left $\mathcal{A}$-module structure such that there is an isometric $\mathcal{A}$-embedding of V into $l^2(\mathcal{A}) \otimes H$ for some Hilbert space H. A *morphism* $f : U \to V$ of Hilbert $\mathcal{A}$-modules is a bounded operator from U to V which commutes with multiplication by $\mathcal{A}$. A morphism $f : U \to V$ is a *weak isomorphism* if its kernel is trivial and its image is dense. A sequence $0 \to U \xrightarrow{j} V \xrightarrow{q} W \to 0$ of Hilbert $\mathcal{A}$-modules is *weakly exact* if j is injective, $\mathrm{clos}(\mathrm{im}(j)) = \ker(q)$ and q has dense image.

A Hilbert $\mathcal{A}$-module V is *finitely generated* if for some positive integer n, there is a surjective morphism $\oplus_{i=1}^n l^2(\mathcal{A}) \to V$. The *dimension* $\dim_{\mathcal{A}}(V)$ of a finitely generated Hilbert $\mathcal{A}$-module is the trace of any projection $pr : \oplus_{i=1}^n l^2(\mathcal{A}) \to \oplus_{i=1}^n l^2(\mathcal{A})$ whose image is isometrically $\mathcal{A}$-isomorphic to V. The notion of dimension can be extended to arbitrary Hilbert $\mathcal{A}$-modules if one allows $\dim_{\mathcal{A}}$ to take value in $[0, \infty]$.

A morphism $f : U \to V$ has a polar decomposition $f = i|f|$ as a product of morphisms. Here $|f| : U \to U$ is a positive operator given by $|f| = \sqrt{f^*f}$ and $i : U \to V$ is a partial isometry which restricts to an isometry between $\ker(f)^{\perp}$ and $\mathrm{clos}(\mathrm{im}(f))$. In particular, if f is a weak isomorphism then i is unitary, and so $\dim_{\mathcal{A}}(U) = \dim_{\mathcal{A}}(V)$.

The von Neumann algebras of interest to us arise from a countable discrete group π. The group von Neumann algebra $\mathcal{N}(\pi)$ is defined to be the algebra of bounded operators on $l^2(\pi)$ which commute with right multiplication by π. Letting e denote the identity element of π, the trace on $\mathcal{N}(\pi)$ is given by $\mathrm{tr}_{\mathcal{N}(\pi)}(f) = \langle f(e), e\rangle$. Then $l^2(\mathcal{N}(\pi))$ is the same as $l^2(\pi)$.

Lemma 1.1. 1. $\dim_{\mathcal{A}}(U) = 0$ *if and only if* $U = 0$.

2. *If* $U \subset V$ *then* $\dim_{\mathcal{A}}(U) \leqq \dim_{\mathcal{A}}(V)$.

3. *If* $U_1 \supset U_2 \supset \ldots$ *is a nested sequence of Hilbert $\mathcal{A}$-submodules of* U *with* $\dim_{\mathcal{A}} U_1 < \infty$ *then*

$$\dim_{\mathcal{A}}\left(\bigcap_{n=1}^{\infty} U_n\right) = \lim_{n\to\infty} \dim_{\mathcal{A}}(U_n).$$

4. *If* $0 \to U \xrightarrow{j} V \xrightarrow{q} W \to 0$ *is weakly exact then*

$$\dim_{\mathcal{A}}(V) = \dim_{\mathcal{A}}(U) + \dim_{\mathcal{A}}(W).$$

Proof. The first three assertions follow from the assumption that $\mathrm{tr}_{\mathscr{A}}$ is a faithful normal trace. For the last assertion, we have canonical weak isomorphisms $U \to \ker(q)$ and $\ker(q)^{\perp} \to W$. As $V = \ker(q) \oplus \ker(q)^{\perp}$, the assertion follows. □

Let $f : U \to V$ be a morphism of Hilbert $\mathscr{A}$-modules. Let $\{E_{\lambda}^{f^* f} : \lambda \in R\}$ denote the (right-continuous) family of spectral projections of the positive operator $f^* f$. In what follows, $|x|$ will denote the norm of an element in a Hilbert $\mathscr{A}$-module and $\|f\|$ will denote an operator norm.

Lemma 1.2. *For $\lambda \geqq 0$, if $x \in U$ is such that $E_{\lambda^2}^{f^* f}(x) = 0$ and $x \neq 0$ then $|f(x)| > \lambda \cdot |x|$. If $E_{\lambda^2}^{f^* f}(x) = x$ then $|f(x)| \leqq \lambda \cdot |x|$.*

Proof. From the definition of the spectral family, we have

$$\langle f^* f(x), x\rangle = \int_0^{\infty} \lambda \, d\langle E_{\lambda}^{f^* f}(x), x\rangle.$$

Since $\langle f^* f(x), x\rangle = |f(x)|^2$, the claim follows. □

Definition 1.3. Define the *spectral density function* $F : [0,\infty) \to [0,\infty]$ of f by

$$F(f,\lambda) = \dim_{\mathscr{A}} \left(\mathrm{im}(E_{\lambda^2}^{f^* f})\right).$$

We say that f is *left-Fredholm* if there is a $\lambda > 0$ such that $F(f,\lambda) < \infty$. □

(To see the relationship with the usual notion of Fredholmness, suppose that $\mathscr{A} = C$. Then f is Fredholm if and only if f and f^* are left-Fredholm, and f is semi-Fredholm if and only if f or f^* is left-Fredholm [3].)

Lemma 1.4. *Let $f : U \to V$ be a left-Fredholm weak isomorphism. Let $L \subset V$ be a Hilbert $\mathscr{A}$-submodule. Then f restricts to a weak isomorphism from $f^{-1}(L)$ to L.*

Proof. From the polar decomposition of f, we may assume that $U = V$ and f is positive. Clearly the restriction of f to $f^{-1}(L)$ is 1-1, and it is enough to show that $f(f^{-1}(L))$ is dense in L. Now L has an orthogonal decomposition of the form $L = \mathrm{clos}(f(f^{-1}(L))) \oplus M$, where M is an $\mathscr{A}$-submodule of L. As $f(f^{-1}(M)) \subset M$ and $f(f^{-1}(M)) \subset f(f^{-1}(L))$, it follows that $f(f^{-1}(M)) = 0$. Thus $M \cap \mathrm{im}(f) = 0$. If we can show that $\dim_{\mathscr{A}} M = 0$ then Lemma 1.1 will imply that $M = 0$, and we will be done. For $\lambda > 0$, consider the map $\pi_{\lambda} : M \to E_{\lambda}^{f}(U)$ given by $\pi_{\lambda}(m) = E_{\lambda}^{f}(m)$. If $m \in \ker(\pi_{\lambda})$ then the spectral theorem shows that $m \in \mathrm{im}(f)$. Thus $\ker(\pi_{\lambda}) = 0$, and Lemma 1.1 implies that $\dim_{\mathscr{A}} M \leqq \dim_{\mathscr{A}}(E_{\lambda}^{f}(U))$. As f is 1-1 and left-Fredholm, Lemma 1.1 implies that $\lim_{\lambda \to 0^+} \dim_{\mathscr{A}}(E_{\lambda}^{f}(U)) = 0$. Thus $\dim_{\mathscr{A}} M = 0$. □

Let $\mathscr{L}(f,\lambda)$ denote the set of all Hilbert $\mathscr{A}$-submodules L of U with the property that if $x \in L$ then $|f(x)| \leqq \lambda \cdot |x|$.

Lemma 1.5. $F(f,\lambda) = \sup\{\dim_{\mathscr{A}}(L) \,:\, L \in \mathscr{L}(f,\lambda)\}$.

Proof. From Lemma 1.2, the image of $E_{\lambda^2}^{f^*f}$ belongs to $\mathscr{L}(f,\lambda)$. Hence

$$F(f,\lambda) \leqq \sup\{\dim_{\mathscr{A}}(L) \,:\, L \in \mathscr{L}(f,\lambda)\},$$

and it remains to show that for all $L \in \mathscr{L}(f,\lambda)$, $\dim_{\mathscr{A}}(L) \leqq \dim_{\mathscr{A}}(\mathrm{im}(E_{\lambda^2}^{f^*f}))$. Lemma 1.2 implies that $\ker(E_{\lambda^2}^{f^*f}|_L)$ is trivial. Hence $E_{\lambda^2}^{f^*f}$ induces a weak isomorphism from L to $\mathrm{clos}(E_{\lambda^2}^{f^*f}(L))$, and the claim follows from Lemma 1.1. □

Lemma 1.6. *Let $f : U \to V$ and $g : V \to W$ be morphisms of Hilbert $\mathscr{A}$-modules. Then*

1. $F(f,\lambda) \leqq F(gf, \|g\| \cdot\lambda)$.
2. $F(g,\lambda) \leqq F(gf, \|f\| \cdot\lambda)$ *if f is left-Fredholm and has dense image.*
3. $F(gf,\lambda) \leqq F(g,\lambda^{1-r}) + F(f,\lambda^r)$ *for all $r \in (0,1)$.*

Proof. 1. Consider $L \in \mathscr{L}(f,\lambda)$. For all $x \in L$, $|gf(x)| \leqq \|g\| \cdot |f(x)| \leqq \|g\| \cdot\lambda\cdot |x|$. This implies that $L \in \mathscr{L}(gf, \|g\| \cdot\lambda)$, and the claim follows.

2. Consider $L \in \mathscr{L}(g,\lambda)$. For all $x \in f^{-1}(L)$, we have $|gf(x)| \leqq \lambda\cdot |f(x)| \leqq \lambda\cdot \|f\| \cdot |x|$, implying $f^{-1}(L) \in \mathscr{L}(gf, \|f\| \cdot\lambda)$. Hence it remains to show $\dim_{\mathscr{A}}(L) \leqq \dim_{\mathscr{A}}(f^{-1}(L))$. Let $p : U \to U/\ker f$ be projection and let $\overline{f} : U/\ker(f) \to V$ be the map induced by f. Clearly $\overline{f}$ is also left-Fredholm. Since p is surjective and $\overline{f}$ is a weak isomorphism, Lemmas 1.1 and 1.4 imply that $\dim_{\mathscr{A}}\left(f^{-1}(L)\right) \geqq \dim_{\mathscr{A}}\left(p(f^{-1}(L))\right) = \dim_{\mathscr{A}}(\overline{f}^{-1}(L)) = \dim_{\mathscr{A}}(L)$.

3. Consider $L \in \mathscr{L}(gf,\lambda)$. Let L_0 be the kernel of $E_{\lambda^{2r}}^{f^*f}|_L$. We have a weakly exact sequence $0 \to L_0 \to L \to \mathrm{clos}(E_{\lambda^{2r}}^{f^*f}(L)) \to 0$. From Lemma 1.2, we have that $|f(x)| > \lambda^r\cdot |x|$ for all nonzero $x \in L_0$. In particular, $f|_{L_0} : L_0 \to \mathrm{clos}(f(L_0))$ is a weak isomorphism, and so Lemma 1.1 implies that $\dim_{\mathscr{A}}(L_0) = \dim_{\mathscr{A}}(\mathrm{clos}(f(L_0)))$. For $x \in L_0$, we have

$$|gf(x)| \leqq \lambda\cdot |x| \leqq \frac{\lambda}{\lambda^r}\cdot |f(x)| = \lambda^{1-r}\cdot |f(x)|.$$

Hence $\mathrm{clos}(f(L_0)) \in \mathscr{L}(g,\lambda^{1-r})$. This shows that $\dim_{\mathscr{A}}(L_0) \leqq F(g,\lambda^{1-r})$. From Lemma 1.1, $\dim_{\mathscr{A}}(\mathrm{clos}(E_{\lambda^{2r}}^{f^*f}(L))) \leqq \dim_{\mathscr{A}}(\mathrm{im}(E_{\lambda^{2r}}^{f^*f})) = F(f,\lambda^r)$ and $\dim_{\mathscr{A}}(L) = \dim_{\mathscr{A}}(L_0) + \dim_{\mathscr{A}}(\mathrm{clos}(E_{\lambda^r}^{f^*f}(L)))$. This implies that $\dim_{\mathscr{A}}(L) \leqq F(g,\lambda^{1-r}) + F(f,\lambda^r)$. □

Definition 1.7. We say that a function $F : [0,\infty) \to [0,\infty]$ is a *density function* if F is monotone non-decreasing and right-continuous. If F and G are two density functions, we write $F \preceq G$ if there are $C > 0$ and $\varepsilon > 0$ such that $F(\lambda) \leqq G(C\cdot\lambda)$ holds for all $\lambda \in [0,\varepsilon]$. As in [18, 37], we say that F and G are *dilatationally equivalent* (in signs $F \simeq G$) if $F \preceq G$ and $G \preceq F$. We say that F is *Fredholm* if there is a $\lambda > 0$ such that $F(\lambda) < \infty$. □

Of course, the spectral density function $F(f,\lambda)$ is a density function, and if f is left-Fredholm then $F(f,\lambda)$ is a Fredholm density function.

Definition 1.8. Let F be a Fredholm density function. The *Betti number* of F is

$$b(F) = F(0).$$

Its *Novikov-Shubin invariant* is

$$\alpha(F) = \liminf_{\lambda\to 0^+} \frac{\ln(F(\lambda)-b(F))}{\ln(\lambda)} \in [0,\infty],$$

provided that $F(\lambda) > b(F)$ holds for all $\lambda > 0$. Otherwise, we put $\alpha(F) = \infty^+$. If f is a left-Fredholm morphism of Hilbert $\mathscr{A}$-modules, we write $b(f) = b(F(f,\lambda))$ and $\alpha(f) = \alpha(F(f,\lambda))$. □

Here ∞^+ is a new formal symbol which should not be confused with ∞. We have $\alpha(F) = \infty^+$ if and only if there is an $\varepsilon > 0$ such that $F(\lambda) = b(F)$ for $\lambda < \varepsilon$. We note that any non-negative real number, ∞ or ∞^+ can occur as the value of the Novikov-Shubin invariant of a spectral density function. We make the following conventions:

Convention 1.9. The ordering on $[0,\infty]\cup\{\infty^+\}$ is given by the standard ordering on R along with $r < \infty < \infty^+$ for all $r \in R$. For all $\alpha, \beta \in [0,\infty]\cup\{\infty^+\}$ we define

$$\frac{1}{\alpha} \leqq \frac{1}{\beta} \Leftrightarrow \alpha \geqq \beta.$$

Given $\alpha,\beta \in [0,\infty]\cup\{\infty^+\}$, we give meaning to γ in the expression

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{\gamma}$$

as follows: If $\alpha,\beta \in R$, let γ be the real number for which this arithmetic expression of real numbers is true. If $\alpha \in R$ and $\beta \in \{\infty,\infty^+\}$, put γ to be α. If $\beta \in R$ and $\alpha \in \{\infty,\infty^+\}$, put γ to be β. If α and β belong to $\{\infty,\infty^+\}$ and are not both ∞^+, put $\gamma = \infty$. If both α and β are ∞^+, put $\gamma = \infty^+$. Given $r \in (0,\infty)$ and $\alpha \in [0,\infty)$, we define $r\alpha \in [0,\infty)$ to be the ordinary product of real numbers, and we put $r\infty = \infty$ and $r\infty^+ = \infty^+$. For example,

$$\frac{1}{\infty} + \frac{1}{\pi} = \frac{1}{\pi}, \quad \frac{1}{\infty^+} + \frac{1}{\pi} = \frac{1}{\pi}, \quad \frac{1}{\infty} + \frac{1}{\infty^+} = \frac{1}{\infty}, \quad \frac{1}{\infty^+} + \frac{1}{\infty^+} = \frac{1}{\infty^+},$$

$$\frac{1}{\alpha} \leqq \frac{1}{\infty} + \frac{1}{4} + \frac{1}{2} \Leftrightarrow \alpha \geqq 4/3 \quad \text{and} \quad \frac{1}{\alpha} \leqq \frac{1}{\infty} + \frac{1}{\infty^+} + \frac{1}{\infty} \Leftrightarrow \alpha \geqq \infty. \qquad \square$$

Here are the basic properties of these invariants.

Lemma 1.10. *Let F and G be density functions and $f : U \to V$ be a morphism of $\mathcal{A}$-Hilbert modules. Assume that G is Fredholm. Then:*

1. *If $F \preceq G$ then F is Fredholm and $b(F) \leqq b(G)$.*
2. *If $F \preceq G$ and $b(F) = b(G)$ then $\alpha(F) \geqq \alpha(G)$.*
3. *If $F \simeq G$ then $b(F) = b(G)$ and $\alpha(F) = \alpha(G)$.*
4. *$\alpha(G(\lambda^r)) = r \cdot \alpha(G(\lambda))$ for $r \in (0, \infty)$.*
5. *$\alpha(G) = \alpha(G - b(G))$.*
6. *If f is left-Fredholm then $b(f) = \dim_{\mathcal{A}}(\ker(f^* f)) = \dim_{\mathcal{A}}(\ker(f))$.*
7. *If f is zero and $\dim_{\mathcal{A}} U < \infty$ then f is left-Fredholm and $\alpha(f) = \infty^+$.*
8. *An endomorphism $f : U \to U$ is an isomorphism if and only if f is left-Fredholm, $b(f) = 0$ and $\alpha(f) = \infty^+$.*
9. *Assume that $i : U' \to U$ is injective with closed image and $p : V \to V'$ is surjective with finite-dimensional kernel. Then f is left-Fredholm if and only $i \circ f \circ p$ is left-Fredholm, and in this case $\alpha(i \circ f \circ p) = \alpha(f)$.*
10. *If F and G are Fredholm then $\alpha(F + G) = \min\{\alpha(F), \alpha(G)\}$.* □

Proof. The assertions 1. to 5. follow directly from the definitions.

6. By definition, $b(f)$ is the von Neumann dimension of $\operatorname{im}(E_0^{f^*f}) = \ker(f^*f)$. As $|f(x)|^2 = \langle f^*f(x), x\rangle$, f and f^*f have the same kernel.

7. If f is zero then $F(f, \lambda) = \dim_{\mathcal{A}} U$ for all nonnegative λ.

8. If f is an isomorphism then the spectrum of f^*f is bounded below by a positive number, and so $F(f, \lambda)$ vanishes for small nonnegative λ. Conversely, suppose that f is left-Fredholm, $b(f) = 0$ and $\alpha(f) = \infty^+$. Then the spectrum of f^*f is contained in $[a, b]$ for some positive real numbers $a \leqq b$, and an inverse of f^*f is given by $\int_a^b \lambda^{-1} dE_\lambda^{f^*f}$. An inverse of f is given by $(f^*f)^{-1} f^*$.

9. By the open mapping theorem, there is a positive constant C such that for all x,

$$C^{-1} \cdot |x| \leqq |i(x)| \leqq C \cdot |x|.$$

Hence $F(f \circ p, \lambda)$ and $F(i \circ f \circ p, \lambda)$ are dilatationally equivalent. Assertion 3.) implies that $i \circ f \circ p$ is left-Fredholm if and only if $f \circ p$ is left-Fredholm, and then $\alpha(i \circ f \circ p) = \alpha(f \circ p)$. We may write p as the composition $j \circ \mathrm{pr}$ of an isomorphism j and a projection pr. Now one easily checks that $F(f \circ j, \lambda)$ and $F(f, \lambda)$ are dilatationally equivalent, and that for all $\lambda \geqq 0$, we have $F(f \circ j, \lambda) + \dim_{\mathcal{A}}(\ker(\mathrm{pr})) = F(f \circ p, \lambda)$. Then assertions 3.) and 5.) prove the claim.

10. As $b(F + G) = b(F) + b(G)$, by assertion 5.) we may assume without loss of generality that $b(F) = b(G) = b(F + G) = 0$. As $F, G \leqq F + G$, assertion 2.) implies that $\alpha(F + G) \leqq \min\{\alpha(F), \alpha(G)\}$. To verify the reverse inequality, we may assume without loss of generality that $\alpha(F) \leqq \alpha(G)$. The cases $\alpha(F) = 0$ and $\alpha(F) = \infty^+$ are trivial, and so we assume that $0 < \alpha(F) \leqq \infty$. Consider any real number α satisfying $0 < \alpha < \alpha(F)$. Then there exists a constant $K > 0$ such that for small positive λ we have $F(\lambda), G(\lambda) \leqq K\lambda^\alpha$, and so $F(\lambda) + G(\lambda) \leqq 2K \cdot \lambda^\alpha$, implying that $\alpha \leqq \alpha(F + G)$. The assertion follows. □

Lemma 1.11. *Let $f : U \to V$ and $g : V \to W$ be morphisms of Hilbert $\mathcal{A}$-modules.*

1. *If gf is left-Fredholm then f is left-Fredholm. If in addition* $\ker(g) \cap \operatorname{im}(f) = \{0\}$ *then*

$$\alpha(f) \geqq \alpha(gf).$$

2. *Suppose that f is left-Fredholm and has dense image. If gf is left-Fredholm then g is left-Fredholm and*

$$\alpha(g) \geqq \alpha(gf).$$

3. *Suppose that f and g are left-Fredholm. Then gf is left-Fredholm. If in addition* $\ker(g) \subset \operatorname{clos}(\operatorname{im}(f))$ *then*

$$\frac{1}{\alpha(gf)} \leqq \frac{1}{\alpha(f)} + \frac{1}{\alpha(g)}.$$

Proof 1. The Fredholmness claim follows from Lemma 1.6. If in addition $\ker(g) \cap \operatorname{im}(f) = \{0\}$ then $\ker(gf) = \ker(f)$ and hence $b(gf) = b(f)$. Now the assertion follows from Lemma 1.6 and Lemma 1.10.2.
2. and 3. The Fredholmness claims follow from Lemma 1.6. We can factorize f as a product of a projection $p : U \to U/\ker(f)$ and an injective morphism $\overline{f} : U/\ker(f) \to V$. From Lemma 1.10.9, $\alpha(\overline{f}) = \alpha(f)$ and $\alpha(g\overline{f}) = \alpha(gf)$, so we may assume without loss of generality that f is injective. Then f induces an injection $\ker(gf) \to \ker(g)$, and Lemma 1.1 gives that $b(gf) \leqq b(g)$.

If f has dense image then Lemma 1.6 gives $F(g,\lambda) - b(g) \leqq F(gf, \|f\| \cdot \lambda) - b(gf)$. Assertion 2.) now follows from Lemma 1.10.2. For assertion 3.), by assumption $\ker(g) \subset \operatorname{clos}(\operatorname{im}(f))$. As $f : U \to \operatorname{clos}(\operatorname{im}(f))$ is assumed to be a weak isomorphism, Lemma 1.4 implies that $b(gf) = b(g) = b(f) + b(g)$. From Lemma 1.6, if $0 < r < 1$ then

$$F(gf,\lambda) - b(gf) \leqq F(f,\lambda^r) - b(f) + F(g,\lambda^{1-r}) - b(g).$$

Parts 2, 4, 5 and 10 of Lemma 1.10 give $\alpha(gf) \geqq \min\{r \cdot \alpha(f), (1-r) \cdot \alpha(g)\}$. Taking inverses gives

$$\frac{1}{\alpha(gf)} \leqq \max\left\{\frac{1}{r \cdot \alpha(f)}, \frac{1}{(1-r) \cdot \alpha(g)}\right\}.$$

We only need to consider the case $\alpha(f), \alpha(g) \in (0,\infty)$, the other cases being now trivial. Since $\frac{1}{r \cdot \alpha(f)}$ (resp. $\frac{1}{(1-r)\cdot\alpha(g)}$) is a strictly monotonically decreasing (resp. increasing) function in r, the maximum on the right side, viewed as a function of r, obtains its minimum precisely when the two functions of r have the same value. One easily checks that this is the case if and only if $r = \frac{\alpha(g)}{\alpha(f)+\alpha(g)}$, and the claim follows. □

Lemma 1.12. *Let $\phi : U_1 \to V_1$, $\gamma : U_2 \to V_1$ and $\xi : U_2 \to V_2$ be morphisms of Hilbert $\mathcal{A}$-modules. Then*

1. $\begin{pmatrix} \phi & 0 \\ 0 & \xi \end{pmatrix}$ *is left-Fredholm if and only if ϕ and ξ are left-Fredholm. In this case,*

$$\alpha\begin{pmatrix} \phi & 0 \\ 0 & \xi \end{pmatrix} = \min\{\alpha(\phi), \alpha(\xi)\}.$$

2. *Suppose that ϕ is invertible. Then $\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix}$ is left-Fredholm if and only ξ is left-Fredholm. In this case, $\alpha\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix} = \alpha(\xi)$.*

3. *If ϕ and ξ are left-Fredholm then $\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix}$ is left-Fredholm. If $\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix}$ is left-Fredholm then ϕ is left-Fredholm.*

4. *If $\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix}$ is left-Fredholm and ξ is injective then $\alpha(\phi) \geqq \alpha\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix}$. If in addition ξ is left-Fredholm then $\left(\alpha\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix}\right)^{-1} \leqq \frac{1}{\alpha(\phi)} + \frac{1}{\alpha(\xi)}$.*

5. *If $\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix}$ is left-Fredholm and ϕ has dense image then ξ is left-Fredholm and*

$$\alpha(\xi) \geqq \alpha\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix} \quad \text{and} \quad \left(\alpha\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix}\right)^{-1} \leqq \frac{1}{\alpha(\phi)} + \frac{1}{\alpha(\xi)}.$$

6. *Suppose that ϕ is left-Fredholm and $\mathrm{clos}(\mathrm{im}(\phi))^{\perp}$ is finite-dimensional. Then ϕ^* is left-Fredholm, $F(\phi, \lambda) - b(\phi) = F(\phi^*, \lambda) - b(\phi^*)$ and $\alpha(\phi) = \alpha(\phi^*)$.*

Proof. 1. follows from Lemma 1.10.10., using $F(\begin{pmatrix} \phi & 0 \\ 0 & \xi \end{pmatrix}, \lambda) = F(\phi, \lambda) + F(\xi, \lambda)$.

2. Apply Lemma 1.10.9 and assertion 1.) to $\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix} = \begin{pmatrix} \phi & 0 \\ 0 & \xi \end{pmatrix} \cdot \begin{pmatrix} 1 & \phi^{-1}\gamma \\ 0 & 1 \end{pmatrix}$.

In what follows, we write $\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix} = gf$, where $g = \begin{pmatrix} 1 & \gamma \\ 0 & \xi \end{pmatrix}$ and $f = \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix}$. From assertion 2.), g is left-Fredholm if and only if ξ is left-Fredholm, and in this case $\alpha(g) = \alpha(\xi)$.

3. If ϕ and ξ are left-Fredholm then Lemma 1.10.8 and assertion 1.) imply that g and f are left-Fredholm. Then Lemma 1.11.3 implies that $\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix}$ is left-Fredholm. If $\begin{pmatrix} \phi & \gamma \\ 0 & \xi \end{pmatrix}$ is left-Fredholm then Lemma 1.11.1 implies that f is left-Fredholm, and hence ϕ is left-Fredholm.

4. If ξ is injective then g is injective. The first inequality now follows from Lemma 1.11.1. If ξ is left-Fredholm then g is left-Fredholm and the second inequality follows from Lemma 1.11.3.

5. If ϕ has dense image then f has dense image and Lemma 1.11.2 implies that g is left-Fredholm. Hence ξ is left-Fredholm, and the first inequality follows from Lemma 1.11.2. The second inequality follows from Lemma 1.11.3.
6. Write ϕ as the composition $U \xrightarrow{p} U/\ker(\phi) \xrightarrow{\overline{\phi}} \operatorname{clos}(\operatorname{im}(\phi)) \xrightarrow{i} V$, where p is projection, i is inclusion and $\overline{\phi}$ is a weak isomorphism. Then $\phi^* = p^* \circ \overline{\phi}^* \circ i^*$. Lemma 1.10.9 shows that ϕ is left-Fredholm if and only if $\overline{\phi}$ is left-Fredholm, and one can check that $F(\phi, \lambda) - b(\phi) = F(\overline{\phi}, \lambda)$. As i^* has finite-dimensional kernel, a similiar statement holds for ϕ^* and $\overline{\phi}^*$. Hence we may assume that ϕ is a weak isomorphism. As $\phi(\phi^*\phi) = (\phi\phi^*)\phi$ and $\phi^*(\phi\phi^*) = (\phi^*\phi)\phi^*$, ϕ and ϕ^* induce injective morphisms $\widehat{\phi}_\lambda : \operatorname{im}(E_\lambda^{\phi^*\phi}) \to \operatorname{im}(E_\lambda^{\phi\phi^*})$ and $\widehat{\phi}_\lambda^* : \operatorname{im}(E_\lambda^{\phi\phi^*}) \to \operatorname{im}(E_\lambda^{\phi^*\phi})$. Using Lemma 1.1, we have that $F(\phi, \lambda) = F(\phi^*, \lambda)$. It follows that ϕ^* is left-Fredholm. □

2. L^2-Betti numbers and Novikov-Shubin invariants of Hilbert $\mathscr{A}$-chain complexes

In this section we introduce and study the L^2-Betti numbers and Novikov-Shubin invariants of chain complexes, and investigate their behaviour with respect to exact sequences and homotopy equivalences.

A Hilbert $\mathscr{A}$-chain complex C is a chain complex of Hilbert $\mathscr{A}$-modules whose differentials are morphisms of such modules, i.e. the differentials are bounded operators compatible with the $\mathscr{A}$-action. It is said to be *finite* if C_n is a finitely generated Hilbert $\mathscr{A}$-module for all integers n and is zero for all but a finite number of integers n. Letting $c_p : C_p \to C_{p-1}$ denote the p-th differential of C, the p-th homology group of C is defined by $H_p(C) = \ker(c_p)/\operatorname{clos}(\operatorname{im}(c_{p+1}))$. Note that we have to quotient by the closure of the image of c_{p+1} if we want to ensure that $H_p(C)$ is a Hilbert space. We say that C is *weakly exact* if its homology groups $H_p(C)$ vanish. We say that C is *exact* if $\ker(c_p) = \operatorname{im}(c_{p+1})$ for all p.

Definition 2.1. Let C be a Hilbert $\mathscr{A}$-chain complex with p-th differential c_p. We say that C is *Fredholm at* p if the induced morphism $\overline{c_p} : C_p/\operatorname{clos}(\operatorname{im}(c_{p+1})) \to C_{p-1}$ is left-Fredholm. We say that C is *Fredholm* if C is Fredholm at p for all p. Suppose that C is Fredholm at p. Its p-th *Betti-number is*

$$b_p(C) = \dim_{\mathscr{A}}(H_p(C)).$$

Its p-th Novikov-Shubin invariant *is*

$$\alpha_p(C) = \alpha(c_p).$$

Put

$$\widetilde{\alpha}_p(C) = \min\{\alpha(c_{p+1}), \alpha(c_p)\}. \qquad \square$$

Note that if C is Fredholm at p then $H_p(C) = \ker(\overline{c_p})$ is finite-dimensional. The Fredholmness condition on C is automatically satisfied if C is finite. This will be the case when one deals with the cellular chain complex of a manifold. When one deals with differential forms on a manifold, the Fredholmness is not automatic and requires some extra analysis. The invariant $\widetilde{\alpha}_p(C)$ corresponds to the notion of Novikov-Shubin invariants as introduced in [37]. However, it turns out to be easier and more efficient to deal with the numbers $\alpha_p(C)$.

We begin by recalling the long homology sequence associated to an exact sequence of Hilbert $\mathcal{A}$-chain complexes $0 \to C \xrightarrow{j} D \xrightarrow{q} E \to 0$. There is a sequence

$$\ldots \xrightarrow{\delta_{p+1}} H_p(C) \xrightarrow{H_p(j)} H_p(D) \xrightarrow{H_p(q)} H_p(E) \xrightarrow{\delta_p} H_{p-1}(C) \xrightarrow{H_{p-1}(j)} \ldots$$

where the boundary operator $\delta_p : H_p(E) \to H_{p-1}(C)$ is defined as follows: Let $x \in \ker(e_p)$ be a representative of $[x]$ in $H_p(E)$. Choose $y \in D_p$ so that $q_p(y) = x$, and $z \in \ker(c_{p-1})$ so that $j_{p-1}(z) = d_p(y)$. Then $\delta_p([x])$ is defined to be the class $[z] \in H_{p-1}(C)$. Note that the homology sequence is always defined, but is generally not weakly exact if one makes no Fredholmness assumptions. The next theorem follows from inspecting the proof of [Theorem 2.1].

Theorem 2.2. 1. *The long homology sequence is weakly exact at $H_p(E)$ if C is Fredholm at p.*

2. *The long homology sequence is weakly exact at $H_p(C)$ if D is Fredholm at $p+1$.*

3. *The long homology sequence is weakly exact at $H_p(D)$ if E is Fredholm at $p+1$.* □

The next theorem is the main result of this section. We mention that one can give examples to show that the inequalities below are sharp.

Theorem 2.3. (Additivity inequalities for the Novikov-Shubin invariants). *Let $0 \to C \xrightarrow{j} D \xrightarrow{q} E \to 0$ be an exact sequence of Hilbert $\mathcal{A}$-chain complexes. Let $\delta_p : H_p(E) \to H_{p-1}(C)$ be the boundary operator in the long weakly exact homology sequence.*

1. *Suppose that C and E are Fredholm at p. Then D is Fredholm at p, δ_p is Fredholm and*

$$\frac{1}{\alpha_p(D)} \leq \frac{1}{\alpha_p(C)} + \frac{1}{\alpha_p(E)} + \frac{1}{\alpha(\delta_p)}.$$

2. *Suppose that C is Fredholm at $p-1$ and D is Fredholm at p. Then E is Fredholm at p, $H_{p-1}(j)$ is Fredholm and*

$$\frac{1}{\alpha_p(E)} \leq \frac{1}{\alpha_{p-1}(C)} + \frac{1}{\alpha_p(D)} + \frac{1}{\alpha(H_{p-1}(j))}.$$

3. *Suppose that D is Fredholm at p and E is Fredholm at $p+1$. Then C is Fredholm at p, $H_p(q)$ is Fredholm and*

$$\frac{1}{\alpha_p(C)} \leqq \frac{1}{\alpha_p(D)} + \frac{1}{\alpha_{p+1}(E)} + \frac{1}{\alpha(H_p(q))}.$$

Proof. 1. The exact sequence $0 \to C \xrightarrow{j} D \xrightarrow{q} E \to 0$ induces the following commutative diagram with exact rows, where $\overline{q_p}$, $\overline{d_p}$ and $\overline{e_p}$ are canonical homomorphisms induced from q_p, d_p and e_p, and i is inclusion:

$$\begin{array}{ccccccccc} 0 & \to & \ker \overline{q_p} & \xrightarrow{i} & D_p/\mathrm{clos}(\mathrm{im}(d_{p+1})) & \xrightarrow{\overline{q_p}} & E_p/\ker(e_p) & \to & 0 \\ & & \partial_p \downarrow & & \overline{d_p} \downarrow & & \overline{e_p} \downarrow & & \\ 0 & \to & C_{p-1} & \xrightarrow{j_{p-1}} & D_{p-1} & \xrightarrow{q_{p-1}} & E_{p-1} & \to & 0. \end{array}$$

To define ∂_p in the above diagram, let $x \in \ker(e_p q_p)$ represent $[x] \in \ker(\overline{q_p})$. Then $d_p(x) = j_{p-1}(y)$ for a unique $y \in C_{p-1}$. We put $\partial_p([x]) = y$. (In fact, $y \in \ker(c_{p-1})$.)

Suppose for a moment that we already know that ∂_p is left-Fredholm. From Lemma 1.10.9, $\overline{e_p}$ is left-Fredholm and $\alpha_p(\overline{e_p}) = \alpha_p(E)$. Lemma 1.12.3 implies that $\overline{d_p}$ is left-Fredholm and hence D is Fredholm at p. As $\overline{e_p}$ is injective, Lemma 1.12.4 gives that

$$\frac{1}{\alpha_p(D)} \leqq \frac{1}{\alpha(\partial_p)} + \frac{1}{\alpha_p(E)}.$$

Hence it remains to show that ∂_p and δ_p are left-Fredholm and that

$$\frac{1}{\alpha(\partial_p)} \leqq \frac{1}{\alpha_p(C)} + \frac{1}{\alpha(\delta_p)}.$$

We construct a sequence which we will show to be weakly exact:

$$C_p \xrightarrow{\overline{j_p}} \ker(\overline{q_p}) \xrightarrow{\widehat{q_p}} H_p(E) \to 0.$$

The map $\overline{j_p}$ is induced from j_p in the obvious way. To define $\widehat{q_p}$, consider an $x \in D_p$ whose class $[x] \in D_p/\mathrm{clos}(\mathrm{im}(d_{p+1}))$ lies in $\ker(\overline{q_p})$. Then $q_p(x)$ is in the kernel of e_p and determines a class $[q_p(x)]$ in $H_p(E)$. Define $\widehat{q_p}([x])$ to be $[q_p(x)]$. One easily checks that $\widehat{q_p} \circ \overline{j_p}$ is zero and $\widehat{q_p}$ is surjective. We will show that $\ker(\widehat{q_p})$ is contained in $\mathrm{clos}(\mathrm{im}(\overline{j_p}))$. Consider $[x] \in \ker(\widehat{q_p})$ with representative $x \in D_p$. Since $[q_p(x)] \in H_p(E)$ is zero, there is a sequence $y_n \in E_{p+1}$ such that in E_p:

$$\lim_{n\to\infty} \left(q_p(x) - e_{p+1}(y_n)\right) = 0.$$

As q_{p+1} is surjective, there is a sequence $\{u_n\}_{n=1}^{\infty}$ in D_{p+1} such that $y_n = q_{p+1}(u_n)$. Thus

$$\lim_{n\to\infty} q_p\left(x - d_{p+1}(u_n)\right) = 0.$$

We write $x - d_{p+1}(u_n) = j_p(w_n) + r_n$, where $w_n \in C_p$ and $r_n \in \operatorname{im}(j_p)^\perp$. Then we obtain $\lim_{n\to\infty} q_p(r_n) = 0$. As the restriction of q_p to $\operatorname{im}(j_p)^\perp$ is an isomorphism, we conclude $\lim_{n\to\infty} r_n = 0$. Thus

$$x = \lim_{n\to\infty} \left(j_p(w_n) + d_{p+1}(u_n)\right),$$

and hence in $D_p/\operatorname{clos}(\operatorname{im}(d_{p+1}))$

$$[x] = \lim_{n\to\infty} \overline{j_p}(w_n).$$

This finishes the proof of weak exactness.

Next, we construct a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & \ker(\widehat{q_p}) & \xrightarrow{i_1} & \ker(\overline{q_p}) & \xrightarrow{\widehat{q_p}} & H_p(E) & \to & 0 \\
 & & \overline{\partial_p}\downarrow & & \partial_p\downarrow & & \delta_p\downarrow & & \\
0 & \to & \operatorname{clos}(\operatorname{im}(c_p)) & \xrightarrow{i_2} & \ker(c_{p-1}) & \xrightarrow{\operatorname{pr}} & H_{p-1}(C) & \to & 0.
\end{array}$$

The maps i_1 and i_2 are the canonical inclusions and the map pr is the canonical projection. The map $\overline{\partial_p}$ is induced from ∂_p, and the fact that its range lies in $\operatorname{clos}(\operatorname{im}(c_p))$ follows from the weak exactness of the preceding sequence. One easily verifies that the diagram commutes. The rows are clearly exact.

Let $\widetilde{\overline{j_p}} : C_p \to \ker(\widehat{q_p})$ be the morphism with dense image induced from $\overline{j_p}$. One easily checks that $\overline{\partial_p} \circ \widetilde{\overline{j_p}} = c_p$. As c_p is left-Fredholm by assumption, Lemma 1.11.1 shows that $\widetilde{\overline{j_p}}$ is left-Fredholm. Lemma 1.11.2 implies that $\overline{\partial_p}$ is left-Fredholm and

$$\alpha(\overline{\partial_p}) \geqq \alpha(c_p) = \alpha_p(C).$$

As $H_p(E)$ is finite-dimensional, δ_p is left-Fredholm. Then from Lemma 1.12.3, ∂_p is left-Fredholm. As $\overline{\partial_p}$ has dense image, Lemma 1.12.5 implies

$$\frac{1}{\alpha(\partial_p)} \leqq \frac{1}{\alpha(\overline{\partial_p})} + \frac{1}{\alpha(\delta_p)}.$$

This finishes the proof of the first assertion of Theorem 2.3.

2. Recall that in general [27, p. 213], the n-th differential of the mapping cylinder of a chain map $g : C \to D$ is defined by

$$\begin{pmatrix} -c_{n-1} & 0 & 0 \\ -id & c_n & 0 \\ g_{n-1} & 0 & d_n \end{pmatrix} : C_{n-1} \oplus C_n \oplus D_n \to C_{n-2} \oplus C_{n-1} \oplus D_{n-1}.$$

There is a canonical map $i : C \to \operatorname{cyl}(g)$ and $\operatorname{cone}(g)$ is defined to be the cokernel of i. That is, the n-th differential of $\operatorname{cone}(g)$ is

$$\begin{pmatrix} -c_{n-1} & 0 \\ g_{n-1} & d_n \end{pmatrix} : C_{n-1} \oplus D_n \to C_{n-2} \oplus D_{n-1}.$$

We define cone(C) to be the mapping cone of the identity map on C, and the suspension ΣC to be the mapping cone of the 0-map on C, i.e. $(\Sigma C)_n = C_{n-1}$.

In our case there is a canonical exact sequence $0 \to D \to \mathrm{cyl}(q) \to \mathrm{cone}(q) \to 0$ and chain homotopy equivalences $E \to \mathrm{cyl}(q)$ and $\Sigma C \to \mathrm{cone}(q)$. We will show later that the numbers $\alpha(c_p)$ are homotopy invariants. So we may assume the existence of an exact sequence $0 \to D \to E \to \Sigma C \to 0$. Moreover, the connecting map from $H_p(\Sigma C)$ to $H_{p-1}(D)$ agrees under these identifications with the map $H_{p-1}(j) : H_{p-1}(C) \to H_{p-1}(D)$. The claim now follows from assertion 1.).

3. Repeat the argument in the proof of assertion 2.), yielding a short exact sequence $0 \to E \to \Sigma C \to \Sigma D \to 0$. □

The dual chain complex C^* is the cochain complex with the same chain modules as C and the adjoints of the differentials of C as codifferentials. The definitions of the Betti numbers and the Novikov-Shubin invariants carry over directly to cochain complexes. The *Laplace operator* $\Delta_p : C_p \to C_p$ is defined to be $c_{p+1}c^*_{p+1} + c^*_p c_p$.

Lemma 2.4. *Let C and D be Hilbert $\mathcal{A}$-chain complexes.*

1. *Δ_p is left-Fredholm if and only if C is Fredholm at p and $p+1$. In this case,*

$$2 \cdot \widetilde{\alpha}_p(C) = \alpha(\Delta_p) \quad \textit{and} \quad b_p(C) = b_p(\Delta_p).$$

2. *C is Fredholm at p if and only if C^* is Fredholm at p. In this case,*

$$\alpha_p(C) = \alpha_p(C^*) \quad \textit{and} \quad b_p(C) = b_p(C^*).$$

3. *$C \oplus D$ is Fredholm at p if and only if C and D are Fredholm at p. In this case,*

$$\alpha_p(C \oplus D) = \min\{\alpha_p(C), \alpha_p(D)\} \quad \textit{and} \quad b_p(C \oplus D) = b_p(C) + b_p(D).$$

1. The Hodge decomposition theorem (see e.g. [31, Theorem 3.7] the proof of which extends to the Fredholm case) gives the claim for the Betti numbers. Moreover, we have the following commutative square with isomorphisms as horizontal morphisms:

$$\begin{array}{ccc}
\ker(c_p)^\perp \oplus \ker(c^*_{p+1})^\perp \oplus \ker(\Delta_p) & \xrightarrow{\cong} & C_p \\
c^*_p c_p \oplus c_{p+1}c^*_{p+1} \oplus 0 \downarrow & & \downarrow \Delta_p \\
\ker(c_p)^\perp \oplus \ker(c^*_{p+1})^\perp \oplus \ker(\Delta_p) & \xrightarrow{\cong} & C_p
\end{array}$$

In what follows, we consider c_p to be an operator from $\ker(c_p)^\perp$ to $\ker(c^*_p)^\perp$, and similarly for c_{p+1}, c^*_p and c^*_{p+1}. Lemmas 1.10.7 and 1.12.1 imply that Δ_p is left-Fredholm if and only if both $c^*_p c_p$ and $c_{p+1}c^*_{p+1}$ are left-Fredholm, and in this case, $\alpha(\Delta_p) = \min\{\alpha(c^*_p c_p), \alpha(c_{p+1}c^*_{p+1})\}$.

In general, as $E^{f^*f}_\lambda = E^{(f^*f)^2}_{\lambda^2}$, Lemma 1.10.4 implies that f is left-Fredholm if and only if f^*f is, and in this case $\alpha(f^*f) = 2 \cdot \alpha(f)$. We have shown in Lemma 1.12.6 that if f is left-Fredholm and its cokernel is finite-dimensional then f^* is left-Fredholm and $\alpha(f) = \alpha(f^*)$. This implies that if c_p is left-Fredholm then $c_p^* c_p$ is left-Fredholm, and in this case $2 \cdot \alpha(c_p) = \alpha(c_p^* c_p)$. Moreover, $c_{p+1} c_{p+1}^*$ is left-Fredholm if and only if c_{p+1} is left-Fredholm, and in this case $2 \cdot \alpha(c_{p+1}) = \alpha(c_{p+1} c_{p+1}^*)$. Now the claim follows.

2. follows from assertion 1.)

3. is a consequence of Lemma 1.12.1. □

We recall that C is said to be *contractible* if C has a chain contraction γ, i.e. a collection of morphisms $\gamma_p : C_p \to C_{p+1}$ such that $\gamma_{p-1} c_p + c_{p+1} \gamma_p = \mathrm{id}$. for all p.

Lemma 2.5. *The following assertions are equivalent for a Hilbert $\mathcal{A}$-chain complex C:*

1. *C is contractible.*
2. *Δ_p is invertible for all p.*
3. *C is Fredholm and for all p, $b_p(C) = 0$ and $\alpha_p(C) = \infty^+$.*

Proof. 1. $\Rightarrow$ 3. Using c_p and γ_{p-1}, we can construct morphisms $\overline{c_p} : C_p/\mathrm{clos}(\mathrm{im}(c_{p+1})) \to C_{p-1}$ and $\overline{\gamma_{p-1}} : C_{p-1} \to C_p/\mathrm{clos}(\mathrm{im}(c_{p+1}))$ such that $\overline{\gamma_{p-1}} \circ \overline{c_p} = \mathrm{id}$. Hence $\overline{c_p}$ induces an invertible operator onto its image. Lemma 1.10.8-9 implies that $\overline{c_p}$ is left-Fredholm, $b_p(\overline{c_p}) = 0$ and $\alpha(\overline{c_p}) = \infty^+$.

3. $\Rightarrow$ 2. From Lemma 2.4, Δ_p is left-Fredholm, $b(\Delta_p) = 0$ and $\alpha(\Delta_p) = \infty^+$ for all p. Now apply Lemma 1.10.8.

2. $\Rightarrow$ 1. Suppose that Δ_p is invertible for all p. Then $\Delta_{p+1}^{-1} \circ c_{p+1}^*$ is a chain contraction of C. □

We now reprove the homotopy invariance of the L^2-Betti numbers and the Novikov-Shubin invariants [12, 14, 18]

Theorem 2.6 (Homotopy invariance). *If $f : C \to D$ is a chain homotopy equivalence then for all $p \in Z$ we have*

$$F(c_p) \simeq F(d_p),\quad b_p(C) = b_p(D),\quad \alpha_p(C) = \alpha_p(D) \text{ and } \widetilde{\alpha}_p(C) = \widetilde{\alpha}_p(D).$$

Proof. There are exact sequences of chain complexes $0 \to C \to \mathrm{cyl}(f) \to \mathrm{cone}(f) \to 0$ and $0 \to D \to \mathrm{cyl}(f) \to \mathrm{cone}(C) \to 0$ with $\mathrm{cone}(f)$ and $\mathrm{cone}(C)$ being contractible. We obtain chain isomorphisms $C \oplus \mathrm{cone}(f) \to \mathrm{cyl}(f)$ and $D \oplus \mathrm{cone}(C) \to \mathrm{cyl}(f)$ by the following general construction for an exact sequence $0 \to C \xrightarrow{j} D \xrightarrow{q} E \to 0$ with contractible E: Choose a chain contraction ε for E and for each p a morphism $t_p : E_p \to D_p$ such that $q_p \circ t_p = \mathrm{id}$. Put

$$s_p = d_{p+1} \circ t_{p+1} \circ \varepsilon_p + t_p \circ \varepsilon_{p-1} \circ e_p.$$

This defines a chain map $s : E \to D$ such that $q \circ s = \mathrm{id}$. Define a chain map $u : D \to C$ by saying that for $x \in D_p$, $u_p(x)$ is the unique $y \in C_p$ such that $x = s_p q_p(x) + j_p(y)$. Then $j + s$ is a chain isomorphism $C \oplus E \to D$, with inverse $u \oplus q$. Since $C \oplus \mathrm{cone}(f)$ and $D \oplus \mathrm{cone}(C)$ are isomorphic and $\mathrm{cone}(f)$ and $\mathrm{cone}(C)$ are contractible, Lemma 2.5 implies that $F(c_p) \simeq F(d_p)$, from which the other assertions follow. □

3. L^2-Betti numbers and Novikov-Shubin invariants of manifolds

In this section we analyse the L^2-Betti numbers and the Novikov-Shubin invariants of compact manifolds.

Throughout this section we will use the following setup: Let M be a compact connected orientable smooth manifold of dimension m with fundamental group π and universal cover $\widetilde{M}$. Suppose that ∂M is the union of two submanifolds $\partial_0 M$ and $\partial_1 M$ such that $\partial(\partial_0 M) = \partial_0 M \cap \partial_1 M = \partial(\partial_1 M)$. We allow that $\partial_0 M$ or $\partial_1 M$ are empty. Let $\widetilde{\partial_0 M}$ denote the preimage of $\partial_0 M$ under the projection $\widetilde{M} \to M$. Let $\mathscr{A}$ be a finite von Neumann algebra, V be a finitely generated Hilbert $\mathscr{A}$-module and $\mu : \pi \to \mathrm{Iso}_{\mathscr{A}}(V)^{op}$ be a right unitary representation of π. In most applications $\mathscr{A}$ will be the von Neumann algebra $N(\pi)$ of π, V will be $l^2(\pi)$ and μ will be the right regular representation.

Let $C(\widetilde{M}, \widetilde{\partial_0 M})$ be the cellular $Z\pi$-chain complex coming from the lift of any CW-decomposition of $(M, \partial_0 M)$ to a π-equivariant CW-decomposition of $(\widetilde{M}, \widetilde{\partial_0 M})$. Note that π acts on the left on $C(\widetilde{M}, \widetilde{\partial_0 M})$, and on the right on V. Let $C(M, \partial_0 M; V)$ denote the Hilbert $\mathscr{A}$-chain complex $V \otimes_{Z\pi} C(\widetilde{M}, \widetilde{\partial_0 M})$. If c denotes the differential of $C(M, \partial_0 M; V)$, define the L^2-homology $H_p(M, \partial_0 M; V)$ with coefficients in V to be the Hilbert $\mathscr{A}$-module $\ker(c_p)/\mathrm{clos}(\mathrm{im}(c_p))$. In this section we will only deal with homology. We note that the corresponding cohomology groups are isometrically isomorphic to the homology groups. Recall that we have defined the L^2-Betti numbers and Novikov-Shubin invariants for chain complexes in Definition 2.1. Since they are homotopy invariants (see Theorem 2.6), the following definition is independent of the choice of the CW-decomposition:

Definition 3.1. Define the p-th *L^2-Betti-number* of $(M, \partial_0 M)$, with coefficients in V, to be

$$b_p(M, \partial_0 M; V) = b_p(C(M, \partial_0 M; V)) = \dim_{\mathscr{A}}(H_p(M, \partial_0 M; V)).$$

Define the p-th *Novikov-Shubin invariant* to be

$$\alpha_p(M, \partial_0 M; V) = \alpha_p(C(M, \partial_0 M; V))$$

and put

$$\widetilde{\alpha}_p(M, \partial_0 M; V) = \widetilde{\alpha}_p(C(M, \partial_0 M; V)).$$

If $V = l^2(\pi)$ then we abbreviate:

$$b_p(M, \partial_0 M) = b_p(M, \partial_0 M; l^2(\pi));$$

$$\alpha_p(M, \partial_0 M) = \alpha_p(M, \partial_0 M; l^2(\pi));$$

$$\widetilde{\alpha}_p(M, \partial_0 M) = \widetilde{\alpha}_p(M, \partial_0 M; l^2(\pi)).$$

We abbreviate $b_p(M, \emptyset)$ by $b_p(M)$, $\alpha_p(M, \emptyset)$ by $\alpha_p(M)$ and $\widetilde{\alpha}_p(M, \emptyset)$ by $\widetilde{\alpha}_p(M)$ □

We refer to $\alpha_p(M, \partial_0 M; V)$ as the Novikov-Shubin invariant, whereas in the previous literature $\widetilde{\alpha}_p(M, \partial_0 M; V)$ is called the Novikov-Shubin invariant. Also, in previous articles the values ∞ and ∞^+ are not distinguished. Moreover, we use the normalization of [24], which differs by a factor of 2 from that used in [14, 18, 37].

We start with Poincaré duality. It gives a $Z\pi$-chain homotopy equivalence

$$\cap [M] : C^{m-*}(\widetilde{M}, \widetilde{\partial_1 M}) \to C_*(\widetilde{M}, \widetilde{\partial_0 M}).$$

Tensoring over $Z\pi$ with V then gives a chain homotopy equivalence of Hilbert $\mathcal{A}$-chain complexes. From Theorem 2.6 and Lemma 2.4 we derive

Proposition 3.2. [Poincaré duality]. 1. $b_{m-p}(M, \partial_1 M; V) = b_p(M, \partial_0 M; V)$.
2. $\alpha_{m+1-p}(M, \partial_1 M; V) = \alpha_p(M, \partial_0 M; V)$.
3. $\widetilde{\alpha}_{m-p}(M, \partial_1 M; V) = \widetilde{\alpha}_p(M, \partial_0 M; V)$. □

Lemma 3.3. *Let $(f, f_0) : (M, \partial_0 M) \to (N, \partial_0 N)$ be a map between pairs such that f and f_0 are n-connected for some $n \geqq 2$. Then*
1. *$b_p(M, \partial_0 M; V) = b_p(N, \partial_0 N; V)$ for $p \leqq n-1$ and $b_n(M, \partial_0 M; V) \geqq b_n(N, \partial_0 N; V)$.*
2. *$\alpha_p(M, \partial_0 M; V) = \alpha_p(N, \partial_0 N; V)$ for $p \leqq n$.*

Proof. Let $C(\widetilde{f}) : C(\widetilde{M}, \widetilde{\partial_0 M}) \to C(\widetilde{N}, \widetilde{\partial_0 N})$ be the $Z\pi$-chain map induced by $\widetilde{f}$. We will abbreviate cyl$(C(\widetilde{f}))$ by cyl and cone$(C(\widetilde{f}))$ by cone. We have the exact sequence

$$0 \longrightarrow C(\widetilde{M}, \widetilde{\partial_0 M}) \xrightarrow{\iota} \text{cyl} \xrightarrow{\text{pr}} \text{cone} \longrightarrow 0$$

Let P be the subcomplex of cone such that $P_i = \{0\}$ for $i \leqq n$, P_{n+1} is the kernel of the $(n+1)$-differential of cone and $P_i = \text{cone}_i$ for $i > n+1$. As cone is n-connected by the Hurewicz theorem, P_{n+1} is finitely-generated stably free, and the inclusion of P into cone is a homotopy equivalence. A chain complex C is *elementary* if it is concentrated in two adjacent dimensions n and $n+1$ and is given there by the same module $C_{n+1} = C_n$, with the identity as the $n+1$-th differential. By possibly adding a finitely-generated free elementary chain complex concentrated in dimensions $n+1$ and $n+2$ to P, we obtain a finite free $Z\pi$-chain complex Q together with a chain homotopy equivalence $g : Q \to$ cone. Let D be the pullback chain complex of $g : Q \to$ cone and the canonical projection cyl $\to$ cone, i.e. the kernel of $g \oplus \text{pr} : Q \oplus \text{cyl} \to \text{cone}$. Then we obtain a short exact sequence

$$0 \to C(\widetilde{M}, \widetilde{\partial_0 M}) \to D \to Q \to 0$$

of finitely-generated free $Z\pi$-chain complexes such that D is chain homotopy equivalent to $C(\widetilde{N}, \widetilde{\partial_0 N})$ and $Q_i = \{0\}$ for $i \leqq n$. By Theorem 2.6, it suffices to prove the claim for $l^2(\pi) \otimes_{Z\pi} C(\widetilde{M}, \widetilde{\partial_0 M})$ and $l^2(\pi) \otimes_{Z\pi} D$. Since these chain complexes have the same chain modules and differentials in dimensions less than or equal to n, the claim follows. □

Corollary 3.4. 1. *The L^2-Betti numbers $b_p(M)$ (respectively the Novikov-Shubin invariants $\alpha_p(M)$) of a compact connected manifold depend only on the fundamental group provided that $p \leqq 1$ (respectively $p \leqq 2$).*

2. *The L^2-Betti numbers $b_p(M)$ and the Novikov-Shubin invariants $\alpha_p(M)$ of a closed connected 3-manifold depend only on the fundamental group.*

3. *The Novikov-Shubin invariants $\alpha_p(M)$ of a closed connected 4-manifold depend only on the fundamental group.*

Proof. The classifying map $M \to B\pi$ for $\pi = \pi_1(M)$ is 2-connected, and $B\pi$ can be chosen to be a CW-complex whose 2-skeleton $B\pi^2$ is finite. Hence Lemma 3.3 implies that $\alpha_p(M) = \alpha_p(B\pi^2)$ (respectively $b_p(M) = b_p(B\pi^2)$) depends only on π provided that $p \leqq 2$ (respectively $p \leqq 1$). (Note that in the proof of Lemma 3.2, one only needs that $C_p(\widetilde{N}, \partial_0\widetilde{N})$ be a finitely generated $Z\pi_1(N)$-module for $p \leqq n$.) The other claims follow from Theorem 3.2 on Poincaré duality. □

Note that the second L^2-Betti number of a closed 4-manifold depends on more than just the fundamental group. For example, by taking repeated connected sums with CP^2 one can increase b_2 by any positive integer.

In the top and bottom dimensions the invariants can be computed completely. We recall that a finitely generated group Γ is said to be *amenable* if there is a π-invariant bounded linear operator $\mu : L^\infty(\Gamma) \to R$ such that

$$\inf\{f(\gamma) : \gamma \in \Gamma\} \leqq \mu(f) \leqq \sup\{f(\gamma) : \gamma \in \Gamma\}.$$

Note that any finitely generated abelian group is amenable and any finite group is amenable. A subgroup and a quotient group of an amenable group are amenable. An extension of an amenable group by an amenable group is amenable. A group containing a free group on two generators is not amenable. A finitely generated group Γ is *nilpotent* if Γ possesses a finite lower central series

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_s = \{1\} \qquad \Gamma_{k+1} = [\Gamma, \Gamma_k].$$

If $\overline{\Gamma}$ contains a nilpotent subgroup Γ of finite index then $\overline{\Gamma}$ is said to be *virtually nilpotent*. Let d_i be the rank of the quotient Γ_i/Γ_{i+1} and let d be the integer $\sum_{i \geqq 1} id_i$. Then $\overline{\Gamma}$ has polynomial growth of degree d [2]. Note that a group has polynomial growth if and only if it is virtually nilpotent [16].

Lemma 3.5. 1. $\alpha_1(M) = \widetilde{\alpha}_0(M)$ *is finite if and only if π is infinite and virtually nilpotent. In this case, $\alpha_1(M)$ is the growth rate of π.*

2. $\alpha_1(M) = \widetilde{\alpha}_0(M)$ *is ∞^+ if and only if π is finite or nonamenable.*

3. $\alpha_1(M) = \widetilde{\alpha}_0(M)$ *is ∞ if and only if π is amenable and not virtually nilpotent.*

4. $b_0(M) = 0$ *if π is infinite and $1/|\pi|$ otherwise.*

5. *If $\partial_0 M$ is not empty then $\alpha_1(M, \partial_0 M; V)$ and $\alpha_m(M, \partial_1 M; V)$ are equal to ∞^+ and $b_0(M, \partial_0 M; V)$ and $b_m(M, \partial_1 M; V)$ are zero.*

6. *If $\partial_0 M$ is empty then $\alpha_m(M; V) = \alpha_1(M; V)$ and $b_m(M; V) = b_0(M; V)$*

Proof. 1. to 3. Since $\alpha_1(M)$ depends only on the fundamental group and there is a closed manifold with π as its fundamental group, we may assume that M is closed. Efremov [14] shows that $\alpha_1(M)$ equals its analytic counterpart. For the analytic counterpart, assertion 1.) is proven in [45] and assertion 2.) is proven in [4]. Assertion 3.) is a direct consequence of 1.) and 2.)

4. is proven in [10, Proposition 2.4].

5. and 6. If $\partial_0 M$ is nonempty then the pair $(M, \partial_0 M)$ is homotopy equivalent to a pair of finite CW-complexes (X, A) such that all of the 0-cells of X lie in A. Hence the cellular $Z\pi_1(M)$-chain complex $C(\widetilde{M, \partial_0 M}; V)$ is $Z\pi_1(M)$-chain homotopy equivalent to a $Z\pi_1(M)$-chain complex which is trivial in dimension 0. Now apply Theorems 2.6 and 3.2. □

For later purposes we will need the following result:

Lemma 3.6. *Let $j : \pi_1(M) \to \Gamma$ be an inclusion of discrete groups. Let $j^* l^2(\Gamma)$ be the unitary representation $\pi_1(M) \to Iso_{N(\Gamma)}(l^2(\Gamma))^{\mathrm{op}}$ obtained from the right regular representation of Γ by composing with j. Then for all p, we have*

1. $b_p(M, \partial_0 M) = b_p(M, \partial_0 M; j^* l^2(\Gamma))$.
2. $\alpha_p(M, \partial_0 M) = \alpha_p(M, \partial_0 M; j^* l^2(\Gamma))$.

Proof. Let $f : \oplus_{i=1}^n Z\pi_1(M) \to \oplus_{i=1}^n Z\pi_1(M)$ be a $Z\pi_1(M)$-linear map. By tensoring with $l^2(\pi_1(M))$ (resp. $j^* l^2(\Gamma)$), we get a morphism of Hilbert $N(\pi_1(M))$ (resp. $N(\Gamma)$)-modules denoted by f_1 (resp. f_2). Let $\{E_\lambda^{f_2^* f_2} : \lambda \in R\}$ denote the spectral family of the self-adjoint operator $f_2^* f_2 : \oplus_{i=1}^n l^2(\Gamma) \to \oplus_{i=1}^n l^2(\Gamma)$ and $\{E_\lambda^{f_1^* f_1} : \lambda \in R\}$ denote the spectral family of $f_1^* f_1 : \oplus_{i=1}^n l^2(\pi_1(M)) \to \oplus_{i=1}^n l^2(\pi_1(M))$. Then $E_\lambda^{f_2^* f_2}$ maps $\oplus_{i=1}^n l^2(\pi_1(M))$ into itself and the restriction of $E_\lambda^{f_2^* f_2}$ to $\oplus_{i=1}^n l^2(\pi_1(M))$ is just $E_\lambda^{f_1^* f_1}$. By [11, Theorem 1, p.97], this implies

$$\begin{aligned} F(f_1, \lambda) &= \operatorname{tr}_{N(\pi_1(M))}\left(E_{\lambda^2}^{f_1^* f_1}\right) = \langle E_{\lambda^2}^{f_1^* f_1}(1), 1\rangle_{l^2(\pi_1(M))} \\ &= \langle E_{\lambda^2}^{f_2^* f_2}(1), 1\rangle_{l^2(\Gamma)} = \operatorname{tr}_{N(\Gamma)}\left(E_{\lambda^2}^{f_2^* f_2}\right) = F(f_2, \lambda), \end{aligned}$$

and the claim follows. □

We now investigate the behaviour with respect to connected sums.

Proposition 3.7. *Let M_1, M_2, ...M_r be compact connected m-dimensional manifolds, with $m \geqq 3$. Let M be their connected sum $M_1 \# \ldots \# M_r$. Then*

1. $b_1(M) - b_0(M) = r - 1 + \sum_{j=1}^r \left(b_1(M_j) - b_0(M_j)\right)$.
2. $b_p(M) = \sum_{j=1}^r b_p(M_j)$ *for* $2 \leqq p \leqq m - 2$.
3. $\alpha_p(M) = \min\{\alpha_p(M_j) : 1 \leqq j \leqq r\}$ *for* $2 \leqq p \leqq m - 1$.
4. *If $\pi_1(M_i)$ is trivial for all i except for $i = i_0$ then $\alpha_1(M) = \alpha_1(M_{i_0})$. Suppose $\pi_1(M_i)$ is trivial for all i except for $i \in \{i_0, i_1\}$, $i_0 \neq i_1$, and that $\pi_1(M_{i_0}) = \pi_1(M_{i_1}) = Z/2$. Then $\alpha_1(M) = 1$. In all other cases $\alpha_1(M) = \infty^+$.*

Proof. We may assume without loss of generality that $r = 2$. The connected sum $M_1 \# M_2$ is obtained by glueing $M_1 \backslash \text{int}(D^m)$ and $M_2 \backslash \text{int}(D^m)$ together along ∂D^m. Since $\partial D^m \to D^m$ is $(m-1)$-connected, the inclusion of $M_j \backslash \text{int}(D^m)$ into M_j is $(m-1)$-connected. Hence the inclusion

$$M_1 \backslash \text{int}(D^m) \cup_{\partial D^m} M_2 \backslash \text{int}(D^m) - \to M_1 \cup_{D^m} M_2$$

is $(m-1)$-connected. Since $M_1 \cup_{D^m} M_2$ is homotopy equivalent to the wedge $M_1 \vee M_2$, from Lemma 3.3 it suffices to prove the claims for $M_1 \vee M_2$.

1. to 3. Let π denote $\pi_1(M_1 \vee M_2) = \pi_1(M_1) * \pi_1(M_2)$. We obtain an exact sequence $0 \to C(*; l^2(\pi)) \to C(M_1; l^2(\pi)) \oplus C(M_2; l^2(\pi)) \to C(M_1 \vee M_2; l^2(\pi)) \to 0$, where $*$ denotes the base point. The long weakly exact Mayer-Vietoris sequence reduces to weak isomorphisms

$$H_p(M_1; l^2(\pi)) \oplus H_p(M_2; l^2(\pi)) \to H_p(M_1 \vee M_2; l^2(\pi)) \ , \ p \geqq 2,$$

and the weakly exact sequence

$$\begin{aligned} 0 \to H_1(M_1; l^2(\pi)) \oplus H_1(M_2; l^2(\pi)) &\to H_1(M_1 \vee M_2; l^2(\pi)) \to l^2(\pi) \\ \to H_0(M_1; l^2(\pi)) \oplus H_0(M_2; l^2(\pi)) &\to H_0(M_1 \vee M_2; l^2(\pi)) \to 0 \end{aligned}$$

We conclude from Lemmas 1.4 and 3.6 that

$$b_1(M_1) + b_1(M_2) - b_1(M_1 \vee M_2) + 1 - b_0(M_1) - b_0(M_0) +$$

$$b_0(M_1 \vee M_2) = 0$$

$$b_p(M_1) + b_p(M_2) = b_p(M_1 \vee M_2) \text{ for } p \geqq 2,$$

from which assertions 1.) and 2.) follow. We obtain assertion 3.) from Theorem 2.3.

4. Since $\alpha_1(M)$ only depends on the fundamental group and $\pi_1(M) = \pi_1(M_1)$ if $\pi_1(M_2)$ is trivial, the first part of the assertion follows. It remains to consider the case when $\pi_1(M_1)$ and $\pi_1(M_2)$ are nontrivial. From Lemma 3.5.2, $\alpha_1(M)$ is ∞^+ if and only if $\pi_1(M)$ is nonamenable. We claim that $\pi_1(M)$ is amenable if and only if $\pi_1(M_1) = \pi_1(M_2) = Z/2$, in which case $\alpha_1(M) = 1$. Namely, suppose that $\pi_1(M)$ is amenable. Then it follows from [10, Theorem 0.2] that $b_1(M) = b_0(M) = 0$. But then assertion 1.) and Lemma 3.5 imply that $| \pi_1(M_i) | = 2$ for $i = 1, 2$. As $Z/2 * Z/2$ is an extension of Z by $Z/2$, it is amenable. Also, there is a two-fold covering of M with the fundamental group of a circle. Hence $\alpha_1(M) = \alpha_1(S^1)$, which is 1 by a simple calculation. □

Next we study manifolds with an S^1-action. Let $(M; \partial_0 M)$ be as above. Suppose that S^1 acts smoothly on M. Let $\phi : \pi_1(M) \to \Gamma$ be an homomorphism such that for one orbit (and hence all orbits) S^1/H in M, the composition of ϕ with the map induced by the inclusion $\pi_1(S^1/H) \to \pi_1(M)$ has infinite image. In particular, the S^1-action has no fixed points. Choose $\mathscr{A}$ to be $N(\Gamma)$ and the representation $\phi^* l^2(\Gamma)$ to be the composition of the regular representation $\Gamma \to \text{Iso}_{N(\Gamma)}(l^2(\Gamma))$ with ϕ. In other words, we are looking at the cover $\overline{M} \to M$ of M associated with ϕ.

Theorem 3.8. (S^1-manifolds). *With the above conditions on the S^1-manifold M, for all $p \geqq 0$ we have:*

1. $b_p(M, \partial_0 M; \phi^* l^2(\Gamma)) = 0$.
2. $\alpha_p(M, \partial_0 M; \phi^* l^2(\Gamma)) \geqq 1$.

Proof. The first assertion was proven in [31, Theorem 3.20].

In what follows we will write $l^2(\Gamma)$ instead of $\phi^* l^2(\Gamma)$, or $j^* \phi^* l^2(\Gamma)$ for j an inclusion. Since we have a smooth S^1-action, M carries a S^1-equivariant CW-structure. This means that we have a filtration

$$\emptyset = M_{-1} \subset M_0 \subset M_1 \subset \ldots M_{m-1} = M$$

such that M_i is obtained from M_{i-1} by attaching a finite number of S^1-equivariant cells $S^1/H \times D^i$ with attaching maps $S^1/H \times S^{i-1} \to M_{i-1}$. Since the S^1-action has no fixed points, the subgroups $H \subset S^1$ are all finite cyclic groups. We will show that

$$\alpha_p(M_i, \partial_0 M \cap M_i; l^2(\Gamma)) \geqq 1 \text{ for } p \leqq i + 1$$

$$\alpha_p(M_i, \partial_0 M \cap M_i; l^2(\Gamma)) = \infty^+ \text{ for } p > i + 1$$

by induction over i, where the representation of $\pi_1(M_i)$ is induced from the inclusion $\pi_1(M_i) \to \pi_1(M)$. The initial step $i = -1$ is trivial. The induction step from $i - 1$ to i is done as follows:

There is an exact sequence of chain complexes

$$0 \to C(M_{i-1}, \partial_0 M \cap M_{i-1}; l^2(\Gamma)) \to C(M_i, \partial_0 M \cap M_i; l^2(\Gamma)) \to$$

$$C(M_i, M_{i-1} \cup (\partial_0 M \cap M_i); l^2(\Gamma)) \to 0.$$

The last chain complex is isomorphic to a direct sum of chain complexes of the form $C(S^1/H \times D^i, S^1/H \times S^{i-1}; l^2(\Gamma))$. Since all isotropy groups H must be finite, such a chain complex looks like $\Sigma^i C(S^1; l^2(\Gamma))$, where $l^2(\Gamma)$ is viewed as a representation space of $\pi_1(S^1)$ by means of an injection $\pi_1(S^1) \to \Gamma$. Lemma 3.6 and a simple calculation of $\alpha_1(S^1)$ show that $\alpha_p(\Sigma^i C(S^1; l^2(\Gamma))) = \alpha(S^1)$ is 1 if $p = i + 1$ and ∞^+ otherwise. Lemma 2.4.3 implies that $\alpha_p(C(M_i, M_{i-1} \cup (\partial_0 M \cap M_i); l^2(\Gamma))$ is also 1 for $p = i + 1$ and ∞^+ otherwise. Upon applying Theorem 2.3.1 to the short exact sequence of weakly acyclic chain complexes above and using the induction hypothesis on M_{i-1}, the claim follows. □

Remark 3.9. If $g : (M, \partial_0 M) \to (N, \partial_0 N)$ is an n-fold finite covering then $b_p(M, \partial_0 M) = n \cdot b_p(N, \partial_0 N)$ and $\alpha_p(M, \partial_0 M) = \alpha_p(N, \partial_0 N)$ for all $p \geqq 0$. Note that the ordinary Betti numbers of a manifold are generally not multiplicative under finite coverings. □

Example 3.10. We state the values of the L^2-Betti numbers and Novikov-Shubin invariants for all compact connected 1- and 2-manifolds. In dimension 1 there are only S^1 and the unit interval I. One easily checks that $b_0(S^1) =$

$b_1(S^1) = 0$ and $\alpha_1(S^1) = 1$. As I is contractible, we have that $b_0(I) = 1$, $b_1(I) = 0$ and $\alpha_1(I) = \infty^+$.

Let F_g^d be the orientable closed surface of genus g with d embedded 2-disks removed. (As any nonorientable compact surface is finitely-covered by an orientable surface, Remark 3.9 shows that it is enough to handle the orientable case.) Using the general formula for the Euler characteristic in terms of L^2-Betti numbers [8]:

$$\chi(M) = \sum_p (-1)^p b_p(M),$$

Lemma 3.5 and the fact that a compact surface with boundary is homotopy-equivalent to a bouquet of circles, one derives:

$$b_0(F_g^d) = \begin{cases} 1 & g = 0, d = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$b_1(F_g^d) = \begin{cases} 0 & g = 0, d = 0, 1 \\ d + 2(g-1) & \text{otherwise.} \end{cases}$$

$$b_2(F_g^d) = \begin{cases} 1 & g = 0, d = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\alpha_1(F_g^d) = \widetilde{\alpha}_0(F_g^d) = \widetilde{\alpha}_1(F_g^d) = \begin{cases} 1 & g = 0, d = 2 \\ 2 & g = 1, d = 0 \\ \infty^+ & \text{otherwise.} \end{cases}$$

$$\alpha_2(F_g^d) = \widetilde{\alpha}_2(F_g^d) = \begin{cases} 2 & g = 1, d = 0 \\ \infty^+ & \text{otherwise.} \end{cases}$$ □

Example 3.11. Suppose that M is a compact connected orientable 3-manifold with finite fundamental group π. We have that $\alpha_p(M) = \infty^+$ for all p. If M is closed then $\widetilde{M}$ is a homotopy sphere, and Remark 3.9 implies that $b_0(M) = b_3(M) = \frac{1}{|\pi|}$ and $b_1(M) = b_2(M) = 0$. If ∂M is nonempty then $\widetilde{M}$ is a connected sum of a homotopy sphere and k 3-disks, for some positive integer k [20]. Then $b_0(M) = \frac{1}{|\pi|}$, $b_2(M) = \frac{k-1}{|\pi|}$ and $b_1(M) = b_3(M) = 0$. □

4. Seifert 3-Manifolds

In this section we compute the L^2-Betti numbers and Novikov-Shubin invariants of Seifert 3-manifolds. We also discuss *Sol* manifolds. We use the definition of *Seifert fibred 3-manifold* , or briefly *Seifert manifold* , given in [41], which we will use as a reference on Seifert manifolds. Recall that a *geometry* on a 3-manifold M is a complete locally homogeneous Riemanian metric on its interior. The universal cover of the interior has a complete homogeneous Riemannian metric, meaning that the isometry group acts transitively [42]. Thurston has shown that there are precisely eight maximal simply-connected 3-dimensional geometries having compact quotients, namely S^3, R^3, $S^2 \times R$

$H^2 \times R$, Nil, $\widetilde{SL_2(R)}$, Sol and H^3. If a closed 3-manifold admits a geometric structure modelled on one of these eight geometries then the geometry involved is unique. In the case of the L^2-Betti numbers, the following result was already given in [6].

Theorem 4.1. *Let M be a closed Seifert 3-manifold. If its fundamental group is infinite then it has vanishing L^2-cohomology. In terms of the Euler class e of the bundle and the Euler characteristic χ of the base orbifold, $\alpha_1(M) = \alpha_3(M)$ is given by*

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	1	3	∞^+
$e \neq 0$	∞^+	4	∞^+

and $\alpha_2(M)$ is given by

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	∞^+	3	1
$e \neq 0$	∞^+	2	1.

Proof. The geometric structure of M is determined as follows: [41, Theorem 5.3]:

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times R$	R^3	$H^2 \times R$
$e \neq 0$	S^3	Nil	$\widetilde{SL_2(R)}$.

If M has a S^3-structure then $\pi_1(M)$ is finite and we can apply Example 3.11.

In all other cases M is finitely covered by the total space $\overline{M}$ of an S^1-principal bundle over an orientable closed surface F. Moreover, $e(M) = 0$ iff $e(\overline{M}) = 0$, and the Euler characteristic χ of the orbifold base of M is negative, zero or positive according to the same condition for $\chi(\overline{M}/S^1)$ [41, p. 426, 427 and 436]. From Remark 3.9, in what follows we may assume without loss of generality that M is $\overline{M}$. Theorem 3.8 implies that $b_p(M) = 0$. If $\chi(F)$ is negative then $\pi_1(F)$ is non-amenable since it contains a free subgroup of rank 2. As $\pi_1(F)$ is a quotient of $\pi_1(M)$, $\pi_1(M)$ is also non-amenable and so $\alpha_1(M) = \infty^+$ by Lemma 3.5. Next, we verify the remaining claims for α_1 and α_2.

R^3: We may assume that $M = T^3$. A direct computation by Fourier analysis gives that $\alpha_p(T^3) = 3$ for all $1 \leqq p \leqq 3$.

$S^2 \times R$: We may assume that $M = S^1 \times S^2$. Now apply Lemma 4.2.

$H^2 \times R$: We may assume that $M = S^1 \times F_g$ for $g \geqq 2$. Now apply Lemma 4.2.

Nil: From [24] we have that $\widetilde{\alpha}_0(M) = 4$ and $\widetilde{\alpha}_1(M) = 2$, and so the claim for α_1 and α_2 follows.

$\widetilde{SL_2(R)}$: A computation using harmonic analysis on $\widetilde{SL_2(R)}$ and the results of [38] gives $\alpha_2(M) = 1$. We will not reproduce the computation here.

The next lemma will finish the proof of Theorem 4.1. □

Lemma 4.2. *Let F_g^d be the (orientable compact connected) surface of genus g with d boundary components. Then*

1. $b_p(S^1 \times F_g^d) = 0$ *for all* p.

2. $$\alpha_1(S^1 \times F_g^d) = \begin{cases} 1 & g = 0, d = 0, 1 \\ 2 & g = 0, d = 2 \\ 3 & g = 1, d = 0 \\ \infty^+ & \text{otherwise} \end{cases}$$

3. $$\alpha_2(S^1 \times F_g^d) = \begin{cases} \infty^+ & g = 0, d = 0, 1 \\ 3 & g = 1, d = 0 \\ 2 & g = 0, d = 2 \\ 1 & \text{otherwise} \end{cases}$$

4. $$\alpha_3(S^1 \times F_g^d) = \begin{cases} 1 & g = 0, d = 0 \\ 3 & g = 1, d = 0 \\ \infty^+ & \text{otherwise} \end{cases}$$

Proof. The claim for the L^2-Betti numbers follows from Theorem 3.8. In the cases $g = 0, d = 0, 1, 2$ and $g = 1, d = 0$, i.e. $S^1 \times S^2$, $S^1 \times D^2$, $S^1 \times S^1 \times I$ and T^3, the claim follows from earlier computations for S^1, T^2 and T^3 (see Example 3.10 and Theorem 4.1). In the remaining cases Example 3.10 gives that $\alpha_p(F_g^d) = \infty^+$ for all p and $b_p(F_g^d) = 0$ for $p \neq 1$. We abbreviate $F = F_g^d$. Let H be the Hilbert chain complex over the von Neumann algebra of $\pi_1(F)$ which is concentrated in dimension 1, and is given there by $\ker(\Delta_1)$, where $\Delta_1 : C_1(F; l^2(\pi_1(F))) \to C_1(F; l^2(\pi_1(F)))$ is the Laplace operator. There is a natural split inclusion $i : H \to C(F; l^2(\pi_1(F)))$. From Lemma 2.5, i is a homotopy equivalence. We have that $C(S^1 \times F, l^2(\pi_1(S^1 \times F)))$ is the Hilbert tensor product of $C(F; l^2(\pi_1(F)))$ and $C(S^1; l^2(\pi_1(S^1)))$, and so is homotopy equivalent to the Hilbert tensor product of H and $C(S^1; l^2(\pi_1(S^1)))$. As the part of H in dimension one is isomorphic to $\oplus_{i=1}^{-\chi(F)} l^2(\pi_1(F))$, this Hilbert tensor product is isometrically isomorphic to the suspension of the direct sum of $-\chi(F)$ copies of $C(S^1; l^2(\pi_1(S^1 \times F)))$. From Lemma 2.4, Theorem 2.6 and Lemma 3.6, the invariants of M are the same as those of the suspension of $C(S^1; l^2(\pi_1(S^1)))$. The claim now follows from Example 3.10. □

Remark 4.3. The fact that the Novikov-Shubin invariants are the same for closed $H^2 \times R$-manifolds and $\widetilde{SL_2(R)}$-manifolds is probably related to the fact that they are $K(\pi, 1)$ manifolds whose universal covers are quasi-isometric [15]

Theorem 4.4. *Let M be a Seifert manifold with nonempty boundary. Then all L^2-Betti numbers vanish. We have that $\alpha_3(M) = \infty^+$, and the other Novikov-Shubin invariants are given by:*

α_1	α_2	
1	∞^+	*M is a solid torus or Klein bottle*
2	2	*M is an I-bundle over T^2 or over a Klein bottle K*
∞^+	1	*otherwise.*

Proof. We have that the boundary of M is compressible iff M is homeomorphic to a solid torus or Klein bottle [41, Corollary 3.3]. The theorem follows in this case from Remark 3.9 and Lemma 4.2 and so we may assume that M has incompressible boundary. As any 2-dimensional orbifold with boundary is finitely covered by a 2-dimensional surface with boundary, we can find a finite cover $\overline{M}$ of M which is homeomorphic to some $S^1 \times F_g^d$, with $d \geqq 1$. From Remark 3.9 and Lemma 4.2, we have to know that M is an I-bundle over T^2 or K iff $F_g^d = S^1 \times I$. This follows from [20, Theorem 10.5]. □

Proposition 4.5. *If M is a closed Sol-manifold then M has vanishing L^2-Betti numbers, $\alpha_1(M) = \infty$ and $\alpha_2(M) \geqq 1$.*

Proof. By taking a finite cover, we may assume that our *Sol*-manifold is a torus bundle over S^1 with hyperbolic glueing map ϕ [41, Theorem 5.3]. Hence $\pi_1(M)$ is a semi-direct product of Z^2 and Z where the action of Z on Z^2 is given by a hyperbolic automorphism of Z^2. Then $\pi_1(M)$ is amenable, as it is an extension of amenable groups. It is easy to see that $\pi_1(M)$ is not virtually nilpotent. Lemma 3.5.3 implies that $\alpha_1(M) = \infty$.

By Example 3.10, $b_p(T^2) = 0$ for all p and $\alpha_p(T^2) = 2$ for $p \in \{1,2\}$. From [30], the L^2-Betti numbers of M vanish. We have a short exact (Wang) sequence of Hilbert chain complexes:

$$0 \to C(T^2; l^2(\pi_1(M))) \xrightarrow{j} C(T^2 \times I; l^2(\pi_1(M))) \xrightarrow{q} C(M; l^2(\pi_1(M))) \to 0.$$

Theorem 2.3.2 gives

$$\frac{1}{\alpha_2(M)} \leq \frac{1}{2} + \frac{1}{2}. \qquad \square$$

5. Analytic L^2-Betti numbers and Novikov-Shubin invariants for manifolds with boundary, and hyperbolic 3-manifolds

In this section we define analytic Novikov-Shubin invariants and L^2-Betti numbers for manifolds with boundary, and show the equivalence between the analytic invariants and the combinatorial invariants of the previous section. As an application, we give a lower bound for the Novikov-Shubin invariants of a compact 3-manifold whose interior admits a complete finite-volume hyperbolic metric.

For closed manifolds, the facts that the analytic L^2-Betti numbers and Novikov-Shubin invariants equal their combinatorial counterparts were proven in [12] and [14]. In order to make the comparisons between the analytic and combinatorial invariants for a compact manifold M with boundary, it will be convenient for us to think of the combinatorial invariants as defined by cellular cochains, instead of cellular chains. In this section, except where otherwise stated, the Novikov-Shubin invariants will be those of the coboundary operator. The smooth forms on $\widetilde{M}$ will be denoted by $C^\infty(\wedge^*(\widetilde{M}))$. Those with compact support will be denoted by $C_0^\infty(\Lambda^*(\widetilde{M}))$. Note that the elements of $C_0^\infty(\Lambda^*(\widetilde{M}))$ do not necessarily vanish on $\partial\widetilde{M}$.

We assume that M has a smooth Riemannian metric and corresponding Levi-Civita connection. We give $\widetilde{M}$ the induced Riemannian metric and Levi-Civita connection. Let d denote the exterior differentiation and let $b : \partial\widetilde{M} \to \widetilde{M}$ denote the boundary inclusion of $\partial\widetilde{M}$ into $\widetilde{M}$. Let ∇ be covariant differentiation on the smooth tensors on $\widetilde{M}$. As before, π denotes the fundamental group of M.

Definition 5.1. Define norms $\| \bullet \|_s$ on the smooth compactly-supported tensors on $\widetilde{M}$ for nonnegative integers s inductively by saying that $\| \bullet \|_0$ is the L^2-norm and

$$\| \omega \|_{s+1}^2 = \| \omega \|_s^2 + \| \nabla\omega \|_s^2 .$$

Let the Sobolev space $\mathcal{H}_s^*(M; l^2(\pi))$ be the Hilbert space completion of $C_0^\infty(\Lambda^*(\widetilde{M}))$ under the norm $\| \bullet \|_s$. □

Put $\mathcal{A} = N(\pi)$. There is a Hilbert $\mathcal{A}$-cochain complex concentrated in dimensions $p-1$, p and $p+1$ given by

$$\begin{aligned} \ldots \to 0 \to \mathcal{H}_2^{p-1}(M; l^2(\pi)) &\xrightarrow{d_2^{p-1}} \mathcal{H}_1^p(M; l^2(\pi)) \\ &\xrightarrow{d_1^p} \mathcal{H}_0^{p+1}(M; l^2(\pi)) \to 0 \to \ldots . \end{aligned} \qquad (1)$$

In Definition 2.1 we introduced the condition of Fredholmness at p, the p-th L^2-Betti number and the p-th Novikov-Shubin invariant of such a Hilbert $\mathcal{A}$-complex. We will show in Theorem 5.7 that the complex (1) is indeed Fredholm at p.

Definition 5.2. The analytic p-th L^2-cohomology of M is the p-th cohomology group of the Hilbert $\mathcal{A}$-cochain complex (1). The analytic p-th L^2-Betti number of M and analytic p-th Novikov-Shubin invariant of M are defined similarly. □

If we put $\widetilde{\alpha}_p(M) = \min(\alpha_p(M), \alpha_{p-1}(M))$ then the application of a Laplace transform to the spectral density function shows that the analytic invariants of the introduction, defined using heat kernels, are the same as those defined here [18, Appendix].

As a topological vector space, $\mathcal{H}_s^*(M; l^2(\pi))$ is independent of the choice of Riemannian metric on M. Given two different Riemannian metrics on M, the identity map on $C_0^\infty(\Lambda^*(\widetilde{M}))$ induces a bounded invertible mapping between the corresponding complexes (1), and in particular a cochain homotopy equivalence. Theorem 2.6 then implies that the analytic L^2-Betti numbers and Novikov-Shubin invariants are independent of the Riemannian metric on M.

Definition 5.3. *Define Sobolev spaces of differential forms with absolute boundary conditions by*

$$\mathcal{H}^p_{2,abs}(M;l^2(\pi)) = \{\omega \in \mathcal{H}^p_2(M;l^2(\pi)) : b^*(*\omega) = b^*(*d\omega) = 0\},$$
$$\mathcal{H}^p_{1,abs}(M;l^2(\pi)) = \{\omega \in \mathcal{H}^p_1(M;l^2(\pi)) : b^*(*\omega) = 0\},$$
$$\mathcal{H}^p_{0,abs}(M;l^2(\pi)) = \mathcal{H}^p_0(M;l^2(\pi)). \quad \square \tag{2}$$

There is a Hilbert $\mathcal{A}$-cochain complex concentrated in dimensions $p-1$, p and $p+1$ given by

$$\ldots \to 0 \to \mathcal{H}^{p-1}_{2,abs}(M;l^2(\pi)) \xrightarrow{d^{p-1}_{2\,abs}} \mathcal{H}^p_{1,abs}(M;l^2(\pi))$$
$$\xrightarrow{d^p_{1\,abs}} \mathcal{H}^{p+1}_{0,abs}(M;l^2(\pi)) \to 0 \to \ldots. \tag{3}$$

Let $\varepsilon > 0$ be small enough that there is a coordinate function $t \in [0,2\varepsilon]$ near $\partial\widetilde{M}$ such that ∂_t is a unit length vector field whose flow generates unit speed geodesics which are normal to the boundary, and $\partial\widetilde{M}$ corresponds to $t=0$. Then a tubular neighborhood of $\partial\widetilde{M}$ is diffeomorphic to $[0,2\varepsilon] \times \partial\widetilde{M}$. A differential form ω on $\widetilde{M}$ can be decomposed near the boundary as

$$\omega = \omega_1(t) + dt \wedge \omega_2(t),$$

where $\omega_1(t)$ and $\omega_2(t)$ are smooth 1-parameter families of forms on $\partial\widetilde{M}$. We can write the condition for ω to be in $\mathcal{H}^p_{1,abs}(M;l^2(\pi))$ as $\omega_2(0,\cdot)=0$, and the condition for ω to be in $\mathcal{H}^p_{2,abs}(M;l^2(\pi))$ as $\omega_2(0,\cdot) = \partial_t\omega_1(0,\cdot) = 0$.

Let $\rho : [0,2\varepsilon] \to R$ be a smooth bump function which is identically one near $t=0$ and identically zero for $t \geqq \varepsilon$. Let $\widehat{\Delta}$ denote the Laplacian acting on forms on $\partial\widetilde{M}$. For $u \geqq 0$, define the operator $e^{-(ue^{1+\widehat{\Delta}})}$ by the spectral theorem.

Definition 5.4. For ω a form on $\widetilde{M}$, restrict ω to $[0,2\varepsilon] \times \partial\widetilde{M}$ and put

$$(K\omega)(t,\cdot) = \rho(t)\int_0^t e^{-(ue^{1+\widehat{\Delta}})}\omega_2(u,\cdot)du. \quad \square \tag{4}$$

We can extend $K\omega$ by zero to become a form on $\widetilde{M}$.

Lemma 5.5. *The map K induces bounded π-equivariant operators*

$$K^p_s : \mathcal{H}^p_s(M;l^2(\pi)) \to \mathcal{H}^{p-1}_{s+1}(M;l^2(\pi))$$

for $s \in \{0,1,2\}$ and

$$K^p_{s,abs} : \mathcal{H}^p_{s,abs}(M;l^2(\pi)) \to \mathcal{H}^{p-1}_{s+1,abs}(M;l^2(\pi))$$

for $s \in \{0,1\}$.

Proof. Whether or not a given distributional form lies in $\mathscr{H}^p_s(M; l^2(\pi))$ is independent of the choice of Riemannian metric. Moreover, the condition for a form to lie in $\mathscr{H}^p_{*,abs}(M; l^2(\pi))$ is the same whether one uses the given metric g_M on M or a metric which becomes the product metric $dt^2 + g_{\partial M}$ near the boundary. Thus we may assume without loss of generality that $[0, 2\varepsilon] \times \partial\widetilde{M}$ has a product metric. Let ω be an element of $\mathscr{H}^p_0(M; l^2(\pi))$. From Definition 5.4, we may assume that ω has support on $[0, 2\varepsilon] \times \partial\widetilde{M}$. Clearly ω_2 lies in $\mathscr{H}^{p-1}_0([0, 2\varepsilon] \times \partial M; l^2(\pi))$. From (4),

$$(\partial_t K\omega)(t, \cdot) = \rho'(t) \int_0^t e^{-(ue^{1+\widehat{\triangle}})} \omega_2(u, \cdot) du + \rho(t) e^{-(te^{1+\widehat{\triangle}})} \omega_2(t, \cdot),$$

and so $\partial_t K$ gives a bounded linear map from $\mathscr{H}^{p-1}_0([0, 2\varepsilon] \times \partial M; l^2(\pi))$ to itself. Let $\widehat{\nabla}$ denote covariant differentiation in the $\partial\widetilde{M}$ directions. Then from (4),

$$(\widehat{\nabla} K\omega)(t, \cdot) = \rho(t) \int_0^t \widehat{\nabla} e^{-(ue^{1+\widehat{\triangle}})} \omega_2(u, \cdot) du.$$

Thus $\widehat{\nabla} K$ is an integral operator with kernel function

$$(\widehat{\nabla} K)(t, u) = \Theta(t-u) \rho(t) \widehat{\nabla} e^{-(ue^{1+\widehat{\triangle}})},$$

where Θ is the Heaviside step function. In order to show that $\widehat{\nabla} K$ is L^2-bounded, it suffices to show that $\int_{[0,2\varepsilon]} \int_{[0,2\varepsilon]} \mathrm{Tr}\left((\widehat{\nabla} K)(t,u)^* (\widehat{\nabla} K)(t,u) \right) dt du < \infty$, as that would imply that $\widehat{\nabla} K$ is actually Hilbert-Schmidt. We have

$$\begin{aligned}
\int_0^{2\varepsilon}\int_0^{2\varepsilon} \mathrm{Tr}\left((\widehat{\nabla} K)(t,u)^* (\widehat{\nabla} K)(t,u) \right) dt du &= \int_0^{2\varepsilon}\int_0^{2\varepsilon} \Theta(t-u) \rho^2(t) \\
&\quad \mathrm{Tr}\left(e^{-(ue^{1+\widehat{\triangle}})} \widehat{\nabla}^* \widehat{\nabla} e^{-(ue^{1+\widehat{\triangle}})} \right) dt du \\
&\leqq \text{const.} \int_0^{2\varepsilon} \mathrm{Tr}\left(\widehat{\nabla}^* \widehat{\nabla} e^{-(2ue^{1+\widehat{\triangle}})} \right) du \\
&\leqq \text{const.} \int_0^{\infty} \mathrm{Tr}\left(\widehat{\nabla}^* \widehat{\nabla} e^{-(2ue^{1+\widehat{\triangle}})} \right) du \\
&= \text{const.} \, \mathrm{Tr}\left(\frac{1}{2} \widehat{\nabla}^* \widehat{\nabla} e^{-(1+\widehat{\triangle})} \right) < \infty.
\end{aligned}$$

Thus ∇K gives an L^2-bounded linear map on $\mathscr{H}^{p-1}_0(M; l^2(\pi))$. It follows that K gives a bounded linear map from $\mathscr{H}^{p-1}_0(M; l^2(\pi))$ to $\mathscr{H}^{p-1}_1(M; l^2(\pi))$. A similar argument shows that K gives a bounded linear map from $\mathscr{H}^p_s(M; l^2(\pi))$ to $\mathscr{H}^{p-1}_{s+1}(M; l^2(\pi))$ for $s = 1, 2$.

As $(K\omega)_2 = 0$, it follows that K maps $\mathscr{H}^p_{0,abs}(M; l^2(\pi))$ to $\mathscr{H}^{p-1}_{1,abs}(M; l^2(\pi))$. Given $\omega \in \mathscr{H}^p_{1,abs}(M; l^2(\pi))$, we have

$$\partial_t (K\omega)_1(0, \cdot) = \partial_t (K\omega)(0, \cdot) = \omega_2(0, \cdot) = 0.$$

Thus K maps $\mathscr{H}^p_{1,abs}(M; l^2(\pi))$ to $\mathscr{H}^{p-1}_{2,abs}(M; l^2(\pi))$. □

Proposition 5.6. *The Hilbert $\mathscr{A}$-cochain complex (1) is Fredholm at p if and only if (3) is Fredholm at p. In this case, they have the same p-th L^2-Betti number and the same p-th Novikov-Shubin invariant.*

Proof. The inclusion i induces a chain map from (3) to (1). We will show that there is a chain map j from the cochain complex (1) to the cochain complex (3) of the form

$$\begin{array}{ccccc}
\mathscr{H}^{p-1}_2(M; l^2(\pi)) & \xrightarrow{d^{p-1}_2} & \mathscr{H}^p_1(M; l^2(\pi)) & \xrightarrow{d^p_1} & \mathscr{H}^{p+1}_0(M; l^2(\pi)) \\
j^{p-1}_2 \downarrow & & j^p_1 \downarrow & & j^{p+1}_0 \downarrow \\
\mathscr{H}^{p-1}_{2,abs}(M; l^2(\pi)) & \xrightarrow{d^{p-1}_{2,abs}} & \mathscr{H}^p_{1,abs}(M; l^2(\pi)) & \xrightarrow{d^p_{1,abs}} & \mathscr{H}^{p+1}_{0,abs}(M; l^2(\pi)),
\end{array}$$

where

$$\begin{aligned}
j^{p-1}_2 &= 1 - d^{p-2}_3 K^{p-1}_2 - K^{p-1}_1 d^{p-1}_2, \\
j^p_1 &= 1 - d^{p-1}_2 K^p_1 - K^{p+1}_0 d^p_1, \\
j^{p+1}_0 &= 1 - d^p_1 K^{p+1}_0.
\end{aligned}$$

Lemma 5.5 implies that that the vertical operators j do indeed map $H^{p+1-s}_s(M; l^2(\pi))$ to $H^{p+1-s}_s(M; l^2(\pi))$ for $s \in \{0, 1, 2\}$. It remains to check that the images actually lie in $H^{p+1-s}_{s,abs}(M; l^2(\pi))$.

Given $\omega \in H^{p+1}_0(M; l^2(\pi))$, it is trivial that $j^{p+1}_0 \omega$ lies in $H^{p+1}_{0,abs}$ $(M; l^2(\pi))$. Given $\omega \in \mathscr{H}^p_1(M; l^2(\pi))$, we have

$$(d^{p-1}_2 K^p_1 \omega)(0, \cdot) = dt \wedge \omega_2(0, \cdot)$$

and

$$(K^{p+1}_0 d^p_1 \omega)(0, \cdot) = 0.$$

Then

$$\left(\omega - (d^{p-1}_2 K^p_1 + K^{p+1}_0 d^p_1)\omega\right)(0, \cdot) = \omega_1(0, \cdot),$$

and so $j^p_1 \omega$ lies in $\mathscr{H}^p_{1,abs}(M; l^2(\pi))$.

Given $\omega \in \mathscr{H}^{p-1}_2(M; l^2(\pi))$, in order to show that $j^{p-1}_2 \omega$ lies in $\mathscr{H}^{p-1}_{2,abs}(M; l^2(\pi))$ it suffices to show that $j^{p-1}_2 \omega$ lies in $\mathscr{H}^{p-1}_{1,abs}(M; l^2(\pi))$ and that $d^{p-1}_2 j^{p-1}_2 \omega$ lies in $\mathscr{H}^p_{1,abs}(M; l^2(\pi))$. As ω also lies in $\mathscr{H}^{p-1}_1(M; l^2(\pi))$, the preceding argument gives that $j^{p-1}_2 \omega$ lies in $\mathscr{H}^{p-1}_{1,abs}(M; l^2(\pi))$. As $d^{p-1}_2 j^{p-1}_2 \omega =$

$j_1^p d_2^{p-1}\omega$ and $d_2^{p-1}\omega$ lies in $\mathscr{H}_1^p(M; l^2(\pi))$, the preceding argument also gives that $d_2^{p-1} j_2^{p-1}\omega$ lies in $\mathscr{H}_{1,abs}^{p-1}(M; l^2(\pi))$.

We now have chain maps i and j. Consider the cochain complexes, concentrated in dimensions p and $p+1$, given by

$$\ldots \to 0 \to \mathscr{H}_1^p(M; l^2(\pi))/\mathrm{clos}(\mathrm{im}(d_2^{p-1})) \xrightarrow{d_1^p} \mathscr{H}_0^{p+1}(M; l^2(\pi)) \to 0 \to \ldots. \tag{5}$$

and

$$\ldots \to 0 \to \mathscr{H}_{1,abs}^p(M; l^2(\pi))/\mathrm{clos}(\mathrm{im}(d_{2,abs}^{p-1})) \xrightarrow{d_{1,abs}^p} \mathscr{H}_{0,abs}^{p+1}(M; l^2(\pi)) \to 0 \to \ldots. \tag{6}$$

Clearly (1) is Fredholm at p if and only if (5) is Fredholm at p, and in this case they have the same p-th Betti number and p-th Novikov-Shubin invariant. A similar statement holds for (3) and (6). Now i and j induce chain maps between (5) and (6), and K and K_{abs} induce chain homotopies between the two compositions and the identity. Thus by Theorem 2.6, the cochain complex (5) is Fredholm if and only if (6) is Fredholm, and in this case they have the same p-th Betti number and p-th Novikov-Shubin invariant. □

Theorem 5.13. *The Hilbert $\mathscr{A}$-cochain complexes (1) and (3) are Fredholm at p. Moreover, the analytic L^2-Betti numbers and Novikov-Shubin invariants of Definition 5.2 are equal to the combinatorial invariants of Section 3, with $\partial_0 M = \emptyset$.*

Proof. The choice of Riemannian metric on M does not affect whether or not (1) is Fredholm. If (1) is Fredholm, its p-th L^2-Betti number and p-th Novikov-Shubin invariant are independent of the choice of Riemannian metric on M. From Proposition 5.6 it suffices to show that (3) is Fredholm at p and that if the metric is a product near the boundary then the p-th L^2-Betti number and p-th Novikov-Shubin invariant agree with the combinatorially defined invariants.

There is an induced Riemannian metric on the double DM, upon which Z_2 acts by isometries. With π still denoting $\pi_1(M)$, there is a π-normal cover of DM, namely the double $D\widetilde{M}$ of $\widetilde{M}$, and it is easy to see that $\mathscr{H}_{s,abs}^p(M; l^2(\pi))$ is isomorphic to $\left(\mathscr{H}_s^p(DM; l^2(\pi))\right)^{Z_2}$, the subspace of $\mathscr{H}_s^p(DM; l^2(\pi))$ which is invariant under the induced Z_2-action, for $s \in \{0,1,2\}$. In particular, (3) is isomorphic to the Z_2-invariant part of the cochain complex $\mathscr{H}_{p+1-*}^*(DM; l^2(\pi))$, restricted to the dimensions $p-1$, p and $p+1$.

It follows from [18], [12] and [14] that $\mathscr{H}_{p+1-*}^*(DM; l^2(\pi))$ is Fredholm at p and that one has equality of the analytic and combinatorial invariants on DM, defined using the π-cover. One can go through the proofs making everything equivariant with respect to the Z_2 action, in order to show that the same is true when one restricts to the Z_2-invariant subspaces. (As in [12] and [14], one first deals with Sobolev spaces of a high enough order that the de Rham map is well-defined. One then shows the analytic invariants are independent of the order of the Sobolev space. In our case, we are finally interested in the

Sobolev space $\mathscr{H}_1^p$. All of these steps will go through equivariantly.) Putting all this together, we have shown Theorem 5.7. □

Now let M be a compact 3-manifold whose interior admits a complete finite-volume hyperbolic metric. If M is closed then we have that $b_*(M; l^2(\pi)) = 0$ [13] and the Novikov-Shubin invariants of the exterior derivative operator are computed in [24] as

$$\alpha_0(M; l^2(\pi)) = \alpha_2(M; l^2(\pi)) = \infty^+, \quad \alpha_1(M; l^2(\pi)) = 1.$$

Suppose that M is not closed. Then it has incompressible torus boundary and the interior M' of M is the union of a compact core and a finite number of hyperbolic cusps (see [44] or [35, p. 52, 54]). Let $i : M \to M'$ be an embedding of M in M' obtained by smoothly truncating the cusps of M' and let M have the induced Riemannian metric. Let $i_1 : M_1 \to M'$ be the embedding of a submanifold (with boundary) M_1 of M' obtained by attaching a collar to M, and let $i_2 : M_2 \to M'$ be the embedding of a submanifold (with boundary) M_2 of M' obtained by attaching a collar to $M' - M$. Then $M_3 = M_1 \cap M_2$ is diffeomorphic to a disjoint union of $I \times T^2$'s (where we take $\{0\} \times T^2$ to be contained in the interior of M_1 and $\{1\} \times T^2$ to be contained in the interior of M_2) and is embedded in M' by a map $i_3 : M_3 \to M'$. Let $i_4 : M_3 \to M_1$ and $i_5 : M_3 \to M_2$ be the obvious embeddings. Put $\pi = \pi_1(M)$.

For each $p \in \{0, 1, 2, 3\}$, consider the following Hilbert cochain complexes concentrated in dimensions $p - 1$, p and $p + 1$:

$$C^*_{(p)} = \mathscr{H}^*_{p+1-*}(M'; l^2(\pi))$$
$$D^*_{(p)} = \mathscr{H}^*_{p+1-*}(M_1; l^2(\pi)) \oplus \mathscr{H}^*_{p+1-*}(M_2; i_2^* l^2(\pi))$$
$$E^*_{(p)} = \mathscr{H}^*_{p+1-*}(M_3; i_3^* l^2(\pi)),$$

with differentials c, d and e given by exterior differentiation. Although M' is noncompact, the Sobolev space $\mathscr{H}^*_s(M'; l^2(\pi))$ can be defined as in Definition 5.1, and is in fact a Sobolev space of differential forms on the hyperbolic 3-space H^3. The complexes $C_{(p)}$ and $E_{(p)}$ are Fredholm at p.

Lemma 5.14. *There is an exact sequence of Hilbert cochain complexes*

$$0 \to C_{(p)} \xrightarrow{j} D_{(p)} \xrightarrow{k} E_{(p)} \to 0, \tag{7}$$

with $j(\omega) = i_1^*(\omega) \oplus i_2^*(\omega)$ *and* $k(\omega_1 \oplus \omega_2) = i_4^*(\omega_1) - i_5^*(\omega_2)$.

Proof. It follows from the definitions that $\ker(j) = 0$, and it is easy to check that $\ker(k) = \operatorname{im}(j)$. To see that k is onto, let $\phi : I \to R$ be a bump function which is identically zero near 0 and identically one near 1. Let $\widetilde{\phi} : \widetilde{M_3} \to R$ denote the composition of the pullbacks of ϕ to M_3 and then to $\widetilde{M_3}$, the preimage of M_3 in H^3. We can think of an element η of $E^*_{(p)}$ as a differential form $\widetilde{\eta}$ on $\widetilde{M_3}$. Then $\widetilde{\phi}\widetilde{\eta}$ extends by zero to a differential form on $\widetilde{M_1}$, which comes from an element ω_1 of $\mathscr{H}^*_{p+1-*}(M_1; l^2(\pi))$. Similarly, we can extend $(\widetilde{\phi} - 1)\widetilde{\eta}$ by zero to a differential form on $\widetilde{M_2}$, which comes from an element ω_2 of $\mathscr{H}^*_s(M_2; i_2^* l^2(\pi))$. Then $k(\omega_1 \oplus \omega_2) = \eta$. □

It follows from Theorem 2.3 that the complex $D_{(p)}$ is Fredholm at p.

Proposition 5.9. *$b_p(E_{(p)}) = 0$, $\alpha_0(E_{(0)}) = \alpha_1(E_{(1)}) = 2$ and $\alpha_2(E_{(2)}) = \infty^+$.*

Proof. As the map $Z^2 = \pi_1(M_3) \to \pi$ is an inclusion, the proof of Lemma 3.6 goes through for the analytic invariants to give that $b_p(E_{(p)}) = b_p(I \times T^2; l^2(Z^2))$ and $\alpha_p(E_{(p)}) = \alpha_p(I \times T^2; l^2(Z^2))$, where the right-hand-sides are defined by Definition 5.2. By the equivalence of the analytic and combinatorial invariants and the homotopy invariance of the combinatorial invariants (Theorem 2.6), these are the same as the invariants of T^2, which were given in Example 3.10. □

Proposition 5.10. *$b_p(C_{(p)}) = 0$, $\alpha_0(C_{(0)}) = \alpha_2(C_{(2)}) = \infty^+$ and $\alpha_1(C_{(1)}) = 1$.*

Proof. As the universal cover of M' is isometrically H^3, this follows from the same calculation in [24] as was cited above for the case of closed hyperbolic 3-manifolds. □

Theorem 5.11. *$\alpha_1(M; l^2(\pi)) \geqq 2/3$.*

Proof. We apply Theorem 2.3 to the exact sequence (7) with $p = 1$. As $H_1(E_{(1)}) = 0$, $\alpha(\delta_1) = \infty^+$. From Proposition 5.10, $\alpha_1(C_{(1)}) = 1$ and from Proposition 5.9, $\alpha_1(E_{(1)}) = 2$. Then Theorem 2.3 gives $\alpha_1(D_{(1)}) \geqq 2/3$. From Lemma 1.10,

$$\alpha_1(D_{(1)}) = \min(\alpha_1(M_1; l^2(\pi)), \alpha_1(M_2; i_2^* l^2(\pi))),$$

from which the assertion of the theorem follows. □

Theorem 5.12. *$b_p(M; l^2(\pi)) = 0$ for all p.*

Proof. We can exhaust $M' = \text{int}(M)$ by a sequence of compact manifolds (with boundary) $\{M_k\}$ which are all diffeomorphic to M. From [9, Theorem 1.1], $b_p(M; l^2(\pi)) = b_p(M_k; l^2(\pi))$ is the von Neumann dimension of the space of L^2 harmonic p-forms on $\widetilde{M'}$. As $\widetilde{M'}$ is H^3, such forms vanish [13]. □

We now revert to letting the $\alpha_p(M)$-invariants refer to boundaries, as in the previous sections, as opposed to coboundaries. The translation is that $\alpha_p(M)$, defined using coboundaries, equals $\alpha_{p+1}(M)$, defined using boundaries.

Theorem 5.13. *$\alpha_1(M) = \alpha_3(M) = \infty^+$.*

Proof. It follows from [47, Proposition 4.1.11] that $\pi_1(M)$ is nonamenable. We derive from Lemma 3.5.2 that $\alpha_1(M) = \infty^+$. As M has nonempty boundary, Lemma 3.5.5 gives that $\alpha_3(M) = \infty^+$. □

In summary, we have shown

Theorem 5.14. *If M is a compact 3-manifold whose interior admits a complete finite-volume hyperbolic structure then M has vanishing L^2-cohomology and $\alpha_1(M) = \alpha_3(M) = \infty^+$. If M is closed then $\alpha_2(M) = 1$ and if M is not closed then $\alpha_2(M) \geqq 2/3$.* □

It will follow from Theorem 0.1.5 that if M is not closed then $\alpha_2(M) \leqq 2$.

6. L^2-Betti numbers and Novikov-Shubin invariants of 3-manifolds

In this section we analyse the L^2-Betti numbers and Novikov-Shubin invariants of compact connected orientable 3-manifolds. It is easy to extend the results to the nonorientable case by means of the orientation covering.

We recall some basic facts about (compact connected orientable) 3-manifolds [20, 41]. A 3-manifold M is *prime* if for any decomposition of M as a connected sum $M_1 \sharp M_2$, M_1 or M_2 is homeomorphic to S^3. It is *irreducible* if every embedded 2-sphere bounds an embedded 3-disk. Any prime 3-manifold is irreducible or is homeomorphic to $S^1 \times S^2$ [20, Lemma 3.13]. One can write M as a connected sum

$$M = M_1 \sharp M_2 \sharp \ldots \sharp M_r$$

where each M_j is prime, and this prime decomposition is unique up to renumbering [20, Theorems 3.15, 3.21]. By the sphere theorem [20, Theorem 4.3], an irreducible 3-manifold is a $K(\pi, 1)$ Eilenberg-MacLane space if and only if it is a 3-disk or has infinite fundamental group.

A properly-embedded orientable connected surface in a 3-manifold is *incompressible* if it is not a 2-sphere and the inclusion induces a injection on the fundamental groups. One says that ∂M is *incompressible in M* if and only if ∂M is empty or any component C of ∂M is incompressible in the sense above. An irreducible 3-manifold is *Haken* if it contains an embedded orientable incompressible surface. If M is irreducible and in addition $H_1(M)$ is infinite, which is implied if ∂M contains a surface other than S^2, then M is Haken [20, Lemma 6.6 and 6.7]. (With our definitions, any properly embedded 2-disk is incompressible, and the 3-disk is Haken.)

Before we prove the main theorem of this paper, we must mention what is known about Thurston's geometrization conjecture for irreducible 3-manifolds with infinite fundamental groups. (Again, our 3-manifolds are understood to be compact, connected and orientable.) Johannson [22] and Jaco and Shalen [21] have shown that given an irreducible 3-manifold M with incompressible boundary, there is a finite family of disjoint, pairwise-nonisotopic incompressible tori in M which are not isotopic to boundary components and which split M into pieces that are Seifert manifolds or are geometrically atoroidal, meaning that they admit no embedded incompressible torus (except possibly parallel to the boundary). A minimal family of such tori is unique up to isotopy, and we will say that it gives a *toral splitting* of M. We will say that the toral splitting is a *geometric toral splitting* if the geometrically atoroidal pieces which do not admit a Seifert structure have complete hyperbolic metrics on their interiors. Thurston's geometrization conjecture for irreducible 3-manifolds with infinite fundamental groups states that such manifolds have geometric toral splittings.

Suppose that M is Haken. The pieces in its toral splitting are certainly Haken. Let N be a geometrically atoroidal piece. The torus theorem says that N is a special Seifert manifold or is homotopically atoroidal i.e. any subgroup of $\pi_1(N)$ which is isomorphic to $Z \times Z$ is conjugate into the fundamental group of a boundary component. Thurston has shown that a homotopically atoroidal

Haken manifold is a twisted I-bundle over the Klein bottle (which is Seifert), or admits a complete hyperbolic metric on its interior.

Thus the case in which Thurston's geometrization conjecture for an irreducible 3-manifold M with infinite fundamental group is still open is when M is a closed non-Haken irreducible 3-manifold with infinite fundamental group which is not Seifert. The conjecture states that such a manifold is hyperbolic.

Our goal is to make general statements about the L^2-Betti numbers and Novikov-Shubin invariants of a 3-manifold. We have already treated the case when the fundamental group is finite in Example 3.11. We will confine ourselves in the sequel to the case when $\pi_1(M)$ is infinite. We will compute the invariants using the putative geometric decomposition of M. As we are studying homotopy invariants which have a simple behaviour with respect to finite coverings, it is enough to assume a weaker condition than that M have a geometric decomposition. Recall from the introduction that we say that a prime 3-manifold is *exceptional* if it is closed and no finite cover of it is homotopy-equivalent to a Haken, Seifert or hyperbolic 3-manifold.

Part 4 of Theorem 0.1 has been proven in Sections 4 and 5. We now prove parts 1a, 2 and 3 of Theorem 0.1. The proof will be by a succession of lemmas. In order to prove the statement about $\alpha_1(M)$, we will show that if $\alpha_1(M) < \infty^+$ then M is one of the special cases listed in the statement of the theorem. The values of $\alpha_1(M)$ in these special cases follow from previous calculations.

Lemma 6.1. *If M is an irreducible Haken manifold with incompressible torus boundary then M has vanishing L^2-cohomology and $\alpha_2(M) > 0$. If $\alpha_1(M) < \infty^+$ then M is one of the special cases listed in Theorem 0.1.2.*

Proof. We know that M has a geometric toral splitting. As a compact connected orientable 3-manifold with torus boundary whose interior has a complete hyperbolic metric is either $T^2 \times I$ or has a complete finite-volume hyperbolic metric [35, p. 52], the pieces in the toral splitting either admit a Seifert structure or have a complete finite-volume hyperbolic metric on their interior. Let s be the number of tori in such a minimal splitting. We will use induction over s. To begin the induction, if $s = 0$ then M is Seifert or hyperbolic and the claim follows from Theorems 4.1, 4.4 and 5.14. The induction step from $s - 1$ to s is done as follows:

Let T^2 be a torus in a minimal family of splitting tori. Depending on whether T^2 is separating or not, we get decompositions $M = M_1 \cup_{T^2} M_2$ or $M = M_1 \cup_{T^2 \times \partial I} T^2 \times I$ by cutting M open along T^2, and an exact sequence $0 \to C(T^2) \to C(M_1) \oplus C(M_2) \to C(M) \to 0$ or $0 \to C(T^2 \times \partial I) \to C(M_1) \oplus C(T^2 \times I) \to C(M) \to 0$ with coefficients in $l^2(\pi_1(M))$. Note that each M_j satisfies the induction hypothesis. Hence $b_p(M_j) = 0$ for all p and $\alpha_2(M_j) > 0$. From Lemma 3.6 and Example 3.10 we have that $b_p(T^2) = 0$ for all p and $\alpha_p(T^2) = 2$ for $p \in \{1,2\}$. The weakly exact Mayer-Vietoris sequence gives that M has vanishing L^2-cohomology, and Theorem 2.3.2 and Lemma 2.4.3 give the inequalities

$$\frac{1}{\alpha_2(M)} \leqq \frac{1}{\alpha_1(T^2)} + \frac{1}{\min\{\alpha_2(M_1), \alpha_2(M_2)\}} \qquad \text{or}$$

$$\frac{1}{\alpha_2(M)} \leqq \frac{1}{\alpha_1(T^2 \times \partial I)} + \frac{1}{\min\{\alpha_2(M_1), \alpha_2(T^2 \times I)\}}.$$

Thus $\alpha_2(M) > 0$.

We also have the exact sequences $0 \to C(M_1) \to C(M) \to C(M_2, T^2) \to 0$ or $0 \to C(M_1) \to C(M) \to C(T^2 \times I, T^2 \times \partial I) \to 0$ with $l^2(\pi_1(M))$ as coefficients. As M_1 has vanishing L^2-cohomology, Theorem 2.3.1 gives that

$$\frac{1}{\alpha_1(M)} \leqq \frac{1}{\alpha_1(M_1)} + \frac{1}{\alpha_1(M_2, T^2)} \qquad \text{or}$$

$$\frac{1}{\alpha_1(M)} \leqq \frac{1}{\alpha_1(M_1)} + \frac{1}{\alpha_1(T^2 \times I, T^2 \times \partial I)}.$$

From Lemma 3.5 we have that $\alpha_1(M_2, T^2) = \alpha_1(T^2 \times I, T^2 \times \partial I) = \infty^+$. This implies in both cases that $\alpha_1(M_1) \leqq \alpha_1(M)$. Hence $\alpha_1(M_1) < \infty^+$, and by symmetry $\alpha_1(M_2) < \infty^+$ in the first case. By the induction hypothesis, M_j must be $T^2 \times I$ or a twisted I-bundle over K. Thus M is either the gluing of two twisted I-bundles over K along their boundaries, or a T^2-bundle over S^1. If M is the gluing of two twisted I-bundles over K over their boundaries then M is double-covered by a T^2-bundle over S^1. In either case, Lemma 6.2 will give that M has the geometric type of some T^2-bundle over S^1. (For later purposes, Lemma 6.2 is stated in greater generality than is needed here.) Then [41, Theorem 5.5] implies that M has a *Sol*, *Nil* or R^3-structure, and is one of the special cases listed. □

Lemma 6.2. *Let $\overline{M}$ be a finite cover of an irreducible closed oriented 3-manifold M with infinite fundamental group. If $\overline{M}$ is homotopy-equivalent to a closed 3-manifold N with a Seifert or Sol-structure then M has the same geometric type as N.*

Proof. From [33, Theorem 3] we have that $\overline{M}$ is irreducible. If N has a Seifert structure then [40, pages 35 and 36] gives that $\overline{M}$ is homeomorphic to N and that M is also a Seifert manifold of the same geometric type. If N has a *Sol*-structure then $\overline{M}$ and N are Haken, and so $\overline{M}$ is homeomorphic to N [20, Theorem 13.6]. It follows from [41, Theorem 5.3] that M has a *Sol*-structure. □

Lemma 6.3. *If M is an irreducible Haken manifold with incompressible boundary then $b_p(M) = 0$ for $p \neq 1$, $b_1(M) = -\chi(M)$ and $\alpha_2(M) > 0$. If $\alpha_1(M) < \infty^+$ then M is one of the special cases listed in Theorem 0.1.2.*

Proof. Because of Lemma 6.1, we may assume that ∂M is nonempty. Let N be $M \cup_{\partial M} M$. Then [46, Satz 1.8] implies that N is irreducible. Clearly N is a closed Haken manifold. From Lemma 6.1 we have that N has vanishing L^2-cohomology and $\alpha_2(N) > 0$. We have the exact sequence $0 \to C(\partial M) \to$

$C(M) \oplus C(M) \to C(N) \to 0$ with coefficients in $l^2(\pi_1(N))$. From Example 3.10 we have that $b_p(\partial M) = 0$ for $p \neq 1$ and $\alpha_p(\partial M) > 0$ for all p. Then we get from the weakly exact Mayer-Vietoris sequence that $b_p(M) = 0$ for $p \neq 1$. From the Euler characteristic formula we derive that $b_1(M) = -\chi(M)$. Theorem 2.3.1 and Lemma 2.4.3 imply that

$$\frac{1}{\alpha_2(M)} \leqq \frac{1}{\alpha_2(\partial M)} + \frac{1}{\alpha_2(N)}$$

and hence $\alpha_2(M) > 0$. Next we prove the claim for $\alpha_1(M)$. Suppose that M does not have a toral boundary. Then ∂M contains a component F_g for $g \geqq 2$. As $\pi_1(F_g)$ is nonamenable and is a subgroup of $\pi_1(M)$, $\pi_1(M)$ is nonamenable and Lemma 3.5.2 implies that $\alpha_1(M) = \infty^+$. Hence the claim follows already from Lemma 6.1. □

Lemma 6.4. *If M is an irreducible Haken manifold and is not a 3-disk, then $b_p(M) = 0$ for $p \neq 1$, $b_1(M) = -\chi(M)$ and $\alpha_2(M) > 0$. If $\alpha_1(M) < \infty^+$ then M is one of the special cases listed in Theorem 0.1.2.*

Proof. Because of Lemma 6.3, we may assume that ∂M is compressible. The loop theorem [20, Theorem 4.2] gives an embedded disk D^2 in M such that D^2 meets ∂M transversally, and $\partial D^2 = D^2 \cap \partial M$ is an essential curve on ∂M. Depending on whether the disk D^2 is separating or not, we get the following two cases:

If D^2 is separating then there are 3-manifolds M_1 and M_2 and embedded disks $D^2 \subset \partial M_1$ and $D^2 \subset \partial M_2$ such that $M = M_1 \cup_{D^2} M_2$. In particular, M is homotopy equivalent to $M_1 \vee M_2$. Since M is prime, M_1 and M_2 are prime. As M_1 and M_2 have nonempty boundary, they are not $S^1 \times S^2$, and so are irreducible. As M is irreducible with infinite fundamental group, it is a $K(\pi, 1)$ Eilenberg-MacLane space. Then the same must be true for M_1 and M_2. If M_i were a 3-disk then the boundary of the embedded 2-disk would not be an essential curve on ∂M. Thus M_1 and M_2 have infinite fundamental groups.

If D^2 is nonseparating then there is a 3-manifold M_1 with embedded $S^0 \times D^2 \subset \partial M_1$ such that $M = M_1 \cup_{S^0 \times D^2} D^1 \times D^2$. The same argument as above shows that M_1 is an irreducible 3-manifold which is a 3-disk or has infinite fundamental group. If it were a 3-disk then M would be $S^1 \times D^2$, which satisfies the claim of the Lemma. So we may assume that M_1 has infinite fundamental group.

We will prove the Lemma using the fact that M is homotopy equivalent to $M_1 \vee M_2$ (respectively $M_1 \vee S^1$). It suffices to verify the claim for M_1 and M_2 (respectively M_1), since the claim for M then follows from the proof of Proposition 3.7. If M_1 and M_2 (respectively M_1) have incompressible boundary then we are done by Lemma 6.3. Otherwise, we repeat the process of cutting along 2-disks described above. This process must stop after finitely many steps. □

Proof of Parts 1a, 2 and 3 of Theorem 0.1 : We have the prime decomposition

$$M = M_1 \# M_2 \# \ldots \# M_r.$$

By assumption, each M_j in the decomposition is nonexceptional. We claim first that if $\pi_1(M_j)$ is finite then $b_1(M_j) = 0$, if $\pi_1(M_j)$ is infinite then $b_1(M_j) = -\chi(M_j)$, and that $\alpha_2(M_j) > 0$. The case of finite fundamental group follows from Example 3.11. From Theorem 2.6 and Remark 3.9 we may assume that if M_j is closed then M_j is Seifert, hyperbolic or Haken. If M_j is closed and Seifert then the result follows from Theorem 4.1. If M_j is closed and hyperbolic then the result follows from Theorem 5.14. If M_j is closed and Haken then the result follows from Lemma 6.1. If M_j has a boundary component which is a 2-sphere then M_j is a 3-disk and the result follows from Example 3.11. If M_j has a nonempty boundary with no 2-spheres then it is Haken and the result follows from Lemma 6.4.

From Lemma 3.5 we have that $b_0(M) = b_3(M) = 0$. From Proposition 3.7.1 we have that

$$b_1(M) = r - 1 + \sum_{j=1}^{r} \left(b_1(M_j) - \frac{1}{|\pi_1(M_j)|} \right).$$

As we have shown that $b_1(M_j) = -\chi(M_j) + \{1 \text{ if } M_j \cong D^3\}$, the claim of Theorem 0.1.1 for $b_1(M)$ follows. The claim for $b_2(M)$ now follows from the Euler characteristic equation. From Proposition 3.7.3 we have $\alpha_2(M) = \min\{\alpha_2(M_j) : j = 1, \ldots r\} > 0$.

From Corollary 3.4.1 we have that $\alpha_1(M) = \alpha_1(P(M))$. Thus, by removing the simply-connected factors, we may assume that $M = P(M)$. Suppose that $\alpha_1(M) < \infty^+$. From Proposition 3.7, we have the possibilities that $r = 1$, or that $r = 2$ and $\pi_1(M_1) = \pi_1(M_2) = Z/2$. If $r = 1$ then $M \cong S^1 \times S^2$ and is one of the special cases listed, or M is irreducible. If M is not closed then it is Haken and Lemma 6.4 implies that it is one of the special cases listed. If M is closed then by assumption a finite cover $\overline{M}$ of M is homotopy equivalent to a Seifert, hyperbolic or Haken manifold N, which must also be closed and orientable. If N is Seifert or hyperbolic then Theorems 4.1, 4.4 and 5.14 imply that N is a closed $S^2 \times R$, R^3, or *Nil* manifold. If N is Haken then Lemma 6.4 implies that N is a closed $S^2 \times R$, R^3, *Nil* or *Sol* manifold. Lemma 6.2 gives that M is of the same geometric type as N, and so is one of the special cases listed.

If $r = 2$, it remains to show that an irreducible (compact connected orientable) 3-manifold M with $\pi_1(M) = Z/2$ is homotopy equivalent to RP^3. This follows from [43, Theorem 1.8]. □

Proof of Theorem 0.1.1b. First, for the group Euler characteristic [5, Section IX.7] to be defined we must show that $\pi_1(M)$ is virtually torsion-free and of finite homological type. Let $\{M_j\}_{j=1}^{s}$ be the prime factors of M with finite fundamental group. Put $\Gamma_1 = \pi_1(M_1) * \ldots * \pi_1(M_s)$ and $\Gamma_2 = \pi_1(M_{s+1}) * \ldots * \pi_1(M_r)$. It is known that Γ_1 has a finite-index free subgroup F and that Γ_2 is torsion-free. Let $\phi : \Gamma_1 * \Gamma_2 \to \Gamma_1$ be the natural homomorphism. Then $\phi^{-1}(F)$ is finite-index in $\pi_1(M)$, and the Kurosh subgroup theorem [20, Theorem 8.3] implies that it is torsion-free. As Γ_1 and Γ_2 have finite homological type, [5, Proposition IX.7.3.e] implies that $\pi_1(M)$ is of finite homological type and that:

$$\chi(\pi_1(M)) = r - 1 + \sum_{j=1}^{r} \chi(\pi_1(M_j)).$$

Thus in order to show that $b_1(M) = -\chi(\pi_1(M))$, it is enough to verify that for each j,

$$-\frac{1}{|\pi_1(M_j)|} - \chi(M_j) + \{1 \text{ if } M_j \cong D^3\} = -\chi(\pi_1(M_j)).$$

As M_j is either a $K(\pi, 1)$ Eilenberg-MacLane space with π infinite, a 3-disk or a closed manifold with finite fundamental group, the equation is easy to verify. The statement for $b_2(M)$ now follows from the Euler characteristic equation. □

We now prove a slightly stronger version of Theorem 0.1.1c.

Proposition 6.5. *Let M be a (compact connected orientable) 3-manifold. If all L^2-Betti numbers of M vanish then M satisfies one of the following conditions:*

1. *M is homotopy equivalent to an irreducible 3-manifold N with infinite fundamental group whose boundary is empty or a disjoint union of tori.*
2. *M is homotopy equivalent to $S^1 \times S^2$ or $RP^3 \# RP^3$.*

If condition 2.) holds, or if condition 1.) holds and N is nonexceptional, then all of the L^2-Betti numbers of M vanish. □

Proof. Suppose that M has vanishing L^2-cohomology. From Example 3.11, $\pi_1(M)$ must be infinite. From Proposition 3.7.1 we have that

$$r - 1 + \sum_{j=1}^{r} \left(b_1(M_j) - \frac{1}{|\pi_1(M_j)|} \right) = 0.$$

Equivalently,

$$\sum_{j=1}^{r} \left(b_1(M_j) - \frac{1}{|\pi_1(M_j)|} + 1 \right) = 1.$$

It follows that the prime decomposition of M must consist of homotopy 3-spheres, 3-disks and either

A. A prime manifold M' with infinite fundamental group and vanishing b_1 or

B. Two prime manifolds M^1 and M^2 with fundamental group $Z/2$.

In case A, M' is $S^1 \times S^2$ or is irreducible. If M' is irreducible and has nonempty boundary then Lemma 6.4 implies that its boundary components must be tori. From the Euler characteristic equation we have that $\chi(M) = 0$, and so no 3-disks can occur in the prime decomposition of M. In case B, we have already shown that M^1 and M^2 are homotopy-equivalent to RP^3. Again, because $\chi(M) = 0$, no 3-disks can occur in the prime decomposition of M. Thus we have shown that if M has vanishing L^2-cohomology then M satisfies one of the two conditions of the corollary.

If M satisfies condition 2. of the corollary then Theorems 2.6 and 4.1 imply that M has vanishing L^2-cohomology. If M satisfies condition 1. of the corollary, from Theorem 2.6 we may assume without loss of generality that $M = N$. We have that its Euler characteristic vanishes. If M has nonempty boundary then Lemma 6.4 implies that it has vanishing L^2-cohomology. If M is closed and nonexceptional then by passing to a finite cover and using Theorem 2.6, we may assume that M is Seifert, hyperbolic or Haken. Theorems 4.1, 4.4 and 5.14 imply that M has vanishing L^2-cohomology. □

We now prove Theorem 0.1.5. Again, we build up to the proof by a sequence of lemmas.

Lemma 6.6. *If M is irreducible and ∂M contains an incompressible torus then $\alpha_2(M) \leqq 2$.*

Proof. From Lemma 6.3 we get $b_2(M)=0$. As T^2 has vanishing L^2-cohomology, the long weakly exact homology sequence of the pair (M, T^2) implies that $H_2(M, T^2; l^2(\pi_1(M)))$ vanishes. We have a short exact sequence $0 \to C(T^2) \to C(M) \to C(M, T^2) \to 0$ and so from Theorem 2.3.3,

$$\frac{1}{\alpha_2(T^2)} \leqq \frac{1}{\alpha_2(M)} + \frac{1}{\alpha_3(M, T^2)}.$$

Proposition 3.2 implies that $\alpha_3(M, T^2) = \alpha_1(M, \partial M - T^2)$. If this is ∞^+ then $\alpha_2(M) \leqq \alpha_2(T^2) = 2$ and we are done. If $\partial M - T^2 \neq \emptyset$ then Lemma 3.5.5 implies that $\alpha_1(M, \partial M - T^2) = \infty^+$. If $\partial M - T^2 = \emptyset$ then Theorem 0.1.2 gives the possible cases in which $\alpha_1(M, \partial M - T^2) < \infty^+$. The only case in which ∂M is a single incompressible torus is when M is a twisted I-bundle over K, and in this case Theorem 4.4 gives that $\alpha_2(M) = 2$. □

Lemma 6.7. *If M is a closed Haken manifold and does not admit an R^3 or Sol structure then $\alpha_2(M) \leqq 2$.*

Proof. If M is Seifert or hyperbolic then the proposition follows from Theorems 4.1 and 5.14. Otherwise, consider the nonempty minimal family of splitting tori. Let T^2 be a member of the minimal family. Cutting M open along T^2 yields decompositions $M = M_1 \cup_{T^2} M_2$ or $M = M_1 \cup_{T^2 \times \partial I} T^2 \times I$, depending on whether T^2 is separating or not. We get the exact sequences $0 \to C(M_1) \to C(M) \to C(M_2, T^2) \to 0$ or $0 \to C(M_1) \to C(M) \to C(T^2 \times I, T^2 \times \partial I) \to 0$ with coefficients in $l^2(\pi_1(M))$. Since $b_1(M) = 0$ (Lemma 6.1), we derive from Theorem 2.3.2 that

$$\frac{1}{\alpha_2(M_2, T^2)} \leqq \frac{1}{\alpha_1(M_1)} + \frac{1}{\alpha_2(M)} \quad \text{or}$$

$$\frac{1}{\alpha_2(T^2 \times I, T^2 \times \partial I)} \leqq \frac{1}{\alpha_1(M_1)} + \frac{1}{\alpha_2(M)}.$$

Suppose that $\alpha_1(M_1) \geqq \infty$. Then we have that $\alpha_2(M) \leqq \alpha_2(M_2, T^2)$ (respectively $\alpha_2(M) \leqq \alpha_2(T^2 \times I, T^2 \times \partial I) = 2$). Proposition 3.2 gives that $\alpha_2(M_2, T^2)$ $= \alpha_2(M_2)$, and we have already proven that this is less than or equal to two. By symmetry, it remains to treat the case when $\alpha_1(M_1), \alpha_1(M_2) < \infty$, (respectively $\alpha_1(M_1) < \infty$). From Theorem 0.1.2, M_1 and M_2 must be I-bundles over K (respectively M_1 must be $I \times T^2$). As before, in either case M carries a *Sol*, *Nil* or R^3-structure. Since $\alpha_2(M) = 2$ in the *Nil* case (Theorem 4.1), the lemma follows. □

Proof of Theorem 0.1.5. From Proposition 3.7.3 we have that

$$\alpha_2(M) = \min\{\alpha_2(M_j) : j = 1, \ldots r\}.$$

Clearly, it is enough to verify the theorem under the assumption that M is prime. As $S^1 \times S^2$ has an $S^2 \times R$-structure, we may assume that M is irreducible. If ∂M contains an incompressible torus then we are done by Lemma 6.6. Suppose that M is closed, has infinite fundamental group and is nonexceptional. Then a finite cover $\overline{M}$, which is closed, orientable and irreducible, is homotopy equivalent to a manifold N which is Seifert, hyperbolic or Haken. If $\alpha_2(M) > 2$ then Theorems 4.1 and 5.14 and Lemma 6.7 imply that N has an R^3, $S^2 \times R$ or *Sol* structure. By Lemma 6.2, M also has such a structure. □

Finally, Theorem 0.1.6 follows from Propositions 3.2 and 3.5.5. □

7. Remarks and conjectures

Conjecture 7.1. *Let M be a compact connected manifold, possibly with boundary. Then*

1. *The L^2-Betti numbers of M are rational. If $\pi_1(M)$ is torsion-free then the L^2-Betti numbers of M are integers.*
2. *The Novikov-Shubin invariants of M are positive and rational.* □

In the case of the L^2-Betti numbers, this seems to be a well-known conjecture. The question of the rationality of the L^2-Betti numbers, for closed manifolds, appears in [1]. Theorem 0.1 shows that Conjecture 7.1.1 is true for the class of 3-manifolds considered there. By Lemma 3.5.1, Conjecture 7.1.2 is trivially true for $\alpha_1(M)$. Theorems ?? and ?? give that it is true for $\alpha_2(M)$ if M is a Seifert 3-manifold. Note that for any positive integer k there are examples of closed manifolds in higher dimensions with $\pi_1(M) = Z$ such that $\alpha_3(M) = \frac{1}{k}$ [24]. Conjecture 7.1 is equivalent to the following purely algebraic conjecture:

Conjecture 7.2. *Let π be a finitely presented group and let $f : \oplus_{i=1}^r Z\pi \to \oplus_{i=1}^r Z\pi$ be a $Z\pi$-homomorphism. We get a bounded π-equivariant operator $\overline{f} : \oplus_{i=1}^r l^2(\pi) \to \oplus_{i=1}^r l^2(\pi)$ by tensoring by $l^2(\pi)$. Then*

1. *The von Neumann dimension of ker$(\overline{f})$ is rational. If π is torsion-free then it is an integer.*
2. *The Novikov-Shubin invariant of $\overline{f}$ is a positive rational number.* □

To see the equivalence between Conjectures 7.1 and 7.2, suppose first that we are given a compact manifold M. Let K be a finite CW-complex which is homotopy equivalent to M. Taking f in Conjecture 7.2 to be the combinatorial Laplacian Δ_p coming from K and using Lemma 2.4, we see that the validity of Conjecture 7.2.1 would imply that Conjecture 7.1.1 holds for M. Taking f to be the differential c_p of the cellular chain complex of $\widetilde{K}$, we see that the validity of Conjecture 7.2.2 would imply that Conjecture 7.1.2 holds for M. It remains to show that Conjecture 7.1 implies Conjecture 7.2. Let X be a finite CW-complex with fundamental group π. Let $f : \oplus_{i=1}^r Z\pi \to \oplus_{i=1}^r Z\pi$ be any $Z\pi$-module homomorphism. For any given $n \geqq 2$, one can attach cells to X in dimensions n and $n+1$ in such a way that the resulting finite CW-complex Y has the same fundamental group as X, and the relative chain complex $C(\widetilde{Y}, \widetilde{X})$ is concentrated in dimensions n and $n+1$ and given there by f [26, Theorem 2.1]. If we choose $n > \dim(X)$ then $b_{n+1}(Y) = b(\overline{f})$ and $\alpha_{n+1}(Y) = \alpha(\overline{f})$. Moreover, there is a compact manifold M, possibly with boundary, which is homotopy equivalent to Y. Since the L^2-Betti numbers and the Novikov-Shubin invariants are homotopy invariants, we get $b_{n+1}(M) = b(\overline{f})$ and $\alpha_{n+1}(M) = \alpha(\overline{f})$. Hence Conjecture 7.1 is equivalent to Conjecture 7.2.

Conjecture 7.2.1 is proven for a large class of groups, which includes elementary amenable groups and free groups, in [23]. It is not hard to see that Conjecture 7.2.2 is true if π is abelian. (A proof of the equivalent Conjecture 7.1.2 in this case appears in [24].) D. Voiculescu informs us that Conjecture 7.2.2 is true when π is a free group. Conjecture 7.2.1 implies a well-known conjecture of algebra.

Conjecture 7.3. *Let π be a finitely-presented group. Then the group ring $Q\pi$ has no zero-divisors if and only if π is torsion-free.* □

The only-if statement is trivial. The if statement would follow from the second conjecture as follows: Let $u \in Q\pi$ be a zero-divisor. We want to show that $u = 0$. We may assume that u lies in $Z\pi$. Let $f : Z\pi \to Z\pi$ be given by right multiplication with u. Since u is a zero-divisor, $\overline{f}$ has a non-trivial kernel. Since the dimension of the kernel of $\overline{f}$ must be a positive number less or equal to the dimension of $l^2(\pi)$, which is 1, it can only be an integer if it is 1. Hence the kernel of $\overline{f}$ is $l^2(\pi)$. This implies that $u = 0$.

Conjecture 7.4. *The second L^2-Betti number of a compact prime 3-manifold vanishes.* □

We have shown in Example 3.11 and Theorem 0.1 that the second L^2-Betti number of a nonexceptional compact prime 3-manifold vanishes. However, there may be a reason why it should vanish which is independent of any geometric decomposition theorem.

Conjecture 7.5. *If M is a closed $K(\pi, 1)$ manifold then its L^2-Betti numbers vanish outside of the middle dimension.* □

Proposition 6.5 implies that a closed $K(\pi, 1)$ 3-manifold of the type considered there has vanishing L^2-Betti numbers, thereby verifying the conjecture.

Conjecture 7.5 includes the unproven conjecture of Singer which states the same for nonpositively-curved manifolds. If $\pi_1(M)$ contains an infinite normal amenable subgroup then the truth of the conjecture follows immediately from [10 Theorem 0.2]. Conjecture 7.5 was emphasized in the case of 4-manifolds in [17]. A consequence would be that if $\dim(M) = 4k+2$ then $\chi(M) \leqq 0$, and if $\dim(M) = 4k$ then $\chi(M) \geqq | \sigma(M) |$.

As mentioned in the introduction, our motivation to study L^2-Betti numbers and Novikov-Shubin invariants comes from the L^2-torsion invariants [6, 24, 29, 31, 32]. Let M be a compact Riemannian manifold whose boundary is decomposed as $\partial M = \partial_0 M \coprod \partial_1 M$, where $\partial_0 M$ and $\partial_1 M$ are disjoint unions of components of ∂M. One can try to define L^2-analogs of the usual Reidemeister and analytic torsions of the pair $(M, \partial_0 M)$. However, one would encounter difficulties in the definitions if the spectrum of the combinatorial or differential-form Laplacian were too thick near zero. A sufficient condition for the L^2-torsions to be well-defined is that the Novikov-Shubin invariants $\alpha_*(M, \partial_0 M)$ are all positive. In this case, we denote the corresponding L^2-torsion invariants by $\rho_{\mathrm{comb}}(M, \partial_0 M)$ and $\rho_{\mathrm{an}}(M, \partial_0 M)$, respectively. Conjecture 7.1.2 would imply that the L^2-torsions are always well-defined. If in addition the L^2-cohomology of $(M, \partial_0 M)$ vanishes then the L^2-Reidemeister torsion is a simple homotopy invariant (and in particular a homeomorphism invariant) and the L^2-analytic torsion is a diffeomorphism invariant.

Conjecture 7.6. *The L^2-Reidemeister and analytic torsions of $(M, \partial_0 M)$ are related by*

$$\rho_{\mathrm{comb}}(M, \partial_0 M) = \rho_{\mathrm{an}}(M, \partial_0 M) + \frac{\ln(2)}{2} \cdot \chi(\partial M). \qquad \square$$

This is the L^2-analog of the Cheeger-Müller theorem for the ordinary Reidemeister and analytic torsions [7, 36], as extended to manifolds with boundary in [25, 28].

Our results show that if M is a 3-manifold of the type considered in Theorem 0.1 then the L^2-torsions are well-defined. Sufficient conditions for the vanishing of the L^2-cohomology are given in Proposition 6.5. If M is a Seifert 3-manifold with infinite fundamental group then its L^2-Reidemeister torsion vanishes [31]. If M is a closed 3-manifold which admits a hyperbolic structure then its L^2-analytic torsion is $-\frac{1}{3\pi}\mathrm{Vol}(M, g_{hyp})$, where g_{hyp} is the unique (up to isometry) hyperbolic metric on M [24, 32].

Conjecture 7.7. *If M is a compact connected 3-manifold with a Thurston geometric decomposition which satisfies one of the conditions of Proposition 6.5 then its L^2-torsion is $-\frac{1}{3\pi}$ times the sum of the (finite) volumes of its hyperbolic pieces.*

As one has a formula for the relationship between the L^2-Reidemeister torsions of the terms in a short exact sequence [31] Conjecture 7.7 would follow from Conjecture 7.6 if one knew that the L^2-torsion of a compact 3-manifold whose interior admitted a complete finite-volume hyperbolic metric were equal to $-\frac{1}{3\pi}$ times the hyperbolic volume of the interior. We note that Conjecture 7.7 would imply that for the manifolds it considers, the L^2-torsion is a universal constant times the simplicial volume discussed in [44].

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