

Basic conjugacy theorems for G_2

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Dedicated to Donald G. Higman

Oblatum 14-V-1994 & 8-IX-1994

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1. Introduction

The main purpose of this article is to prove a basic conjugacy result (1.1) for subsets of the Lie group $G_2(\mathbf{C})$ and give a fast classification (1.4) of the conjugacy classes of finite subgroups of $G_2(\mathbf{C})$.

An obvious difficulty in dealing with $G_2(\mathbf{C})$ is that it is defined by degree 3 conditions (making it the automorphism group of an algebra, or of a cubic form). In contrast, the symplectic and orthogonal groups are generally easier to work with; these groups are defined by quadratic conditions.

Our main idea is to embed $G_2(\mathbf{C})$ into a group of type E_8 , where conjugacy phenomena in $G_2(\mathbf{C})$ acquire useful additional structures. Experience shows that E_8 “completes” many themes found in the exceptional groups.

Suppose G is a group, H a subgroup. Two subsets H_1 and H_2 are *strongly fused in H with respect to G* if they are conjugate in G and, whenever $x \in G$ satisfies $H_1^x = H_2$, there are $y \in H$ and $c \in C_G(H_1)$ so that $x = cy$.

We comment that we do not require reductive groups to be connected. It has been known for a long time that orthogonal and symplectic groups strongly control fusion of its subsets (Section 2).

Regard $G_2(\mathbf{C})$ as a subgroup of $O(7, \mathbf{C})$ and let V be the natural 7-dimensional module for $O(7, \mathbf{C})$. The first result gives an affirmative answer to a question of J.-P. Serre; we thank him for his persistence.

1.1 Theorem 1. $G_2(\mathbf{C})$ strongly controls fusion of its subsets with respect to $GL(V)$.

Theorem 1 can be implies the equivalence of the following properties, for a reductive group S and homomorphisms α and β to $G_2(\mathbf{C})$: (a) α and β are conjugate by an element of $G_2(\mathbf{C})$; (b) the embeddings of S in $GL(V)$, given by α and β , are equivalent representations; (b') the embeddings of S in $GL(W)$, given by action on an arbitrary module W for $G_2(\mathbf{C})$, are equivalent representations; (c) for every $x \in S$, x^α and x^β are conjugate in $GL(V)$; (c') for every semisimple element $x \in S$, x^α and x^β are conjugate in $GL(V)$. The equivalence of (b) and (b') is based on the fact that V generates the representation ring, as a lambda-ring. The analogous statements for subsets may be wrong. For example, in a group with just one conjugacy class of involutions (like $G_2(\mathbf{C})$), any two pairs of involutions form a pair of elementwise conjugate subsets; they are not conjugate if they generate nonisomorphic dihedral groups.

Conjugacy in $GL(n, \mathbf{C})$ for finite groups is decidable by character theory and for connected reductive groups it is decidable by weight theory. The following is an immediate consequence of Theorem 1.

1.2 Corollary 1. Two embeddings of a finite group in $G_2(\mathbf{C})$ are conjugate if and only if they are conjugate in $GL(V)$ if and only if they afford the same character on V .

A p -local subgroup of a group, for some prime number p , is the normalizer of a nonidentity p -subgroup. A local subgroup is a p -local subgroup for some p . In $G_2(\mathbf{C})$, all maximal local subgroups are known. An irreducible linear group $S \leq GL(V)$ is *imprimitive* if there is a nontrivial decomposition $V = \bigoplus_i V_i$ where the V_i are permuted by S ; otherwise it is called *primitive*. A finite subgroup F of a connected Lie group G is *Lie primitive* if whenever H is a closed Lie subgroup such that $F \leq H \leq G$, then H is finite or $H = G$.

1.3 Theorem 2. A finite subgroup of $G_2(\mathbf{C})$ lies in a local subgroup or is irreducible on the 7-dimensional module.

The above involves a criterion (6.1) for subgroups of $\text{Spin}(7, \mathbf{C})$ to be in a $G_2(\mathbf{C})$ subgroup. The next result gives a new classification of finite subgroups of $G_2(\mathbf{C})$ which is faster than the first such classification [CoWa].

1.4 Corollary 2. A finite subgroup of $G_2(\mathbf{C})$

(1) is in one of the following rank 2 reductive subgroups: a torus normalizer, a central product $SL(2, \mathbf{C}) \circ SL(2, \mathbf{C})$ or a subgroup $SL(3, \mathbf{C}) : 2$; or

(2) is isomorphic to one of the following irreducible linear groups (primitive linear groups, except for the first one and the second, which is a 2-local subgroup):

$$GL(3, 2), \quad 2^3 \cdot GL(3, 2), \quad G_2(2), \quad GL(3, 2) : 2,$$

$$PSU(3, 3) \cong G_2(2)', \quad SL(2, 8), PSL(2, 13).$$

Also, there is just one $G_2(\mathbb{C})$ -conjugacy class of each isomorphism type of finite group in (2). All the groups in (2) are Lie primitive [Gr][CoGr] and all groups in (1) and the second in (2) are local.

We have an analogue of Theorem 1 for the exceptional group $F_4(\mathbb{C})$. Its usefulness is unclear since there does not seem to be a simple criterion for conjugacy in $E_6(\mathbb{C})$.

1.5 Theorem 3. *Consider a natural containment $F \leq H$, where F has type F_4 and H is simply connected of type E_6 . Then, F strongly controls fusion of its subsets with respect to H .*

As a caution, we give a few negative results on control of fusion for exceptional Lie groups. For each $r \in \{2, 3, 4\}$, there is a pair of nonconjugate elementary abelian 3-groups of order 3^r in $E_8(\mathbb{C})$ with the property that all nonidentity elements are in the same conjugacy class, $3B$ (consequently, the associated pair of abstract embeddings of a group of order 3^r have the same character on the adjoint module). For $r = 2$, both groups are toral. For each $r \geq 3$, one member of each pair is toral in $E_8(\mathbb{C})$ and the other is nontoral. See [Gr] (13.2). This example shows that no module for $E_8(\mathbb{C})$ is good for strong control of fusion. For a second example, look in $2E_7(\mathbb{C})$ to see that the numbers of classes of elements of orders 2, 3 and 4 are 3, 5, 11; these numbers for $E_8(\mathbb{C})$ are 2, 4 and 7 [CoGr, Gr] (and these classes do not fuse in $GL(248, \mathbb{C})$).

Since submitting this paper, the articles [La1, La2] have come to our attention. In [La1], Larsen gives a proof of our Corollary 1; his method is quite different. Serre independently found a proof in a similar spirit. In [La2], Larsen proves that if a finite group has two homomorphisms to the Lie group G such that every element maps to a pair of G -conjugate elements, then the homomorphisms are G -conjugate for $G = G_2(\mathbb{C})$ but not in general for $F_4(\mathbb{C})$, $E_6(\mathbb{C})$, $E_7(\mathbb{C})$ and $E_8(\mathbb{C})$ (the negative result for $E_8(\mathbb{C})$ had been known; A. Borovik gave an example in 1989; also see the previous paragraph).

2. Conjugacy in classical groups

We begin with a few results on conjugacy in automorphism groups of bilinear forms. Related results have appeared other places (e.g. [Fr], [Ma1]). The following proof of strong control of fusion (2.3) comes from ideas in the proof

of Theorem 7.3 in [T]. In this section, V denotes a vector space over a field F of characteristic not 2 for which every finite degree extension is closed under taking square roots.

2.1 Lemma. *Let A be a finite dimensional algebra over F . Every invertible element of A is a square.*

Proof. Since A is Artinian, may assume that A is local with nilpotent maximal ideal, I . Let $y \in A$ be invertible. We may assume that $A = F[y]$, whence A/I is a field. The hypotheses on F allow us to assume that $y = 1 + u$, for $u \in I$. There is a square root of the form $1 + \frac{u}{2} + \cdots$ (in $(\mathbf{Z}[\frac{1}{2}].1_F)[u]$).

2.2 Corollary. *If $y \in GL(V)$, there is a polynomial P so that $P(y)^2 = y$.*

Proof. Apply Lemma 1 to $A := F[y]$.

2.3 Theorem. *Let V be a vector space and suppose that f is a nondegenerate alternating or symmetric bilinear form on V . Then $\text{Aut}(f)$ strongly controls fusion of its subsets with respect to $GL(V)$.*

Proof. There is an antiinvolution $t \mapsto t'$ on $\text{End}(V)$ which is defined by the formula $f(at, b) = f(a, bt')$, for $a, b \in V$. Let S and S^* be subsets of $\text{Aut}(f)$ and $g \in GL(V)$ so that $S^g = S^*$. Then, $y := gg'$ is hermitian (i.e., $y = y'$). By Lemma 2, there is a polynomial P so that $x := P(y)$ satisfies $x^2 = y$. Since $y = y', x = x'$. Now, given $s \in S, s^g \in \text{Aut}(f)$, which is equivalent to $s^g(s^g)' = 1$, or $sgg's' = gg'$, which means that $y = gg' \in C(s)$ since $s' = s^{-1}$. At once, $x \in C(S)$. Now, set $h := x^{-1}g$. We have $hh' = x^{-1}gg'x'^{-1} = x^{-1}yx^{-1} = 1$, and so $h \in \text{Aut}(f)$ and $g = xh$, as required.

2.4 Remarks. (i) (2.3) may be false if the form is degenerate. Consider the example of a vector space V with basis e_1, e_2 and bilinear form f defined by $f(e_i, e_j) = \delta_{1,i}\delta_{1,j}$; and with two groups $\langle g \rangle, \langle -g \rangle$ of order 2 acting on V , with g fixing e_1 and negating e_2 .

(ii) Two conjugate embeddings of a set into $O(n, \mathbf{C})$ are conjugate under $SO(n, \mathbf{C})$ if and only if there is an orthogonal transformation of determinant -1 centralizing one of the subsets. This hypothesis is automatically satisfied if n is odd.

(iii) We ask if there are reasonable criteria for $G: 2$ strongly to control fusion of its subsets with respect to some larger group; here, $G: 2$ is $GL(n, \mathbf{C})$ extended by the graph automorphism. The case $n = 3$ presents itself in the $G_2(\mathbf{C})$ situation. The same could be asked for G of type D_4 or E_6 extended by a group of graph automorphisms.

3. Natural embeddings of Chevalley groups

3.1 Notation. (See [Gr].) We let $E \cong E_8(\mathbf{C})$ and let $G \times F$ be a subgroup of E with $G \cong G_2(\mathbf{C})$ and $F \cong F_4(\mathbf{C})$. We also let $D \cong \mathrm{HSpin}(16, \mathbf{C})$ be the subgroup of type D_8 which is the centralizer of an involution $z \in E$ of type $2B$. Let V_{16} be the 16-dimensional natural module for $\mathrm{SO}(16, \mathbf{C})$. The groups $D \cong \mathrm{HSpin}(16, \mathbf{C})$ and $\mathrm{SO}(16, \mathbf{C})$ are nonisomorphic binary quotients of $\mathrm{Spin}(16, \mathbf{C})$; thus, an element of $\mathrm{SO}(16, \mathbf{C})$ corresponds to a pair of elements of $D \cong E$.

3.2 Notation. Let L be a quasisimple group of Lie type, T a maximal torus and Γ a type of indecomposable root system. A subgroup M of L is called a *standard subgroup* of type Γ (with respect to T) if the root system of M has type Γ and there is a subgroup M_0 of L generated by a set of standard root groups so that $M = M_0$ or M is the fixed point subgroup of a standard graph automorphism of M_0 . A *natural* subgroup of type Γ in L is a conjugate of a standard one.

3.3 Lemma. *Let $g \in W$, a Weyl group for an indecomposable root system.*

(i) *If V is the rational vector space containing the root lattice, then g is a product of reflections for roots in $[V, g]$. (Here, we identify a Cartan subalgebra with its dual and speak of roots as elements of this subalgebra.)*

(ii) *If g has a single eigenvalue -1 and the remaining eigenvalues equal to 1, g is a conjugate of a fundamental reflection.*

Proof. See [Ca], 2.5.5.

3.4 Lemma. *Let L be any quasisimple Lie group. Suppose that K is a closed quasisimple subgroup of L , that H is the subgroup of K generated by standard root groups of L with respect to some maximal torus and that $\mathrm{rank} H = \mathrm{rank} K$.*

(i) *If U is a maximal torus of H , $C(U) = UC(H)$.*

(ii) *There is a maximal torus T of E containing U such that $N_K(U) \leq N_E(T)$.*

(iii) *Suppose that A is a connected reductive subgroup of L containing K . Then, there is a central factor Y of A containing K with H as a subgroup generated by standard root groups of Y .*

(iv) *The set of Y of a fixed type which arise as in (iii) form an orbit under $C(H)$, provided that (a) in a root system of type Y , all root subsystems of the type of K form an orbit under the Weyl group of Y ; and (b) in a root system of type L , all root subsystems of the type of Y form an orbit under the Weyl group of L .*

Proof. By hypothesis, there is a maximal torus T of L which normalizes the given set of root groups of L . Set $U := T \cap H$, a maximal torus of H . Since all roots in H are roots for T , (i) follows.

(ii) In all cases, $N_H(U)$ is normal in $N_K(U)$ and the quotient is metacyclic, so $N_K(U)$ normalizes a torus in $C(U)^\circ = C(N_H(U))$ [BS].

(iii) If R is a maximal torus of A containing U , R normalizes each root group of H . Thus, the set of standard root groups of H may be expanded to a set of standard root groups of A . Each root group of $HZ(A)/Z(A)$ must be in a simple direct factor of $A/Z(A)$ since it is normalized by a maximal torus of $A/Z(A)$. Quasisimplicity of K finishes the proof.

(iv) This follows from (i) and the conjugacy of maximal tori in a Lie group.

3.5 Lemma. *A subgroup of E of type E_n is natural.*

Proof. Given an E_6 type subgroup X of E , we argue that it is a natural one. Expand a maximal torus \tilde{T} of X to T , a maximal torus of E . Then $N_X(\tilde{T})' \cong \tilde{T}.\Omega^-(6, 2)$ must act trivially on the reductive group $C(\tilde{T})^\circ/\tilde{T}$ because it has rank at most 2 so involves only tori or factors of type A_1 or A_2 . It follows that T is normalized by $N_X(\tilde{T})'$ and that there is a rank 2 torus U in T which is an $N_X(\tilde{T})$ -invariant complement to \tilde{T} and such that $N_X(\tilde{T})$ is involved in the Weyl group of a component Y of $C(U)$; such a component must have type among ADE . Therefore, Y is a natural subgroup of E of type E_n ; since $\text{rank } C(U)' \leq 6$, $n = 6$. The argument is similar for $n = 7$ and is trivial for $n = 8$.

3.6 Lemma. *Let A be a connected reductive subgroup of E containing F properly. Then*

(i) $F \leq A \leq F \times G$ or A contains a natural E_6 subgroup which contains F ;

(ii) if A does not contain F as a central factor, A is a natural subgroup of type E_n , for some $n \in \{6, 7, 8\}$.

(iii) the set of such type E_n subgroups, as in (ii); forms an orbit under $G = C(F)$;

(iv) any two F_4 subgroups of an E_n subgroup are conjugate.

Proof. Let U be a maximal torus of F ; expand it to a maximal torus \tilde{U} of A , then expand \tilde{U} to T , a maximal torus of E . We have $C(U)$ of type T_4D_4 . Let $\mathfrak{a}, \mathfrak{u}, \mathfrak{t}$ be the Lie subalgebras associated to A, U and T , respectively. We may assume that $N_F(U)$ normalizes T . Let $W_F \leq W_A \leq W_E$ be the natural containment of Weyl groups. Then $\dim [\mathfrak{t}, W_F] = 6$.

(i, ii) Consider $\mathfrak{a}_0 := \mathfrak{a} \cap [\mathfrak{t}, W_F]$. Since W_F acts irreducibly as Σ_3 on the 2-dimensional space $[\mathfrak{t}, W_F]/\mathfrak{u}$, \mathfrak{a}_0 is \mathfrak{u} or $[\mathfrak{t}, W_F]$. Suppose $\mathfrak{a}_0 = \mathfrak{u}$. Then, (3.3) implies that the short reflections of W_F are reflections of W_A and so every root of F is a root for A . Since a root system of type F_4 is not in a subsystem an

indecomposable root system of higher rank, we conclude that F is a central factor of A and so $F \leq A \leq F \times G$.

We now assume that $\mathfrak{a}_0 = [\mathfrak{t}, W_F]$. We deduce from (3.5.i) that there are roots of A corresponding to points of $\mathfrak{a}_0 \setminus [\mathfrak{t}, W_F]$. Let Δ be the set of roots of A in \mathfrak{a}_0 . Then, Δ has rank 6 and its span contains $[\mathfrak{t}, W_F]$. Since its Weyl group contains W_F , the only possibility is for Δ to have type E_6 . We have proved that a natural F_4 subgroup is contained in the natural E_6 subgroup of A determined by the maximal torus T ; the set of such E_6 subgroups forms one orbit under conjugation by $C(U)$. By (3.4), A has a central factor which is natural of type E_n and so (i) and (ii) (and (iii) for $n = 6$) are proved.

(iii) This follows since an E_6 subgroup is a component in the centralizer of a unique class of elements of order 3 in E or in an E_7 subgroup and an E_7 subgroup is a component in the centralizer of a unique class of elements of order 2 in E .

(iv) Let X be an arbitrary F_4 subgroup and F the standard one. We may assume that both are in Y , a standard E_6 subgroup. Weight theory for F_4 shows that the only irreducibles of dimension at most 27 are those of dimension 1 and 26. Let M be a 27-dimensional irreducible for Y . Then, both X and F fix a 1-space in M pointwise. The analysis of elementary abelian 3-subgroups in Y (see [Gr] (1.8) Table II) shows that we may assume $X \cap F$ contains a nontoral elementary abelian subgroup of order 27. The action of this group stabilizes a unique 1-space, which must be the 1-space fixed by both X and F ; its stabilizer is an algebraic group proper in Y so, by (i), $X = F$.

3.7 Lemma. *Let W be a Weyl group of rank n and type B, D, E or F . Then, (i) for any integer $m \in \{4, 5, \dots, n\}$, W acts transitively on subsystems of type D_m ; (ii) if W has type B or F , W acts transitively on subsystems of type B_{m-1} , for any $m \in \{4, 5, \dots, n\}$.*

Proof. For type B or D , this is an exercise and for F_4 it is trivial since the root system is the union of three maximal subsystems of type B_4 which form an orbit under W . For type E_n , use Witt's theorem and the fact that W acts as the full orthogonal group on the root lattice modulo 2.

3.8 Lemma. *Suppose that an indecomposable root system Φ has a subsystem Δ of type D_4 which is closed under sums (i.e. $(\Delta + \Delta) \cap \Phi \subseteq \Delta$). Then Φ has type D_n , $n \geq 4$, B_n , $n \geq 5$, F_4 or E_n , $n \in \{6, 7, 8\}$.*

Proof. One just eliminates types A and C .

3.9 Proposition. (i) *Let H be a natural D_4 subgroup of E . If A is a connected, reductive subgroup of E containing H , then A has a quasisimple central factor, Y , which contains H as a subgroup generated by a subset of standard set of root groups and which has type B_n , $n \geq 4$, D_n , $n \geq 4$, F_4 or E_n , $n \in \{6, 7, 8\}$.*

(ii) *The set of Y as in (i) of a given type form an orbit under $C(H)$.*

Proof. (i) (3.4) and (3.8).

(ii) (3.7) and (3.4.iv).

3.10 Lemma. (i) *Let L be a natural G_2 -subgroup of F . Then, $C_F(L) \cong \text{PSL}(2, \mathbf{C})$.*

(ii) *There is one conjugacy class of subgroups of F isomorphic to $2^3 \cdot \text{GL}(3, 2)$. The centralizer of one of these is conjugate to $C_F(L)$ as in (i).*

(iii) *There is one conjugacy class of subgroups in F isomorphic to G_2 , whence there is in E one conjugacy class of subgroups of the form $L_1 \times L_2$, where $L_1 \cong L_2$ and one of the L_i is a conjugate of G .*

(iv) *If L is as in (i) and $K := C_E(G \times L) = C_F(L)$, then $C_E(K) = [G \times L]\langle p \rangle$, where p is an involution interchanging G and L under conjugation.*

Proof. (i) By definition, $L \leq J$, a natural type D_4 -subgroup, so $J \cong \text{Spin}(8, \mathbf{C})$. If z is an involution of $Z(J) \cong 2^2$, z is in the E -class $2B$ and so we may assume that $L \leq J \leq D$ (3.1). Let G be as in (3.1). The action of $L \times G$ on V_{16} has decomposition into irreducibles of the shape $1 \otimes 7 + 7 \otimes 1 + 1 \otimes 1 + 1 \otimes 1$ and summands may be chosen to be nonsingular. Therefore, there is a 1-torus in $C_F(L)$, so this reductive group is positive dimensional. We have $C(L) \cap N_F(J) \cong \text{Alt}_4$. Since $Z(J)$ is selfcentralizing in $C_F(L)$, we conclude that $C_F(L) \cong \text{PSL}(2, \mathbf{C})$.

(ii) Let $M \leq F$, $M \cong 2^3 \cdot \text{GL}(3, 2)$. Then, all involutions of $O_2(M)$ are in the same F -class. By [Gr], this class must be $2A$ and $O_2(M)$ is nontoral. There is a unique class of nontoral maximal elementary abelian 2-groups in F and if R is one such containing $O_2(M)$, $R \cong 2^5$ and all three elements of $R \cap 2B$ lie in a four-group P in R . All complements to P in R are conjugate in $N_F(R)$. We conclude that $C_F(O_2(M)) \cap N_F(R) = O_2(M) \times Z$, where $P = O_2(Z)$ and $Z \cong \Sigma_4$. From here, it is not hard to show, arguing as in (i), that $C_F(O_2(M))/O_2(M) \cong \text{PSL}(2, \mathbf{C})$ since P maps to a selfcentralizing fours group in the quotient. Once we notice that M acts trivially on this copy of $\text{PSL}(2, \mathbf{C})$, we get (ii).

(iii) Let H be a subgroup of F isomorphic to $G_2(\mathbf{C})$. Let R be an eights group in H ; $N_H(R) \cong 2^3 \cdot \text{GL}(3, 2)$. We have $N_F(R) = S \times N_H(R)$, where $S \cong \text{PSL}(2, \mathbf{C})$, by (ii). This means that the action of $S \times N_H(R)$ on the 26-dimensional irreducible module for F has the irreducible decomposition $3 \otimes 7 + X \otimes 1 + 1 \otimes 1 + 1 \otimes 1$, where X is a 3-dimensional irreducible or a 3-dimensional trivial module. We now claim that W , the sum of the three 7-dimensional modules for $N_H(R)$, is a 21-dimensional irreducible for H . There are just four H -irreducibles of dimensions at most 26, namely the modules of dimensions 1, 7, 14 and 21 (use the dimension formula, p. 140 [Hum]). The claim follows and so W^\perp , the orthogonal complement of W , which is the 5-dimensional space of fixed points for $N_H(R)$, is also an H -submodule. In this 5-space, we take the 2-dimensional fixed point subspace, W_0 of a fours group in S ; since the stabilizer of W_0 in F is a natural subgroup of type D_4 , extended by graph automorphisms, we have an embedding of H into a natural

D_4 -subgroup; Since $H \times S$ does not embed, X is a 3-dimensional irreducible. Thus, H is natural, by highest weight theory and (2.3).

(iv) Let U be a fours group in K . Then $N_K(U) \cong \text{Sym}_4$ and U is $2B$ -pure (this follows from [Gr](7.3)). Thus, $G \times L$ is in the type D_4^2 -subgroup $N_E(U)$ where we can see the involution p in $C_E(N_K(U)) \cong G_2(\mathbf{C}) \text{ wr } 2$. Since p acts on $K \cong \text{PSL}(2, \mathbf{C})$ and centralizes $N_K(U)$, it centralizes K .

3.11 Lemma. *Let A be a reductive subgroup of E which contains G .*

(i) *Then, G is contained in a central quasisimple factor Y of A and either (a) Y is a natural subgroup of type G_2 or B_3 ; or (b) G is contained in a natural subgroup X of type D_4 in Y ; also Y has type as in (3.9.i).*

(ii) *The set of such Y in (i) of a given type is an orbit under $C(G)$ and so is the set of X in (b).*

Proof. (i) Use (3.4). We may suppose that the subsystem of long roots of A has type among ADE and rank $n \geq 3$. Since a type A_2 subgroup of G is normalized by an element of G inducing its graph automorphism, we eliminate type A for $n \geq 4$. If the type is $A_3 = D_3$, $G \leq Y$ implies that Y has type B_3 .

Now suppose that $n \geq 4$. Then (3.8) implies that Y has type B, D, F or E . As in the proof of (3.4), we study the containment of Cartan subalgebras $\mathfrak{u} \leq \mathfrak{t}$ for $U \leq T$, maximal tori of G and Y , respectively, and the action of W_G on \mathfrak{t} , the Lie algebra for T . We have $\dim[\mathfrak{t}, W_G] \in \{2, 3, 4\}$. If 2 or 3, we use (3.3.i) to deduce that Y has two root lengths hence Y must have type B_n for $n \geq 4$ or F_4 and $\dim[\mathfrak{t}, W_G] = 3$. Thus, the intersection of the root system for T with $[\mathfrak{t}, W_G]$ is a root system of type B_3 ; but then from (3.7) we deduce an embedding of G in a natural subgroup of type D_4 in Y , contradiction. We conclude that $\dim[\mathfrak{t}, W_G] = 4$ and that G lies in some subgroup Y of type B_4, D_4 or F_4 . If Y has type B_4 or D_4 , the conclusion follows from weight theory since the only dimensions for a G -irreducible constituent on the standard module for Y are 1 and 7. If Y has type F_4 , use (3.10.iii).

(ii) The conjugacy statement is proved as follows. Weight theory for G tells us that there is just one nontrivial irreducible of degree at most 8, namely the 7-dimensional one. Any two embeddings of G in a group of type B_3 or D_4 are conjugate, by (2.4). Now use (3.7) and the fact that in $C_E(U)$, where U is a maximal torus of G , all maximal tori are conjugate (3.4).

4. Proof of Theorem 1

(4.1) **Notation.** We need to compare two occurrences of $G_2(\mathbf{C})$, namely the group G in (3.1) and the subgroup $J \cong G_2(\mathbf{C})$ of $O(7, \mathbf{C})$, as in the hypothesis of Theorem 1. Consider V to be an orthogonal direct summand of V_{16} (3.1). The isometry of V into V_{16} gives an embedding of $J \leq \text{SO}(7, \mathbf{C})$ into $\text{SO}(16, \mathbf{C})$. Since J is simply connected, we deduce from (3.1) well defined

embeddings of J into $\text{Spin}(16, \mathbf{C})$ and into $H \text{Spin}(16, \mathbf{C}) \cong D = C(z) \leq E$; in the notation of (3.1), we take G as the image in the latter group.

(4.2) We assume that G has an embedding in $GL(V)$ such that conjugation by an element of $GL(V)$ gives a bijection b of S to S^* . Then, since any two embeddings of G in $GL(V)$ are equivalent, by highest weight theory, b is realized by such a conjugation for any embedding of G in $GL(V)$.

(4.3) Define $Z := \langle z \rangle$. Suppose that $p \in GL(V)$ conjugates S to S^* . We now use (4.2), (2.4) and (2.5.ii) to get $g \in C(B)$ such that $s^g \equiv s^p \pmod{Z}$, for all $s \in S$. We have $(SZ)^g = S^*Z$; we might guess that $S^g = S^*$, but this is so only for certain g . To prove that p factors suitably, we prove that g does.

We know (9.8) that the intersection $Y := G \cap G^g$ is one of: $G; P'$, where P is a parabolic subgroup of G ; or L° , where L is a natural $SL(3, \mathbf{C})$: 2-subgroup of G . For all such Y , there is one G -conjugacy class of algebraic subgroups isomorphic to Y . Since $Y^{g^{-1}} \leq G$, there is $u \in G$ so that $Y^{g^{-1}u} = Y$.

For all these groups, Y , the natural map of $N_G(Y)$ to $\text{Aut}(Y)$ is onto (9.8.ii). So, there are $y \in N_G(Y)$ and $c \in C_{C(B)}(Y)$ so that $g^{-1}u = cy$, i.e., $g = uy^{-1}c^{-1}$. Since $yu^{-1} \in G$, we have a factorization of the required kind if $c \in C(S^*)$, but this may not be the case. We have $S^* \leq Y^* := G \cap ZG^g$. We assume that $c \notin C(S^*)$, whence $c \notin C(Y^*)$ and $Y < Y^*$. Then, as Y^* maps isomorphically into $C(B)/Z \cong SO(7, \mathbf{C})$ and the image is the intersection of two $G_2(\mathbf{C})$ -subgroups of $C(B)/Z$ (since $Y^*Z = ZG \cap ZG^g$), we see from the list (9.12) that Y^* is a natural $SL(3, \mathbf{C})$: 2-subgroup of G and $Y = (Y^*)^\circ$. We conclude that $a^c = a$ if $a \in Y$ and $a^c = az$ if $a \in Y^* \setminus Y$. This means that the image of c in $SO(7, \mathbf{C})$ is an involution of the form $\text{diag}(1, -1, -1, -1, -1, -1, -1)$, fixing elementwise the fixed points on V of Y . Although c does not centralize S^* , it does centralize S^* modulo Z . So, taking images of $g = (uy^{-1})c^{-1}$ in $C(B)/Z$, we have the required factorization of the conjugating element. Note that replacing g by gc , we retain the condition $s^g \equiv s^p \pmod{Z}$ and moreover have $G \cap G^g \cong SL(3, \mathbf{C}) : 2$ and $S^g = S^*$.

(4.4) *Remark.* In step (4.3), if the element g satisfies $S^g = S^*$, we have a different proof of a suitable factorization. Let $C := C_E(S)$ and $C^* := C_E(S^*)$. Both algebraic groups C and C^* contain F ; in fact F is contained in a Levi factor L of C and in a Levi factor L^* of C^* . Each of L° , $(L^*)^\circ$ contains F in a unique central factor of type F_4 or E_n , by (3.3). By conjugacy of Levi factors in $C^* = C^g$ and (3.5.iv), there is $h \in C(S^*)$ so that $F^{gh} = F$. So, $gh \in N_E(F) = G \times F$ and there are $x \in G$ and $y \in F$ such that $gh = xy = yx$ and $g = x(yh^{-1})$, as required. This kind of argument recurs in Section 7.

5. Proof of Theorem 2

This section is independent of the proof of Theorem 1. We use the notation of (4.1) here.

(5.1) We assume that S is a finite subgroup of J not in a local subgroup and which is reducible on V , the 7-dimensional module.

(5.2) We now shift to K , the compact form of J . Any compact subgroup of J is conjugate to a subgroup of K . On the associated real form of V , K operates transitively on 1-spaces and the stabilizer in K of a 1-space is $H \cong SU(3, \mathbf{R}):2$. We deduce that if S has a 1-dimensional irreducible, it is in the standard A_2 -subgroup of J containing H ; it is isomorphic to $SL(3, \mathbf{C}):2$, a local subgroup, contradiction. An alternate argument: (avoiding compactness) uses (9.8) to get a finite subgroup in a natural $SL(3, \mathbf{C}):2$ or in the derived group of a parabolic associated to the long root of the Dynkin diagram; but then a finite subgroup is conjugate to a subgroup of the Levi factor, which is in a natural $SL(3, \mathbf{C}):2$ -subgroup.

(5.3) We now assume that no irreducible submodule is 1-dimensional. Since S is assumed to be not in a local subgroup, the socle of S (i.e., the product of the minimal normal subgroups) is a direct product of nonabelian simple groups; denote the socle by X .

(5.4) We claim that any irreducible constituent of S has dimension at least 3. Suppose otherwise. Then, there is a 2-dimensional submodule, say U , and X acts on U trivially since $GL(2, \mathbf{C})$ has no finite simple subgroups. After conjugation, if necessary, we deduce that X is contained in a natural $SL(3, \mathbf{C})$ as above. By Blichfeldt's Theorem (see Section 8), X is isomorphic to one of Alt_5 , $GL(3, 2)$, whose outer automorphism groups are cyclic. Since S embeds in $\text{Aut}(X)$, S/X is cyclic, whence S fixes a 1-space in U , contradiction.

(5.5) We now take S -irreducible submodules U and W , of dimensions 3 and 4, respectively. Both subspaces are nonsingular. Define t_0 to be the involution on V which is 1 on U and -1 on W . We extend t_0 to an orthogonal transformation on V_{16} via trivial action on the orthogonal complement of V .

(5.6) Let t be one of the two elements in E corresponding to t_0 in the sense of (3.1); then $|t| = 2$ [Gr], (2.8.b) and $Z := \langle t, z \rangle$ is a four group with distribution AAB [Gr](1.4)(2.14). We shall prove that t or tz is in G .

Since $\langle S, Z \rangle = Z \times S$ and $C(S)$ contains F , whence by (3.6), $C(S)^\circ$ has a component Y which is a natural subgroup of type F_4 or E_n . Since $(C(S) \cap C(Z))^\circ = B$, the only possibility is that $Y = F$ (reason: otherwise, by (3.6.i), Y contains a natural E_6 subgroup, whose centralizer is a natural A_2 subgroup of G ; but then S is in a local subgroup, contradiction).

We know that $C_F(z)$ is contained in $C_D(S)'$, a group of type B_4 . The structures of involution centralizers in F (types B_4 and A_1C_3 [Gr](2.14); the latter is not in a group of type B_4) imply that $C_F(z) = C_D(S)'$ is a group of type B_4 and that there is no fours group with centralizer of type B_4 . Consequently, z induces on F an inner automorphism of order 2. The definition of t_0 makes it clear that $[t, C_F(z)] = 1$. Therefore, the image of Z in $\text{Aut}(F)$ has order 2 and the kernel of the action has order 2. Replacing t with tz if necessary, we may assume that $t \in G = C(F)$. We now have that $S \times \langle t \rangle$ is in $C_G(t) \cong SL(2, \mathbf{C}) \circ SL(2, \mathbf{C})$, a local subgroup, contradiction. Theorem 2 follows.

6. Proof of Corollary 2

We continue to use the notations of (4.1).

6.1 Proposition. *Let R be a reductive subgroup of $B^* := C_E(B)^\circ \cong \text{Spin}(7, \mathbb{C})$.*

(i) *R is contained in a B^* -conjugate of G if and only if $C_E(R)^\circ$ contains a conjugate of F by B^* .*

(ii) *If R acts irreducibly on V , the only possible fixed point subalgebras for its action on the Lie algebra of type E_8 are of type B_4 and F_4 (of respective dimensions 36 and 52); R is in a G_2 subgroup of $O(7, \mathbb{C})$ if and only if the fixed point subalgebra has type dimension 52.*

Proof. (i) Trivial. (ii) We claim that, if R is irreducible on $V \cong \mathbb{C}^7$, then $C_E(R)$ is a natural B_4 or F_4 . If not, it contains a natural D_5 (3.8) and so its centralizer is contained in a natural D_3 subgroup of B^* ; but then R fixes a 1-space in V , contradiction. Thus, (ii) follows.

6.2 Remarks. (i) For an irreducible finite subgroup, if we get the traces of its elements on the adjoint module for E_8 and compute the inner product with the trivial character, (6.1.ii) implies that the only possible multiplicities are 36 or 52, which correspond to fixed point subalgebra of type B_4 or F_4 respectively. We get R in a G_2 subgroup if and only if the multiplicity is 52. Note that since all involutions of G are of type $2A$, $z \notin G$.

(ii) The spectrum of a semisimple element of D on the adjoint module of E is obtained straightforwardly by extending the relevant character on a D_8 lattice to an E_8 overlattice. Therefore, if R is a finite subgroup of D , the centralizer condition in (i) is checked mechanically by taking the inner product of the trivial character of R with the restriction to R of the adjoint character of E .

(iii) It would be interesting to find a simple criterion, *internal* to $GL(7, \mathbb{C})$ or $O(7, \mathbb{C})$, to decide containment of a subgroup in a conjugate of $G_2(\mathbb{C})$.

6.3. The calculations The discussion of Section 5 shows that to treat the reducible case, it suffices to survey finite subgroups of $SL(3, \mathbb{C}) : 2$ and $SL(2, \mathbb{C}) \circ SL(2, \mathbb{C})$, which can be obtained from well-known results; see Section 8. So, we go over the groups in Wales's list of irreducible finite groups, given in the Section 8. In [CoWa], the authors could have quoted [Wa] but instead do analysis for 7-dimensional finite linear groups in the more restricted case that they lie in a $G_2(\mathbb{C})$ -subgroup. Our method for settling existence and uniqueness up to conjugacy for the larger list of candidates from [Wa] is so short that we present it in full.

Uniqueness of an embedding of finite subgroup up to conjugacy follows from applying Theorem 1 and the fact that, in all cases on Wales' list, the set of irreducible degree 7 characters of the finite group R which give embeddings form an orbit under $\text{Aut}(R)$. Of course, Theorem 1 implies strong control of fusion in the case of two embeddings with the same characters.

6.3.1 Nonembedding results. An irreducible subgroup is, modulo scalars, a group X as in the Section 8. Since $G_2(\mathbb{C})$ preserves an algebra structure on the 7 dimensional module, the only scalar transformation it may contain is the identity.

The 2-rank of $G_2(\mathbb{C})$ is 3 and the p -rank is 2 if p is an odd prime; also, any finite p -subgroup is abelian for $p \geq 5$ [Gr].

For odd p , elementary abelian groups in $G_2(\mathbb{C})$ are toral and this means that p -local subgroups are reducible on V ; therefore, no irreducible, imprimitive subgroup of $GL(V)$ with an abelian normal p -subgroup is in $G_2(\mathbb{C})$. Consequently, all imprimitive groups on Wales's list are eliminated except for $PSL(2, 7)$ and $2^3 \cdot GL(3, 2)$.

We eliminate $X \cong \text{Alt}_8$, Sym_8 and $\text{Sp}(6, 2)$ by 2-rank (theirs are trivially seen to be at least 4, 4 and 6, respectively) and all groups containing the non-abelian group 7^{1+2} .

6.3.2 Embedding results. The remaining nonsolvable groups on Wales's list embed in a way unique up to conjugation. Existence of embeddings may be verified by a sequence of inner product computations to execute the procedure (6.2.i, ii). This was done for the determinant 1 irreducible 7-dimensional representations of $PGL(2, 7)$, $PSL(2, 13)$ and $G_2(2) \cong PSU(3, 3) : 2$. For $SL(2, 8)$, there are four degree 7 irreducibles; the rational one does not lead to an embedding in $G_2(\mathbb{C})$ but any irrational one does (the three such form an orbit under $\text{Aut}(SL(2, 8)) \cong SL(2, 8) : 3$).

Verifications are straightforward except that when we determine the pairs of elements (more precisely, of spectra of elements) in E corresponding to the elements of our finite subgroup (see (3.1) and (6.2)), it was not obvious in certain cases which spectrum of the pair was the right one for an element of G . We now discuss these cases.

We note the two pairs of elements in E corresponding to the two classes of outer elements of order 8 from $PGL(2, 7)$; each pair consists of elements of E of order 8 with two different traces, namely 78 and 14. The respective inner products with the trivial character are 36 and 52. If Y is the preimage in E of this linear group, $Y \cong 2 \times PGL(2, 7)$ (since the outer involutions lift to involutions). Conjugating if necessary, we arrange for Y' to be in $G = C(F)$. The argument of (5.6) may be used here to see that Y induces a group of order 2 on F (generated by conjugation with z) and that $Y \cap G \cong PGL(2, 7)$. (This argument is repeated below for $G_2(2)$; we solve the $PGL(2, 7)$ problem by quoting the $G_2(2)$ result and the fact that $PGL(2, 7) \leq G_2(2)$).

In the case of $G_2(2)'$, there are characters of degrees 6 and 7 but only the real character of degree 7 is possible here. At five classes, we have the above ambiguity. Use of the power maps settles the choices, starting with classes of elements of order 8, and we are led to dimension 52.

This degree 7 character of $G_2(2)'$ extends to a determinant 1 character for $G_2(2)$. Recall that the Schur multiplier of $G_2(2)'$ is 1 [Gr 72] but that the nonperfect group $G_2(2)$ does have nonsplit central extensions. The spectrum of outer elements of order 2 in $G_2(2)$ is $\{1^3, -1^4\}$, so the corresponding group in

$\text{Spin}(7, \mathbb{C})$ is $G_2(2) \times 2$. Conjugating if necessary, we arrange for $X \cong G_2(2)'$ to be in $G = C(F)$. Then our above $G_2(2)$ subgroup of $SO(7, \mathbb{C})$ gives us a group $Y > X$ of B^* such that $Y \cong G_2(2) \times 2$. The argument of (5.6) may be used here to see that Y induces a group of order 2 on F (generated by conjugation with z) and that $Y \cap G \cong G_2(2)$.

The imprimitive cases in Wales's list are easy to handle. The $PSL(2, 7)$ subgroup is just the commutator subgroup of the above $PGL(2, 7)$ and the $2^3 \cdot GL(3, 2)$ subgroup is the normalizer of an elementary abelian group of order 8; this subgroup is well-known and is discussed, for instance, in [Gr] (1.8) Table II.

We now summarize the results.

Table 1. $E_8(\mathbb{C})$ -Spectra for elements of finite primitive subgroups of $GL(7, \mathbb{C})$ which are in $G_2(\mathbb{C})$

Notation. A *spectrum sequence* gives the respective multiplicities of $\zeta^0, \zeta^1, \dots, \zeta^{n-1}$ of an group element of order n in some representation of the group. For an element of order n in G , we list its trace and (r, s, \dots) , its spectrum sequence on the 7 dimensional module; below it, we give $[a, b, \dots]$, its spectrum on the E_8 adjoint module ($G < E$ as in (3.1)); The enriched character tables and notations for algebraic integers and conjugation come from [Atlas]; = is a horizontal ditto mark.

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$$PGL(2, 7) \cong GL(3, 2) : 2.$$

1A	2A	3A	4A	7A	7B
7	-1	1	-1	0	0
(7)	(3, 4)	(3, 2, 2)	(1, 2, 2, 2)	(1, 1, 1, 1, 1, 1, 1)	(=)
[248]	[136, 112]	[134, 57, 57]	[82, 56, 54, 56]	[80, 28, 28, 28, 28, 28, 28]	[=]
	2B	6A	8A	8B	
	-1	-1	1	1	
	(3, 4)	(1, 1, 1, 2, 1, 1)	(1, 1, 1, 1, 0, 1, 1, 1)	(=)	
	[136, 112]	[80, 29, 28, 54, 28, 29]	[80, 28, 27, 28, 2, 28, 27, 28]	[=]	

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$$PSL(2, 13).$$

1A	2A	3A	6A	7A
7	-1	1	-1	0
(7)	(3, 4)	(3, 2, 2)	(1, 1, 1, 2, 1, 1)	(1, 1, 1, 1, 1, 1, 1)
[248]	[136, 112]	[134, 57, 57]	[80, 29, 28, 54, 28, 29]	[80, 28, 28, 28, 28, 28, 28]

$7B$	$7C$	$13A$	$13B$
0	0	$-b13$	*
(=)	(=)	(0, 0, 1, 3, 4, 9, 10, 12)	(*)
[=]	[=]	[80, 27, 1, 27, 27, 1, 1, 1, 1, 27, 27, 1, 27]	[*]

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$SL(2, 8)$.

$1A$	$2A$	$3A$	$7A$	$7B$	$7C$
7	(-1)	1	0	0	0
(7)	(3, 4)	(3, 2, 2)	(1, 1, 1, 1, 1, 1, 1)	(=)	(=)
[248]	[136, 112]	[134, 57, 57]	[80, 28, 28, 28, 28, 28, 28]	[=]	[=]

$9A$	$9B$	$9C$
$-y^9$	*	**
(1, 1, 1, 0, 1, 1, 0, 1, 1)	(=)	(=)
[80, 2, 28, 27, 27, 27, 27, 28, 2]	[*]	[**]

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$G_2(2)'$.

$1A$	$2A$	$3A$	$3B$	$4A$	$4B$
7	-1	-2	1	3	3
(7)	(3, 4)	(1, 3, 3)	(3, 2, 2)	(3, 2, 0, 2)	(=)
[248]	[136, 112]	[86, 81, 81]	[134, 57, 57]	[134, 56, 2, 56]	[=]

$4C$	$6A$	$7A$	$7B$
-1	0	0	-1
(1, 2, 2, 2)	(1, 2, 1, 0, 1, 2)	(1, 1, 1, 1, 1, 1, 1)	(=)
[82, 56, 54, 56]	[82, 54, 27, 4, 27, 54]	[80, 28, 28, 28, 28, 28, 28]	[=]

$8A$	$8B$
-1	2
(1, 1, 0, 1, 2, 1, 0, 1)	(1, 1, 0, 1, 2, 1, 0, 1)
[80, 28, 1, 28, 54, 28, 1, 28]	[80, 28, 1, 28, 54, 28, 1, 28]

$12A$	$12B$
0	0
(1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1)	(1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1)
[80, 27, 0, 2, 27, 27, 2, 27, 27, 2, 0, 27]	[80, 27, 0, 2, 27, 27, 2, 27, 27, 2, 0, 27]

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7. Proof of Theorem 3

Suppose that S and S^* are subsets of F which are conjugate by $g \in H$, where H is a natural $3E_6(\mathbf{C})$ -subgroup containing F . Now imitate the argument in (4.4), reversing the roles of G and F , and using (3.11). One step must be modified; in (4.4), there is a unique central factor of the Levi factor containing a conjugate of F , whereas in this case, possibly $C^* := C_E(S^*)$ contains $G \times \hat{G}$, a direct product of two E -conjugates of G (three factors is not possible (3.10)). To carry out the argument, we need two such direct products to be conjugate in C^* . Let $J := C_E(G \times \hat{G})$, an adjoint group of type A_1 . Then, J contains S and S^* and $C_E(J)$ is a wreath product of G with a cyclic group of order 2 (3.10.iv). The desired conjugacy follows.

Notice that the situation treated in (4.3) does not arise here.

8. Appendix: Assumed classifications

First, we recall two old results of H.F. Blichfeldt [B]:

The classification of finite subgroups of $SL(2, \mathbf{C})$. The conjugacy classes consist of two infinite families (cyclic groups and generalized quaternion groups) and the three finite groups $SL(2, 3)$, $SL(2, 3) \cdot 2 \cong 2 \cdot \text{Sym}_4$ and $SL(2, 5)$.

The classification of finite irreducible subgroups of $SL(3, \mathbf{C})$. Such a finite group is solvable or is conjugate to one of SX , where S is a group of scalars and X is isomorphic to one of Alt_5 , $3 \cdot \text{Alt}_6$, $PSL(2, 7)$.

Secondly, we recall the theorem of David Wales [Wa]:

The classification of finite primitive subgroups of $GL(7, \mathbf{C})$. The conjugacy classes of primitive linear groups are represented, modulo scalars, by one of:

- (1) a subgroup of $7^{1+2} : SL(2, 7)$ containing $O_7(7^{1+2} : SL(2, 7))$.
- (2) $PSL(2, 13)$, $PSL(2, 8)$, $PSL(2, 8) : 3$, Alt_8 , Sym_8 ,
- (3) $PGL(2, 7)$, $PSU(3, 3)$, $PU(3, 3) \cong G_2(2)$, $\text{Sp}(6, 2)$. There are additional irreducible subgroups, namely the imprimitive groups:
- (4) $PSL(2, 7)$; any finite irreducible subgroup of $GL(7, \mathbf{C})$ with a non-central normal abelian subgroup (this family includes the nonsplit extension $2^3 \cdot GL(3, 2)$).

For additional background on linear groups, see the summary in [F].

9. Appendix: Cosets of $G_2(\mathbf{C})$ in $SO(7, \mathbf{C})$ and $\text{Spin}(7, \mathbf{C})$

In this appendix, we prove transitivity of the automorphism group of split Cayley algebras on each sphere in the trace 0 part and we deduce the double coset structure of such an automorphism group in the associated orthogonal and spin groups.

Transitivity of $\text{Aut}(\mathbf{O})$ on vectors of a given norm in V

9.1 Notation. (See [J].) Let K be a field of characteristic not 2 and \mathbf{O} a Cayley (octonian) algebra over K . Let V be the subspace of trace 0 vectors (this is the orthogonal complement of $1 \in \mathbf{O}$; see (9.2) for further notation on \mathbf{O}).

We prove that if $G = \text{Aut}(\mathbf{O})$ is transitive on all nonempty sets of the form $\{x \in V | (x, x) = a\}$, for $a \in K$, provided V has singular vectors. We prove this by using results on Cayley algebras from [J]. Two Cayley algebras which are not division algebras are isomorphic and have useful degree 2 matrix subalgebras; this is the situation if a Cayley algebra has nontrivial singular vectors of trace 0.

9.2 Notation. (See [J].) $x \mapsto \bar{x}$, Cayley conjugation; N , the norm; tr , the trace; they are related by the formulas $N(x) = x\bar{x} = \bar{x}x$, $\text{tr}(x) = x + \bar{x}$, $x^2 - \text{tr}(x)x + N(x) = 0$ and have the properties $N(xy) = N(x)N(y)$, $\text{tr}(xy) = \text{tr}(yx)$ and $(x, y) := \frac{1}{2}(x\bar{y} + y\bar{x})$ is a symmetric, nondegenerate bilinear form; $\text{tr}(xy) = -\text{tr}(x\bar{y}) = -\text{tr}(\bar{x}y)$ if $x, y \in V$ and $x^2 = 0$ if $x \in V$ is singular.

9.3 Lemma. *Suppose that the trace 0 element $x \neq 0$ is singular. Choose $y \in V$ so that $(x, y) = -1$ and, furthermore, that y is singular. Then, the subalgebra generated by x and y is a degree 2 matrix algebra. Furthermore, the correspondence*

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad xy \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad yx \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

extends to an isomorphism of algebras.

Proof. For any element $p \in \mathbf{O}$, the subalgebra generated by p is at most 2 dimensional (use the rule $p^2 - \text{tr}(p)p + N(p) = 0$). Now, use the fact that any 2-generator subalgebra is associative and the formula $xy + yx = 1$ to get that the algebras, A , generated by x and y is spanned by $1, x, y, xy$ and so has dimension at most 4. Since clearly $\dim A \geq 3$, we need to see that $\dim A > 3$.

We suppose that $\dim A = 3$ and seek a contradiction. Since the commutator $c := xy - yx$ has trace 0, $c \in \text{span}\{x, y\}$. By associativity of the symmetric form, $(x, xy) = (x, yx) = (y, xy) = (y, yx) = 0$ and so $(x, y) = -1$ implies that $c = 0$ and xy is a multiple of 1, say $xy = yx = \lambda$. We have $\lambda^2 = (xy, xy) = (x, y(xy)) = (x, y(yx)) = (x, y^2x) = (x, 0) = 0$, whence $xy = yx = 0$ and $1 = xy + yx = 0$, contradiction.

Thus, $\dim A = 4$ and $\{x, y, xy, yx\}$ is a basis for A . Since $xy + yx = 1$, it is straightforward to verify that A is isomorphic to the algebra of degree 2 matrices over K via the stated correspondence.

9.4 Notation. Let $a \in K$, $S(a) := \{0 \neq x \in V | Nx = a\}$.

9.5 Lemma. *If V has singular vectors, they span V .*

Proof. Let $S := \text{span } S(0)$ and assume that $0 \neq S \neq V$. Let $x \in S(0)$. Since $V \setminus S$ spans V , there is $u \in V \setminus S$ so that $(x, u) = 1$. Then, $y := \frac{(u, u)}{2}x - u$ is singular and in $V \setminus S$, contradiction.

9.6 Proposition. *If V has singular vectors and $S(a) \neq \emptyset$, then G acts transitively on $S(a)$.*

Proof. Given $u \in S(a)$, let y be a singular vector satisfying $(u, y) = 1$. Set $x := \frac{a}{2}y - u$, a singular vector; $(x, y) = 1$. Both $\{u, y\}$ and $\{x, y\}$ generate the same four dimensional subalgebra, isomorphic to degree 2 matrices over K . Now, let u' be another element of $S(a)$; choose y' and define x' analogously. There is an automorphism of \mathbf{O} which carries x, y to x', y' , respectively ([J], Theorem 3) and it must carry u to u' since it preserves the inner product.

9.7 Proposition. *Let H be a connected reductive algebraic group and M a finite dimensional irreducible module. Let U be a maximal unipotent subgroup of H . Then, $\dim C_M(U) = 1$. If there is an invariant symmetric bilinear form on M , $C_M(U)$ is singular.*

Proof. Since M is a highest weight module, the first statement follows from the fact that the highest weight space is 1-dimensional. The second statement follows since the stabilizer of a nonsingular subspace in the orthogonal group preserves a complement, whereas this is not the case for the (upper triangular) action of U on M .

9.8 Corollary. (i) *If K is algebraically closed, V has singular vectors and G acts transitively on $S(a)$, for all $a \in K$. Therefore, G has one orbit on nonsingular 1-spaces, and the stabilizer is a group of the form $SL(3, K) : 2$; the stabilizer of a nonsingular vector is a natural $SL(3, K)$ -subgroup. The stabilizer of a singular 1-space is a parabolic subgroup, P whose Levi factor has type T_1A_1 , with semisimple part a fundamental $SL(2, K)$ associated to a long root. The subgroup of it stabilizing a nontrivial vector in that 1-space is P' .*

(ii) *If Y is a stabilizer, every automorphism of Y or Y° (as an algebraic group) is induced by an element of $N_G(Y)$.*

Proof. (i) All is clear except possibly for the statement about the stabilizer of a singular 1-space. Consider a natural A_1^2 subgroup of G , say LM , where L, M are fundamental $SL(2, K)$ -subgroups associated to orthogonal long and short root, respectively. This subgroup is $C(z)$, for some involution z , and on V , z has spectrum $\{1^3, -1^4\}$. On the -1 -eigenspace, LM acts faithfully as $SO(4, K)$ and on the $+1$ -eigenspace, L acts trivially and M acts as $SO(3, K)$. Since L fixes pointwise the invariant 1-space in the $+1$ -eigenspace, its stabilizer, R , contains a maximal unipotent ((9.6), (9.7)) and L so is contained in a maximal parabolic, P and contains P' . Since we can see the action of a maximal torus in

G as a subgroup of a natural $SL(3, K)$ acting dually on a pair of 3-dimensional isotropic subspaces, we conclude that $R = P'$.

(ii) For an $SL(3, \mathbb{C})$: 2-subgroup, this is obvious. Suppose that $Y = P'$. Then, $Y = Y^\circ$ has the structure: the unipotent radical $R(Y)$ has nilpotence class 3 with descending central factors of shapes 2, 1, 2 where 2 and 1 denote the irreducible for the Levi factor $L \cong SL(2, K)$ of dimension 2 and 1, respectively. If a is an automorphism, we may assume, by conjugacy of Levi factors and the fact that $\text{Aut}(L) = \text{Inn}(L)$, that a centralizes L and so is scalar on the top factor of the descending central series. Since there is an element of the 1-torus $C_P(L)$ acting the same way on the top factor, we may assume that a is trivial on the top factor and so $[R(Y), a]$ is 1 or is the center of $R(Y)$ (because $[a, L] = 1$). Let $U := C_{R(Y)}(L)$, a root group for a short root perpendicular to the long root associated to a root group of L . By Schur's Lemma, there is just a 1-dimensional space of L -invariant homomorphisms from the top factor to the bottom factor, so we may replace a by ab , where b is conjugation by an element of U , to get $[R(Y), a] = 1$. Since $[Y, a] = 1$, we are done.

Double cosets of $G_2(\mathbb{C})$ in $SO(7, \mathbb{C})$ and $\text{Spin}(7, \mathbb{C})$

The following procedure offers an interpretation of the spaces of cosets and double cosets of G in $H \leftarrow SO(7, \mathbb{C})$ and $S := \text{Spin}(7, \mathbb{C})$. We thank J.-P. Serre for describing it. We take the field to be the complex numbers, \mathbb{C} ; the results (9.6) and (9.8) apply.

9.9 Notation. We consider V the subspace 1^\perp in the Cayley numbers, or octonians; we have $\mathbf{O} = \mathbb{C} \cdot 1 \oplus V = \{(p, q) | p \in \mathbb{C} \cdot 1, q \in V\}$. Let X be the set of unit vectors in \mathbf{O} , where we use the usual octonion norm; so $(p, q) \in X$ if and only if $p^2 + Nq = 1$. We take the natural action of H on \mathbf{O} and get an action of S by lifting H to $\text{Spin}(8, \mathbb{C})$, then applying triality, θ . Since θ centralizes G , we may view S as an overgroup of G in $SO(8, \mathbb{C})$. The orbits of G on X are distinguished by their inner products with $(1, 0)$, i.e., by the first coordinate of $(p, q) \in X$, when $p \neq \pm 1$; when $p = \pm 1$, there are two orbits.

9.10 Lemma. S acts transitively on X , with point stabilizer G .

Proof. The stabilizer of $(1, 0) \in X$ in G since G stabilizes 1 and is a maximal algebraic subgroup of H . Let J be the stabilizer in S of a point $x = (u, v) \in X$ (9.8). Assume that $u \neq \pm 1$; then v is nonsingular and $J \neq G$. Note that $J \cap G \cong SL(3, \mathbb{C})$.

We want to prove that $J \cong G$. Since $\dim X = 1$ and $\dim S = 21$, we have $\dim J \geq 21 - 7 = 14$, so $S > J > J \cap G$. Let L be a Levi factor of J containing $J \cap G$. Since on a half spin module for S , the weight 0 does not occur, $\text{rank } L = 2$ and so $L^\circ = J \cap G$ or $L = L^\circ = J$ has type G_2 . Assume the former. Thus, the unipotent radical $R(J)$ of J has dimension at least $14 - 8 = 6$. It follows that J is Q' , where Q is the parabolic subgroup of S with a Levi

factor of type A_2 and is the stabilizer of a maximal isotropic subspace on the natural 7-dimensional representation of S . In \mathbf{O} , there is a unique 1-space fixed by Q' and it is singular (9.7), a contradiction since x is nonsingular. So, $J \cong G$.

Since a nontrivial action of $G_2(\mathbf{C})$ on an 8-space is a unique sum of 1 and 7 dimensional irreducibles, it follows that we have a bijection between pairs of elements $\{x, -x\}$ from X and subgroups of S isomorphic to G . Since $-1 \in S$, transitivity of S on X follows.

9.11 Notation. By (9.10), we identify X with the space of right cosets $\{Gx|x \in S\}$. An orbit of G thus corresponds to a double coset GxG . We may view the orbits as a set of the form $\{(p, q)|p \in \mathbf{C}1, q \in V|p^2 + Nq = 1\}$, with p fixed; when $p \neq \pm 1$, this set is an orbit for G due to transitivity of G on the spheres of a give radius in V (9.7). When $p = \pm 1$, there are two orbits, according to whether $q = 0$ or is a nonzero singular point.

9.12 Remark. For $s \in S$, the possibilities for $G \cap G^s$ are G, P' and L° (9.7). We use the map $X \rightarrow X/\{\pm 1\}$ and $H \cong S/\{\pm 1\}$ to get an action of H . Now identify G with its image in H . The intersections $G \cap G^h$, for $h \in H$, look the same as within S except that we get both L and L° , for the nonsingular case. The reason for the difference is that L acts on sets $\{q, -q\}$, for $q \neq 0$ in the L -invariant 1-space; the action has kernel L° ; so if we take $(p, q) \in X$, as in (9.11), it is stabilized by L modulo the action of -1 if and only if $p = 0$; thus, $G \cap G^s$ is conjugate to L if $p = 0$ and to L° if $p \neq 0$.

Acknowledgements. We thank Gopal Prasad and J.-P. Serre for helpful discussions. For financial support during the time this article was written, we thank the National Science Foundation (grant DMS-9304279), the University of Michigan, the University of British Columbia, the University of Chicago, the Eidgenossische Technische Hochschule and the Ellentuck Fund of the Institute for Advanced Study.

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