A Bound for the Fixed-Point Index
of an Area-Preserving Map
with Applications to Mechanics*

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Area-preserving maps and flows play an essential role in the study of motions of mechanical systems, especially in celestial mechanics (see [1, 14]). Since one is often interested in the behavior of an area-preserving map around a fixed point and in the number and type of critical points and periodic orbits of an area-preserving flow, the *fixed-point index* of a map and the *index of a singularity* or a closed orbit of a flow can yield much information about the map or flow. For example, the fact that the index of an isolated singularity of an area-preserving flow can never be greater than +1 has aided in setting a lower bound for the number of stationary points of certain area-preserving flows. It has also been a useful necessary condition for a flow to be area-preserving. For these reasons, the conjecture that the fixed-point index of an area-preserving homeomorphism of a 2- manifold is always less than or equal to +1 has drawn attention. In this paper, we answer this conjecture in the affirmative for smooth maps and then put this bound to work to show that certain maps must have at least two fixed points and certain flows at least two periodic orbits. An important application is the following generalization of a famous theorem of Liapunov: a Hamiltonian vectorfield on M^4 must have *two* distinct one-parameter families of periodic orbits around a non-degenerate minimum (or maximum) of the Hamiltonian, even when the pure imaginary characteristic exponents are in resonance.

1. The Index for Flows and Transformations

Let $\dot{x} = G(x)$ be a continuous vectorfield on a manifold M^n with isolated singularity x_0. Since we are interested only in local behavior,

we can assume that $x_0 \in U$ an open ball in $\mathbb{R}^n$, that $G: U \to \mathbb{R}^n$, and that $U \cap G^{-1}(0) = \{x_0\}$. The following are equivalent definitions of the index of vectorfield G at x_0:

a) The local degree of G at x_0, i.e., if U is a ball of radius r and $S^{n-1} = \partial U$, the index is the degree of the map $\bar{G}: S^{n-1} \to S^{n-1}$ by $\bar{G}(x) = \dfrac{r\,G(x)}{|G(x)|} + x_0$, for $x \in \partial U$;

b) the intersection number of $G(U)$ and the zero section in the tangent bundle of M;

c) if G is smooth, the index is $\displaystyle\sum_{x \in G^{-1}(b)} \{$sign det $DG(x)\}$, where b is a regular value of G near 0;

d) in 2-dimensions, the index is $1 + \frac{1}{2}[E - H]$ where E is the number of elliptic regions about x_0 and H is the number of hyperbolic regions, or is $+1$ if x_0 is a rotation point for the flow [Poincaré Index Formula].
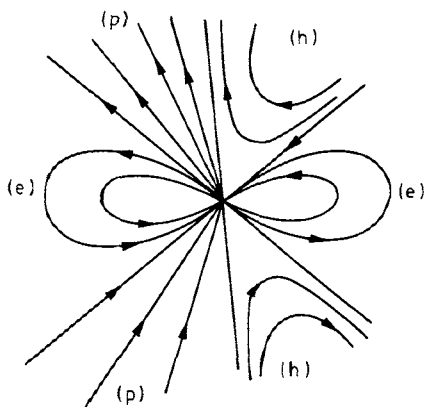

Fig. 1. Flow of G about x_0 with elliptic regions (e), hyperbolic regions (h), and parabolic regions (p)

On the other hand, let $f: M \to M$ be a continuous homeomorphism with isolated fixed point m_0. The following are equivalent definitions of the index of f at m_0:

e) The local degree of $1_M - f$ at m_0, i.e., if m_0 corresponds to the origin in ball U of radius r, and if it is the only fixed point in U, then the index is the degree of the mapping $S^{n-1} \to S^{n-1}$, given by $x \mapsto r\dfrac{x - f(x)}{|x - f(x)|}$, where $\partial U = S^{n-1}$;

f) the intersection number of the graph of f with the diagonal at (m_0, m_0) in $M \times M$;

g) The Lefschetz number of $(f|V)_*$: $H(V; Q) \to H(f V; Q)$, where m_0 is the only fixed point in V.

Among the many references for these definitions are [4–8], and [12].

2. The Index for an Area-Preserving Flow

Proposition 1. *If p is an isolated stationary point of an area-preserving flow X, the index of X at p is $\leq +1$.*

There are two quick ways of verifying this proposition. One can use definition d) above and observe that an area preserving flow cannot have any elliptic or parabolic regions about a stationary point. Alternatively, one can note that in a ball U about p with coordinates (x, y) there is a real-valued function $H(x, y)$ such that the vectorfield X is $\dot{x} = \dfrac{\partial H}{\partial y}$, $\dot{y} = -\dfrac{\partial H}{\partial x}$, (see [1]) and then follow the argument in Hartman [7, p. 173].

Note that Proposition 1 holds also for gradient vector fields since the vectorfield $\dot{x} = \dfrac{\partial H}{\partial y}$, $\dot{y} = -\dfrac{\partial H}{\partial x}$ is perpendicular at each point to the corresponding gradient vector field $\dot{x} = \dfrac{\partial H}{\partial x}$, $\dot{y} = \dfrac{\partial H}{\partial y}$.

3. Parameterizing Diffeomorphisms by Vector Fields

Let $C^k(\mathbb{R}^n, \mathbb{R}^n)$ be the space of C^k maps $\mathbb{R}^n \to \mathbb{R}^n$ with the fine C^k topology ([11]), $k \geq 1$. Let $D^k(\mathbb{R}^n, \mathbb{R}^n)$ be the open subspace consisting of the C^k diffeomorphisms, with the identity map $I \in D^k(\mathbb{R}^n, \mathbb{R}^n)$. Since $f \to \frac{1}{2}(I + f)$ is a continuous map of $D^k(\mathbb{R}^n, \mathbb{R}^n)$ into $C^k(\mathbb{R}^n, \mathbb{R}^n)$, there is a neighborhood $\mathcal{U}$ of I in $D^k(\mathbb{R}^n, \mathbb{R}^n)$ such that if $f \in \mathcal{U}, \frac{1}{2}(I + f) \in D^k(\mathbb{R}^n, \mathbb{R}^n)$. Similarly, let $\mathcal{V}$ be a neighborhood of the zero map 0 in $C^k(\mathbb{R}^n, \mathbb{R}^n)$ such that if ξ is in $\mathcal{V}$, $I + \frac{1}{2}\xi$ and $I - \frac{1}{2}\xi$ are in $D^k(\mathbb{R}^n, \mathbb{R}^n)$. Let $\mathcal{A} = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^n \mid f \in \mathcal{U}\}$, an open neighborhood of the diagonal Δ in $\mathbb{R}^n \times \mathbb{R}^n$.

Proposition 2. a) *Consider maps $F: \mathcal{U} \to C^k(\mathbb{R}^n, \mathbb{R}^n)$ and $G: \mathcal{U} \to D^k(\mathbb{R}^n, \mathbb{R}^n)$ defined by $F(f) = (I - f) \circ [\frac{1}{2}(I + f)]^{-1}$ and $G(\xi) = (I - \frac{1}{2}\xi) \circ (I + \frac{1}{2}\xi)^{-1}$. Then $F: \mathcal{U} \to \mathcal{V}$, $G: \mathcal{V} \to \mathcal{U}$, and F and G are inverses to each other.*

b) *Consider the linear map $H: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n = T\mathbb{R}^n$ by $H(x, y) = [\frac{1}{2}x + \frac{1}{2}y, x - y]$. H takes the graph of an f in $\mathcal{U}$ onto the graph of $F(f)$ in $\mathcal{V}$.*

c) *Suppose that $f: \mathbb{R}^n \to \mathbb{R}^n$; that $f(x_0) = x_0$; that for some neighborhoods U and V about x_0, $\frac{1}{2}(I + f): U \to V$ is a surjective diffeomorphism; and that $(x, f(x)) \in \mathcal{A}$ for all $x \in U$. Then $F(f): V \to \mathbb{R}^n$ is defined and $F(f)(y) = x - f(x)$, where $y = \frac{1}{2}(x + f(x))$.*

x is a fixed point of f if and only if $y=\frac{1}{2}(x+f(x))$ is a zero of $F(f)$; and the fixed point index of x for f is equal to the index of y as a stationary point for the vector field $F(f)$. If $n=2$ and $k\geq 1$ and if $f\colon U\to\mathbb{R}^2$ is area-preserving, then $F(f)\colon V\to\mathbb{R}^2$ is divergence-free.

d) Likewise, if $\xi\colon\mathbb{R}^n\to\mathbb{R}^n$, $\xi(y_0)=0$, and $I+\frac{1}{2}\xi\colon V\to W$ is a surjective diffeomorphism for neighborhoods V and W of y_0 with $(y,\xi(y))\in H(\mathscr{A})$ for $y\in V$, then $G(\xi)\colon W\to\mathbb{R}^n$ is defined and $G(\xi)(x)=y-\frac{1}{2}\xi(y)$ where $x=y+\frac{1}{2}\xi(y)$. y a zero of G of index j implies that x is a fixed point of $G(\xi)$ of index j. If $n=2$ and $k\geq 1$ and $\xi\colon V\to\mathbb{R}^2$ is divergence-free, then $G(\xi)$ is area-preserving.

Proof. a) To see that $G\circ F$ is the identity, let $z\in\mathbb{R}^n$. Then,

$$
\begin{aligned}
\big(z,(G\circ F)(f)(z)\big)&=\big(z,G(\xi_f)(z)\big)\\
&=\big(z,(I-\tfrac{1}{2}\xi_f)\circ(I+\tfrac{1}{2}\xi_f)^{-1}z\big)\\
&=\big(\omega+\tfrac{1}{2}\xi_f(\omega),\,\omega-\tfrac{1}{2}\xi_f(\omega)\big),\quad\text{where }\omega=(I+\tfrac{1}{2}\xi_f)^{-1}z,\\
&=\big(\tfrac{1}{2}(y+f(y))+\tfrac{1}{2}(y-f(y)),\,\tfrac{1}{2}(y+f(y))-\tfrac{1}{2}(y-f(y))\big),\\
&\hspace{4cm}\text{where }\omega=\tfrac{1}{2}(y+f(y))\\
&=(y,f(y)).
\end{aligned}
$$

So, $z=y$ and $(G\circ F)(f)(z)=f(z)$.

Similarly, $F(G(\xi))=\xi$.

b) $H(x,f(x))=\big[\tfrac{1}{2}(x+fx)),\,x-f(x)\big]$
$$=\big[y,(I-f)\circ[\tfrac{1}{2}(I+f)]^{-1}y\big],\quad\text{where }\tfrac{1}{2}(x+f(x))=y.$$

c) Clearly, $F(f)(y)=0$ if and only if $x=f(x)$, where $y=\frac{1}{2}(x+f(x))$. H takes the graph of $f\,|\,U$ onto the graph of $F(f)\,|\,V$. H also takes the graph of I to the graph of 0. Using definition b) for the index of a zero of a vector field, definition f) for the index of a fixed point of a map, and the fact that H preserves intersection numbers, one finds that the index of a zero of $F(f)$ is the same as the index of the corresponding fixed point of f. For the last sentence of c), we use the fact that a transformation f of the plane is area-preserving if and only if its Jacobian, $\det Df(x)$, is $+1$ for all x; and the fact that $\operatorname{div}\xi=\operatorname{trace}D\xi$. So, if $\xi=F(f)$,

$$
D\xi(x)=D(I-f)_{[\frac{1}{2}(I+f)]^{-1}x}\circ D[\tfrac{1}{2}(I+f)]^{-1}(x)
$$
$$
=D(I-f)_y\circ\big[D(\tfrac{1}{2}(I+f))_y\big]^{-1},\quad\text{where }y=[\tfrac{1}{2}(I+f)]^{-1}(x)
$$
$$
=\begin{bmatrix}1-\dfrac{\partial f_1}{\partial y_1} & -\dfrac{\partial f_1}{\partial y_2}\\[2mm] -\dfrac{\partial f_2}{\partial y_1} & 1-\dfrac{\partial f_2}{\partial y_2}\end{bmatrix}\circ\dfrac{1}{\det D[\tfrac{1}{2}(I+f)]_y}\begin{bmatrix}\dfrac{1}{2}\left(1+\dfrac{\partial f_2}{\partial y_2}\right) & -\dfrac{1}{2}\dfrac{\partial f_1}{\partial y_2}\\[3mm] -\dfrac{1}{2}\dfrac{\partial f_2}{\partial y_1} & \dfrac{1}{2}\left(1+\dfrac{\partial f_1}{\partial y_1}\right)\end{bmatrix}
$$

$$= \frac{1}{\det D\left[\frac{1}{2}(I+f)\right]_y} \begin{bmatrix} \frac{1}{2}\left(1 - \det Df(y) + \frac{\partial f_1}{\partial y_2} - \frac{\partial f_1}{\partial y_1}\right) & -\frac{\partial f_1}{\partial y_2} \\ -\frac{\partial f_2}{\partial y_1} & \frac{1}{2}\left(1 - \det Df(y) + \frac{\partial f_1}{\partial y_1} - \frac{\partial f_2}{\partial y_2}\right) \end{bmatrix}.$$

So, $\operatorname{div} \xi(x) = \operatorname{trace} D\xi(x) = 0$ for all x if and only if $1 = \det Df(y)$ for all y.

d) The first part of d) is proven as the first part of c) was. For the last sentence, if $f = G(\xi)$,

$$Df(x) = D(I + \tfrac{1}{2}\xi)_{(I - \frac{1}{2}\xi)^{-1}x} \circ D(I - \tfrac{1}{2}\xi)^{-1} x$$
$$= D(I + \tfrac{1}{2}\xi)_y \circ [D(I - \tfrac{1}{2}\xi)_y]^{-1}, \quad \text{where } y = (I - \tfrac{1}{2}\xi)^{-1} x.$$

So,

$$\det Df(x) = \frac{\det D(I + \tfrac{1}{2}\xi)_y}{\det D(I - \tfrac{1}{2}\xi)_y} = \frac{1 + \tfrac{1}{2}\operatorname{div}\xi(y) + \tfrac{1}{4}\det D\xi(y)}{1 - \tfrac{1}{2}\operatorname{div}\xi(y) + \tfrac{1}{4}\det D\xi(y)}.$$

$\det Df(x) = 1$ for all x if and only if $\operatorname{div}\xi(y) = 0$ for all y.   Q.E.D.

Proposition 2 describes a "coordinate chart" from a neighborhood of the identity in the group of diffeomorphisms of $\mathbb{R}^n$ to a neighborhood of zero in the Banach space of smooth vector fields on $\mathbb{R}^n$. (This chart is not smooth since F and G involve inversion.) For $n = 2$, this chart shows that the space of area-preserving diffemorphisms of the plane is a sub-manifold modelled on the subspace of divergence-free vectorfields. Both of these results are special cases of much more general results in symplectic geometry in the work of Weinstein [18, 21]. A *symplectic manifold* M is one that supports a closed non-degenerate 2-form Ω. Ω non-degenerate means that $\tilde{\Omega}: TM \to T^*M$ by $\tilde{\Omega}(X) = X \lrcorner \Omega$ is an isomorphism. If (M, Ω) is a symplectic manifold, there is a natural symplectic 2-form Ω_1 on TM and a natural symplectic 2-form Ω_2 on $M \times M$. In [18], Weinstein describes a diffeomorphism F of a neighborhood of the diagonal Δ in $(M \times M, \Omega_2)$ onto a neighborhood of the zero section Z in (TM, Ω_1). F is symplectic in that $F^*\Omega_1 = \Omega_2$. As a result, F maps the graphs of symplectic diffeomorphisms f of M (i.e., $f: M \to M$ with $f^*\Omega = \Omega$) near the identity onto the graphs of infinitesimally symplectic vectorfields X (i.e., $X: M \to TM$ such that $L_X\Omega = 0$) near the zero vector field. In two dimensions, symplectic means area-preserving if one chooses Ω to be the area element.

Meyer uses such a parametrization of symplectic diffeomorphisms by symplectic (locally Hamiltonian) vector fields to study generic bifurcations of periodic points of area-preserving maps in [9] and to write canonical forms for symplectic and Hamiltonian matrices in [10].

4. The Fixed Point Index of an Area-Preserving Map

In this section, we compute the possibilities for the index of a fixed point of C^1 area-preserving map f of a 2-manifold. Since this is purely a local problem, we can suppose without loss of generality that U is a convex neighborhood of the origin 0 in $\mathbb{R}^2$, that $f: U \to \mathbb{R}^2$ has 0 as its only fixed point, and that $\det Df(x) = 1$ for all x in U.

Since $\det Df(0)$ is the product of the eigenvalues λ, μ of $Df(0)$, there are three possibilities for these eigenvalues: a) $0 < |\lambda| < 1 < |\mu|$ (hyperbolic case), b) $|\lambda| = |\mu| = 1$ but $\lambda \neq 1$ (elliptic case), and c) $\lambda = \mu = 1$ (parabolic case).

Proposition 3. *With $f: U \to \mathbb{R}^2$ as above, the index of 0 in case* a) *is* ± 1 *and the index in case* b) *is* $+1$.

Proof. To compute the index, we use the fact that the index is a homotopy invariant. More precisely, if f_t is a continuous one-parameter family of mappings of U into $\mathbb{R}^2$ for $0 \leq t \leq 1$ with 0 an isolated fixed point for each f_t, the index of 0 for f_0 equals the index of 0 for f_1. (See Dold [4].)

Let $f_\varepsilon(x) = \dfrac{1}{\varepsilon} f(\varepsilon x)$ for $0 < \varepsilon \leq 1$; $f_0(x) = Df(0) x$. If $f(x) = Df(0) x + R(x)$

is the Taylor expansion of f where $R = o(|x|)$, then $f_\varepsilon(x) = Df(0) x + S(\varepsilon, x)$, where S is C^1 in ε and x and $S(0, x) \equiv 0$. So, f_ε is a homotopy of f for $0 \leq \varepsilon \leq 1$. $f_\varepsilon(x) = x$ for $x \in U$ if and only if $f(\varepsilon x) = \varepsilon x$ if and only if $x = 0$, provided 1 is not an eigenvalue of $Df(0)$. So, in cases a) and b), the index of f at 0 is the same as the index of $Df(0)$ at 0.

In case a), $Df(0)$ moves points as in Fig. 2 with a possible flip about the origin. Here, E_λ is the eigenspace of λ and E_μ the eigenspace of μ. Using definition e) for the fixed point index, one computes easily that the index of 0 in case a) is -1 if $Df(0)|E_\mu$ preserves the natural orientation of E_μ and is $+1$ if $Df(0)|E_\mu$ reverses this orientation. For a more general theorem, see Proposition 4.11 in Smale [17].

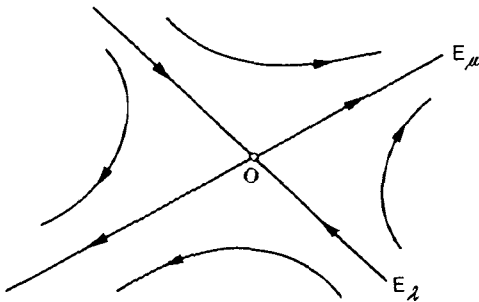
Fig. 2

In case b), $Df(0)$ has the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for some $\theta\in(0,2\pi)$ or is similar to $\begin{pmatrix} -1 & \eta \\ 0 & -1 \end{pmatrix}$. Using the same techniques as in case a), one computes that the index of 0 for the rotation matrix is $+1$. By letting $\eta\to 0$ in $\begin{pmatrix} -1 & \eta \\ 0 & -1 \end{pmatrix}$ and using the invariance of index under homotopy, one finds that the index of 0 is also $+1$ in this situation. $\square$

We come now to the main goal of this paper.

Theorem 1. *Let $f: N\to N$ be a C^k area-preserving transformation of a 2-manifold N, $k\geqq 1$. If p is an isolated fixed point of f, then the index of f at p is $\leqq +1$.*

Proof. As described in the first paragraph of this section, we are led to consider $f: U\to\mathbb{R}^2$ where U is a neighborhood of 0 in $\mathbb{R}^2$. By Proposition 3, we need only investigate the case where $Df(0)=\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$. Now, $D(\frac{1}{2}(I+f))(0)=\frac{1}{2}(I+Df(0))=\begin{pmatrix} 1 & \eta/2 \\ 0 & 1 \end{pmatrix}$. By the inverse function theorem, there are neighborhoods U_1 and V such that $f\,|\,U_1$ is a diffeomorphism and $\frac{1}{2}(I+f)\,|\,U_1\colon U_1\to V$ is a surjective diffeomorphism, where $(x,f(x))\in\mathcal{A}$ (of Proposition 2) for $x\in U_1$. Using Proposition 2, construct divergence-free vector field $F(f)$ on V. Since the fixed-point index of f at 0 is equal to the index of 0 as a singularity of $F(f)$, this fixed-point index is $\leqq +1$ by Proposition 1. $\square$

5. Elementary Applications

As discussed in the introduction, the fixed point index is an important tool in obtaining lower bounds for the number of fixed points of maps of certain manifolds M – mainly because of the fact that if $f: M\to M$ is a map homotopic to identity with isolated fixed points or ξ is a vectorfield on M with isolated singularities, then the Euler characteristic is equal to the sum of the fixed-point indices of f and to the sum of the indices of the singularities of ξ (see [4, 6, 8, 12]). For example, since the Euler characteristic of the 2-sphere S^2 is $+2$, every map $f: S^2\to S^2$ homotopic to identity must have a fixed point and every vector field on S^2 must have a zero. Proposition 1 and Theorem 1 yield the following strengthening of this result in the area-preserving case, since an area-preserving map has degree 1 and so by Hopf's Theorem is homotopic to identity.

Theorem 2. *If $f: S^2\to S^2$ is a C^k area-preserving map of the 2-sphere, $k\geqq 1$, then f has at least 2 fixed points. If ξ is a gradient, Hamiltonian, or divergence-free vector field on S^2, then ξ has at least two singularities.*

If one could prove the analogue of Theorem 2 for C^0 maps, then one could show that an area preserving map g of the open *two* disk onto itself must have a fixed point, without using the techniques of Brouwer for mappings of the plane. One would construct from g an area-preserving map $\hat{g}$ of S^2 with a fixed point at infinity and use the C^0 analogue of Theorem 2 to point out that $\hat{g}$ must have another fixed-point.

As another simple application of Theorem 1, we obtain the following clarification of Poincaré's Geometric Theorem.

Theorem 3. *Let T be a C^1 area-preserving transformation of the annulus that leaves the boundary circles C_1 and C_2 invariant with $T \mid C_i$ a simple rotation of the circle. If T has a fixed point of index $-n$, then T has at least $n+1$ fixed points.*

(Birkhoff [3] has shown that if T rotates the boundary circles in opposite directions, then T must have at least one fixed point.)

Proof. The sum of the indices throughout the annulus is zero. To see this, let D_1 and D_2 be the 2-disks bounded by the simple closed curves C_1 and C_2 respectively, with $D_1 \subset D_2$. Construct $g \colon D_1 \to D_1$ with a single fixed point such that $g \mid C_1 = T \mid C_1$, thus extending T to a map $\hat{T} \colon D_2 \to D_2$. Using definition e) of the index as in part a) of Proposition 2, one notes that the sum of the fixed-point indices of $g \colon D_1 \to D_1$ is $+1$, as is the corresponding sum for $\hat{T} \colon D_2 \to D_2$. Therefore, the sum of the fixed-point indices in $\overline{D_2 - D_1}$, the annulus, is zero. But by Theorem 1, each index in the annulus is $\leqq +1$. $\square$

6. Families of Periodic Orbits for Hamiltonian Systems

Our main tool in this section will be the index for periodic orbits of a smooth vector-field X as described and utilized by Fuller [5]. Let M be a smooth manifold with $\phi \colon M \times (-\infty, \infty) \to M$ the flow of X on M. Let Ω be an open set in $M \times (0, \infty)$ such that

a) $\bar{\Omega}$ is compact in $M \times (0, \infty)$;

b) if $(x, t) \in \bar{\Omega} \smallsetminus \Omega$, then $\phi(x, t) \neq x$;

c) if $(x, t) \in \bar{\Omega}$, $X(x) \neq 0$.

For such Ω, Fuller defines a rational number $i(\Omega)$, the *index of Ω*, with the following properties:

1) if $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \phi$, and Ω_1 and Ω_2 satisfy a), b), c), then $i(\Omega) = i(\Omega_1) + i(\Omega_2)$,

2) if X_0 and Ω_0 are continuously deformed to X_α and Ω_α such that each Ω_α satisfies a), b), c) for X_α, $\alpha \in [0, 1]$, then $i(\Omega_\alpha)$ is independent of α.

3) Suppose $\Omega = U \times \left(kt_0 - \dfrac{t_0}{2}, \; kt_0 + \dfrac{t_0}{2}\right)$ where k is an integer, U is an open set in M that contains only one periodic orbit γ; and γ has least period t_0. Let D be an $(n-1)$-disk meeting γ transversally at x_0, with $D \cap \gamma = \{x_0\}$. Let T, mapping a neighborhood of x_0 in D into D, be the Poincaré first-return map for X. Then, $i(\Omega)$ equals $1/k \times$ (fixed-point index of T^k at x_0).

4) If $\Omega = U \times (t_1, t_2)$ and $i(\Omega) \neq 0$, then U contains a periodic orbit of period $\tau \in (t_1, t_2)$.

Fuller applies his index in [5] to prove the following theorem of Seifert.

Proposition 4 (see Seifert [13]). *Let X be a vector field on S^3 with every orbit a non-trivial periodic orbit of least period 2π and with the orbit space homeomorphic to S^2. Let $\Omega = S^3 \times (\pi, 3\pi)$. Then $i(\Omega) = +2$. Therefore, perturbations of X have at least one periodic orbit.*

Fuller's proof involves i) constructing a vectorfield G on S^2 where flow fixes the north and south poles and sends the other points down the meridians, ii) lifting G to $\hat{G}$ on S^3, using the fact that S^2 is the orbit space of X, iii) noting that $X_\varepsilon = X + \varepsilon \hat{G}$ has only two periodic orbits, both of index one, for $\varepsilon \neq 0$, small, iv) using the invariance of $i(\Omega)$ under homotopy.

Let us now apply Fuller's index to the search for families of periodic orbits of Hamiltonian dynamical systems. Let $H: M^4 \to \mathbb{R}$ be a smooth (Hamiltonian or energy) function on a four-dimensional phase space, with $p \in M^4$ an isolated critical point. Choose canonical coordinates about p so that we can view H as defined on a neighborhood of $\mathbf{0}$ in $\mathbb{R}^4$ with $H(\mathbf{0}) = 0$ and $\operatorname{grad} H(\mathbf{0}) = 0$. Let J be the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Then, the corresponding Hamiltonian equations for a system with energy function H is $\dot{x} = J \circ \operatorname{grad} H(\mathbf{x}) \equiv X_H(\mathbf{x})$. Suppose the characteristic multipliers of X_H at $\mathbf{0}$, i.e., the eigenvalues of the linear map $DX_H(\mathbf{0})$, are pure imaginary: $\pm i\omega_1$ and $\pm i\omega_2$, where ω_1 and ω_2 are real and $|\omega_1| < |\omega_2|$. In this case, by a standard diagonalization technique [14, §15], there are canonical coordinates (x_1, x_2, y_1, y_2) in U such that

$$(*) \qquad H(x_1, x_2, y_1, y_2) = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \frac{\omega_2}{2}(x_2^2 + y_2^2) + R(\mathbf{x}, \mathbf{y})$$

where R is $0(|(\mathbf{x}, \mathbf{y})|^2)$. The linearized equations, $\dot{\mathbf{x}} = DX_H(\mathbf{0})\mathbf{x}$, are the Hamiltonian equations for the Hamiltonian

$$(\ast) \qquad H_0(x_1, x_2, y_1, y_2) = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \frac{\omega_2}{2}(x_2^2 + y_2^2),$$

the quadratic part of H. These linear equations are:

$$\dot{x}_1 = \omega_1 x_1, \qquad \dot{x}_2 = \omega_2 y_2, \qquad \dot{y}_1 = -\omega_1 x_1, \qquad \dot{y}_2 = -\omega_2 x_2.$$

If we choose coordinates $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ in U where $z_k = x_k + i y_k$, $k = 1, 2$, then $\dot{\mathbf{x}} = DX_H(\mathbf{0})x = X_{H_0}(\mathbf{x})$ becomes $\dot{z}_k = i\omega_k z_k$ with solutions

$$z_1(t) = z_1 e^{-i\omega_1 t}, \qquad z_2(t) = z_2 e^{-i\omega_2 t}.$$

Here, the (generalized) eigenspace $z_2 = 0$ for $\pm i\omega_1$ is a one-parameter family of periodic orbits of $\dot{z} = DX_H(\mathbf{0})z$ with period $\left|\dfrac{2\pi}{\omega_1}\right|$ and the (generalized) eigenspace $z_1 = 0$ of $\pm i\omega_2$ is a one-parameter family of periodic orbits of period $\left|\dfrac{2\pi}{\omega_2}\right|$: A famous theorem of Liapunov (see [14]) asserts that there still exists a smooth one-parameter family of orbits for the non-linear equation $\dot{\mathbf{x}} = X_H(\mathbf{x})$ near $z_1 = 0$ with period near $\left|\dfrac{2\pi}{\omega_2}\right|$; and if $\dfrac{\omega_2}{\omega_1}$ *is not an integer*, there is another such family near $z_2 = 0$ with period $\left|\dfrac{2\pi}{\omega_1}\right|$.

In Theorem 4 below, we use Theorem 1 and Fuller's index to show the existence of this second family even if $\dfrac{\omega_2}{\omega_1}$ is an integer (even 1), provided that the Hessian of H at $\mathbf{0}$ is positive (or negative) definite. Weinstein [19, 20] has used much more complex techniques to obtain the definitive results in this situation. Berger [2] has used techniques of the calculus of variations to exhibit families of periodic orbits in similar situations. On the other hand, for each integer $n \geqq 1$, there exists $H : \mathbb{R}^4 \to \mathbb{R}$ with $H(\mathbf{0}) = 0$, $DH(\mathbf{0}) = 0$, $D^2H(\mathbf{0})$ indefinite, $\dfrac{\omega_2}{\omega_1} = -n$, and X_H having only *one* family of periodic orbits [16].

Theorem 4. *Let $H : M^4 \to \mathbb{R}$ be a smooth (C^2) Hamiltonian function on a 4-dimensional phase space with $p \in M$ a non-degenerate minimum (or maximum) of H and $H(p) = c_0$. Then, for c near c_0, the Hamiltonian system $\dot{\mathbf{x}} = J \circ \mathrm{grad}\, H(\mathbf{x})$ has at least two periodic orbits on each energy surface $H^{-1}(c)$.*

Proof. As above, choose canonical coordinates (z_1, z_2) in $\mathbb{C} \times \mathbb{C}$ about **p** so that **p** corresponds to 0 and in this neighborhood

$$H(z_1, z_2) = \frac{\omega_1}{2} z_1 \bar{z}_1 + \frac{\omega_2}{2} z_2 \bar{z}_2 + R(z_1, z_2), \qquad |\omega_1| \leq |\omega_2|,$$

as in (∗). Since **p** is a non-degenerate minimum (maximum), $D^2 H(0)$ is positive (negative) definite and ω_1 and ω_2 are both positive (negative). Without loss of generality, we assume **p** is a minimum. (Otherwise, work with $-H$.) Consider the one-parameter family of Hamiltonians

$$H_\varepsilon(\mathbf{z}) = \frac{1}{\varepsilon^2} H(\varepsilon \mathbf{z}) \quad \text{for } 0 < \varepsilon \leq 1,$$

$$H_0(\mathbf{z}) = \frac{\omega_1}{2} z_1 \bar{z}_1 + \frac{\omega_2}{2} z_2 \bar{z}_2 \quad \text{for } \varepsilon = 0.$$

Since $R(z_1, z_2)$ is $0(|\mathbf{z}|^2)$, $H_\varepsilon(z)$ is smooth in ε. Let X_{H_ε} be the Hamiltonian vector field $\dot{z} = J \circ \operatorname{grad} H_\varepsilon(\mathbf{z})$. For $\varepsilon \neq 0$, multiplication by ε takes integral curves of X_{H_ε} on $H_\varepsilon^{-1}(1)$ onto integral curves of $X_H \equiv X_{H_1}$ on $H^{-1}(\varepsilon^2)$. So a periodic orbit for X_{H_ε} on $H_\varepsilon^{-1}(1)$ corresponds to periodic orbit of X_H on $H^{-1}(\varepsilon^2)$. [We are using the fact that H_ε is constant on integral curves of X_{H_ε}.]

Consequently, using the homotopy H_ε allows us to view X_H as a perturbation of the linear system $X_{H_0}(\mathbf{z}) = DX_H(0)(\mathbf{z})$. X_{H_0} has solution $(z_{10} e^{-i\omega_1 t}, z_{20} e^{-i\omega_2 t})$. If $\frac{\omega_2}{\omega_1}$ is not an integer, Liapunov's Theorem asserts that X_H has 2 families of periodic orbits, parametrized by energy. [In this case, on $H_0^{-1}(c)$ the periodic orbits

$$\left\{ \left(0, \left|\frac{2c}{\omega_2}\right|^{\frac{1}{2}} e^{-i\omega_2 t}\right) \,\Big|\, t \in \mathbb{R} \right\} \quad \text{and} \quad \left\{ \left(\left|\frac{2c}{\omega_1}\right|^{\frac{1}{2}} e^{-i\omega_1 t}, 0\right) \,\Big|\, t \in \mathbb{R} \right\}$$

have non-zero Fuller index and so are preserved in perturbing X_{H_0} to X_{H_ε}.]

We treat the case where $\frac{\omega_2}{\omega_1}$ is an integer. Since $X_{\tilde{H}}$ has the same phase portrait as X_H where $\tilde{H} = \frac{1}{\omega_1} H$, we can suppose that $\omega_1 = 1$ and $\omega_2 = \omega \in \mathbb{N}$. $H_0^{-1}(\frac{1}{2}) = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \,|\, |z_1|^2 + \omega |z_2|^2 = 1\}$ is diffeomorphic to the 3-sphere S^3. Each orbit $(z_{10} e^{-it}, z_{20} e^{-i\omega t})$ has least period 2π, except the orbit $\gamma_0 = \left\{ \left(0, \sqrt{\frac{1}{\omega}} e^{-i\omega t}\right) \,\Big|\, t \in \mathbb{R} \right\}$ which has *least* period $\frac{2\pi}{\omega}$.
Let T_a be the torus $\left\{ (z_1, z_2) \in H_0^{-1}(\frac{1}{2}) \,\Big|\, |z_1| = a^{\frac{1}{2}} \left(\text{and thus } |z_2|^2 = \frac{1-a}{\omega}\right) \right\}$, $0 < a < 1$. $H_0^{-1}(\frac{1}{2})$ is the disjoint union of the T_a's for $0 < a < 1$ and the orbits

γ_0 and γ_1, where $\gamma_1 = \{e^{-it}, 0) | t \in \mathbb{R}\}$. Each T_a is invariant under the flow of X_{H_0} on $H_0^{-1}(\frac{1}{2})$.

We want to compute $i(\Omega)$ where $\Omega = H_0^{-1}(\frac{1}{2}) \times \left(2\pi - \dfrac{\pi}{\omega}, 2\pi + \dfrac{\pi}{\omega}\right)$.

For $\omega = 1$ (characteristic exponents of X_H equal in pairs), $i(\Omega) = +2$ by the calculation of Proposition 4. So in this case, using 2) of the properties of $i(\Omega)$, $i(\Omega_\varepsilon) = +2$ where $\Omega_\varepsilon = H_\varepsilon^{-1}(\frac{1}{2}) \times (\pi, 3\pi)$; and X_{H_ε} must have a periodic orbit of period near 2π for ε small. Using the fact that the Poincaré first-return map on an energy surface of a Hamiltonian system on M^4 is area-preserving [1, 14], property 3) of the Fuller index, and Theorem 1, we see that the index of this orbit is at most $+1$. Since $i(\Omega_\varepsilon) = +2$, there must be another periodic orbit on $H_\varepsilon^{-1}(\frac{1}{2})$.

To compute $i(\Omega)$ when $\omega > 1$, consider the smooth map $g : H_0^{-1}(\frac{1}{2}) \to \mathbb{R}$ by $g(z_1, z_2) = |z_1|^2$. Let Y be the gradient vector field of g on $H_0^{-1}(\frac{1}{2})$. The circles γ_0 and γ_1 are circles of critical points of Y and the flow of Y takes torus T_a to torus $T_{a'}$ where $0 < a < a' < 1$. Now, consider vector field $F_\varepsilon = X_{H_0} + \varepsilon Y$ on S^3. For $\varepsilon \neq 0$, the orbits γ_0 and γ_1 are the only periodic orbits of F_ε and the flow of F_ε sends points from γ_0 toward γ_1 (i.e., γ_0 is a source and γ_1 a sink). So, $i(\Omega)$ satisfies a), b), c) for each F_ε. By property 2) of i, $i(\Omega)$ for X_{H_0} equals $i(\Omega)$ for F_ε. Let $U_1 = \{(z_1, z_2) \in H_0^{-1}(\frac{1}{2}) \mid |z_1| < \frac{1}{2}\}$ and $U_2 = $ the interior of its complement in $H_0^{-1}(\frac{1}{2})$. For vector field F_ε,

$$i(\Omega) = i\left(U_1 \times \left(2\pi - \frac{\pi}{\omega}, 2\pi + \frac{\pi}{\omega}\right)\right) + i\left(U_2 \times \left(2\pi - \frac{\pi}{\omega}, 2\pi + \frac{\pi}{\omega}\right)\right)$$

by properties 1) and 4) of $i(\Omega)$

$$= +1 + \frac{1}{\omega}$$

by property 3) and the fact that the fixed point index of a contraction or its inverse is $+1$. Finally, for $\varepsilon > 0$ and small, $H_\varepsilon^{-1}(\frac{1}{2})$ is a 3-sphere and $\Omega_\varepsilon = H_\varepsilon^{-1}(\frac{1}{2}) \times \left(2\pi - \dfrac{\pi}{\omega}, 2\pi + \dfrac{\pi}{\omega}\right)$ satisfies a), b), c) for X_{H_ε}. Therefore, $i(\Omega_\varepsilon) > 1$. Again using the fact that Poincaré map for X_{H_ε} on $H_\varepsilon^{-1}(\frac{1}{2})$ is area-preserving, property 3) of i, and Theorem 1, we see that there must be at least two periodic orbits on $H_\varepsilon^{-1}(\frac{1}{2})$. These correspond to periodic orbits of X_H on $H^{-1}\left(\dfrac{\varepsilon^2}{2}\right)$.

As Alan Weinstein and Jean Martinet have pointed out to me, one can construct an area-preserving Poincaré map about a periodic orbit of a volume-preserving vector field X on a manifold M. If η is the volume form on M and $L_X \eta = 0$, one puts the volume form $X \lrcorner \eta$ on a codimension one transverse disk. Therefore, the above techniques and Theorem 1 yield the following strengthening of Proposition 4.

Theorem 5. *Let F be a vectorfield on S^3 equivalent to the vectorfield $z_1 = -iz_1$, $z_2 = -ikz_2$ on the 3-sphere $\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_1|^2 + k|z_2|^2 = \frac{1}{2}\}$, where k is a positive integer. Then, divergence-free perturbations of F have at least two distinct periodic orbits.*

7. No Bounds in Higher Dimensions

The following computations, worked out with Charles Titus, show that there are no restrictions on the index for volume preserving or even symplectic transformations and flows in higher dimensions. Once again, put coordinates $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ on $\mathbb{R}^4$. Consider the Hamiltonian function

$$H(z_1, z_2) = -\tfrac{1}{2}(z_1^n + \bar{z}_1^n + z_2^m + \bar{z}_2^m),$$

where n and m are integers greater than 1. The corresponding Hamiltonian vector-field is

$$\dot{z}_1 = -2i\frac{\partial H}{\partial \bar{z}_1} = in\bar{z}_1^{n-1}$$

$$\dot{z}_2 = -2i\frac{\partial H}{\partial \bar{z}_2} = im\bar{z}_2^{m-1},$$

or $(\dot{z}_1, \dot{z}_2) = G(z_1, z_2)$. A simple computation shows that the determinant of the Jacobian of G will have the same sign in a neighborhood of 0 for each m, n. By definition c) of the index of a singularity of a vector field, the index of 0 for $\dot{z} = G(z)$ is equal to the cardinality of $G^{-1}(c)$ for regular values c near 0, i.e., to $(m-1)(n-1)$. The time-one map for the flow of this vector field will be volume-preserving on $\mathbb{R}^4$ and have 0 as an isolated fixed point of index $(m-1)(n-1)$.

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