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Biinvariant Operators on Nilpotent Lie Groups

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The purpose of this article is to prove P-convexity for biinvariant differential operators on connected simply connected nilpotent Lie groups. More precisely, we show that for any compact subset K of a connected simply connected nilpotent Lie group N, and for any non-zero biinvariant differential operator P on N, there is a compact subset $L \supset K$ with the property that whenever the support of Pu is contained in L for a C^∞ function of compact support u on N, then the support of u is contained in L. I am grateful to M. Duflo, to A. Cerezo, and to F. Rouvière for several helpful discussions.

Solubility properties of biinvariant operators have been considered by several authors. S. Helgason [6] proves local solvability of biinvariant operators on semisimple Lie groups. Rais [8] proves the existence of a fundamental solution for a biinvariant operator on a connected simply connected nilpotent Lie group. Duflo and Rais [4] prove the local solvability of biinvariant operators on a solvable Lie group and Rouvière [9] proves semi-global solvability for biinvariant operators on simply connected solvable groups. Finally, Duflo [3] proves local solvability of biinvariant operators on any Lie group whatsoever.

Semi-global solvability is in general false even for noncompact simple groups as was demonstrated by A. Cerezo and F. Rouvière [2]. Finally, even local solvability of left invariant operators is frequently false as was shown by L. Hormander, c.f. [6] and independently by A. Cerezo and F. Rouvière [1]. From our result and that of Rais [8] or Rouvière [9], we conclude the global solvability of biinvariant operators on simply connected nilpotent Lie groups, i.e. that for any C^∞ function f and nonzero biinvariant operator P on a simply connected nilpotent Lie group N, there exists a C^∞ function u on N such that $Pu=f$. For simply connected abelian Lie groups, this reduces to the theorem of Malgrange and Ehrenpreis that constant coefficient differential operators on R^n are globally solvable, c.f. [11]. Thus our Theorem 2 can be regarded as a generalization of the Malgrange-Ehrenpreis theorem.

Henceforward N will denote a connected simply connected nilpotent Lie group, and $\mathfrak{N}$ its Lie algebra. We write $\exp: \mathfrak{N} \to N$ for the exponential map of $\mathfrak{N}$

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onto N, which is known to be an analytic diffeomorphism and log: $N \to \mathfrak{N}$ will denote the analytic diffeomorphism inverse to exp. We recall that the center of N is connected and simply connected. Since a connected and simply connected abelian Lie group has a natural translation invariant convex structure, we may define a subset S of N to be C-convex if its intersection with every coset of the center $C(N)$ of N is convex or in other words if $x^{-1}(S \cap x\,C(N))$ is a convex subset of $C(N)$ for every $x \in N$.

The support of a function will mean the set of points where it is non-zero (this is a departure from the usual usage). When we say a function has compact support, we mean that its support is contained in some compact set (this is the usual usage). Supp f will denote the support of f, a complex valued C^∞ function. Z will denote a central one parameter subgroup of N, and $\varkappa$ will denote a generator of the Lie algebra of Z. Thus $\varkappa$ is a biinvariant vector field on N. We denote a Haar measure on Z by $d\mu(z)$. If f is a C^∞ function of compact support on N then $\tilde{f}$ will denote the function on N/Z defined by $\tilde{f}(xZ) = \int_Z f(xz)\,d\mu(z)$. We note that if $\tilde{f} \equiv 0$ then there is a C^∞ function u of compact support on N such that $\varkappa u = f$. We denote the natural projection of N onto N/Z by π and remark that the inverse image under π of a C-convex subset of N/Z is C-convex. P will denote a biinvariant differential operator on N. We shall identify the algebra of left invariant differential operators on N with the complexified universal envelopping algebra $U(\mathfrak{N})$ of $\mathfrak{N}$. Following Trèves [10], we say that a subset S of N is P-full if whenever $Pu = f$ is a C^∞ function of compact support whose support is contained in S, and u has compact support, then the support of u is contained in S. Since P is biinvariant, any (left or right) translate of a P-full set is P-full. A C-convex set is $\varkappa$-full for any biinvariant vector field $\varkappa$.

A C^∞ function f on N will be called Z-invariant if $f(xz) = f(x)$ for all $x \in N$ and all $z \in Z$. When P is biinvariant differential operator on N, then the action of P on Z-invariant functions defines a differential operator on N/Z, denoted $\tilde{P}$. By "differentiating under the integral", we have $\widetilde{Pu} = \tilde{P}\tilde{u}$ for a C^∞ function u of compact support on N.

We begin with some preparatory lemmas.

Lemma 1. *Let Z be a central one parameter subgroup of N, and let $\varkappa$ be a generator of the Lie algebra of Z. Let D be a left invariant differential operator on N which annihilates all Z-invariant functions. Then $D = D_1 \circ \varkappa$ where D_1 is some left invariant operator on N. If D is biinvariant, so is D_1.*

Proof. Let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ be a basis of the Lie algebra of N/Z and let $x_1, x_2, \ldots, x_n, \varkappa$ be a basis of the Lie algebra $\mathfrak{N}$ of N such that the projection of x_i onto the Lie algebra of N/Z is $\tilde{x}_i$. The Poincaré-Birkhoff-Witt theorem implies that monomials of the form $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \varkappa^k$, with $a_1, a_2, \ldots, a_n$, k non-negative integers form a basis of the vector space of left invariant differential operators on N so that we may write

$$D = \sum_{(a_1 \ldots a_n, k)} C_{(a_1, a_2 \ldots a_n, k)} x_1^{a_1} \ldots x_n^{a_n} \varkappa^k$$

where the sum runs over $(n+1)$-tuples of non-negative integers, and all but finitely many of the $C_{(a_1 \ldots a_n, k)}$ are zero. The action of D on Z-invariant functions defines an operator $\tilde{D}$ on N/Z and we have

$$0=\tilde{D}=\sum_{\substack{(a_1\dots a_n,k)\\ k=0}} C_{(a_1,\dots,a_n,k)}\tilde{x}_1^{a_1}\tilde{x}_2^{a_2}\dots\tilde{x}_n^{a_n}.$$

The Poincare-Birkhoff-Witt theorem now implies that $C_{(a_1,a_2,\dots,a_n,k)}=0$ whenever $k=0$ so that we may write

$$\begin{aligned} D&=\sum_{\substack{(a_1\dots a_n,k)\\ k>0}} C_{(a_1\dots a_n,k)}x_1^{a_1}\dots x_n^{a_n}z^k\\ &=\sum_{\substack{(a_1\dots a_n,k)\\ k>0}} C_{(a_1\dots a_n,k)}x_1^{a_1}\dots x_n^{a_n}z^{k-1}\circ z=D_1\circ z\end{aligned}$$

where $D_1=\sum_{\substack{(a_1\dots a_n,k)\\ k>0}} C_{(a_1\dots a_n,k)}x_1^{a_1}\dots x_n^{a_n}z^{k-1}$.

Now suppose D is biinvariant and let ρ_g denote right translation by $g\in N$. We then have

$$D_1\circ z=D=\rho_g D=\rho_g(D_1\circ z)=(\rho_g D_1)\circ(\rho_g z)=\rho_g D_1\circ z$$

so that $(\rho_g D_1-D_1)\circ z=0$.

But the universal envelopping algebra has no divisors of zero and $z\neq 0$ so $\rho_g D_1-D_1=0$ and $\rho_g D_1=D_1$. Therefore D_1 is biinvariant.

Lemma 2. *If u has compact support on N, then $\pi\operatorname{supp} z u=\pi\operatorname{supp} u$.*

Proof. Since $\operatorname{supp} z u\subset\operatorname{supp} u$, clearly $\pi\operatorname{supp} z u\subset\pi\operatorname{supp} u$. Now let $x\in\operatorname{supp} u$ so that $\pi(x)\in\pi\operatorname{supp} u$. Define

$$\phi\colon Z\to\mathbb{C}\quad\text{by } \phi(z)=u(xz).$$

ϕ is a non-zero function on Z of compact support so $z\phi$ is non-zero of compact support on Z. But $zu(xz)=z\phi(z)$ so zu is not identically zero on xZ so $\pi(x)\in\operatorname{supp} z u$.

Proposition 1. *If K is a P-full set in N/Z, then $L=\pi^{-1}(K)$ is a P-full set in N.*

Proof. Let b be a smooth function of compact support on Z with $\int_z b(z)\,d\mu(z)=1$.

Let $\sigma\colon N/Z\to N$ be a continuous map satisfying $\pi\circ\sigma=\mathrm{Id}_{N/Z}$.

For any complex function f of compact support on N define $f^*\colon N\to\mathbb{C}$ by

$$f^*(x)=f(x)-\tilde{f}(\pi(x))\cdot b(x\cdot(\sigma(\pi(x)))^{-1}).$$

Then

$$\begin{aligned}\int f^*(xz)\,d\mu(z)&=\int f(xz)\,d\mu(z)-\int\tilde{f}(\pi(xz))\cdot b(xz\cdot(\sigma(\pi(xz)))^{-1})\,d\mu(z)\\ &=\tilde{f}(\pi(x))-\int\tilde{f}(\pi(x))\cdot b(xz(\sigma(\pi(x)))^{-1})\,d\mu(z)\\ &=\tilde{f}(\pi(x))-\tilde{f}(\pi(x))\int b(xz(\sigma(\pi(x)))^{-1})\,d\mu(z)\\ &=0.\end{aligned}$$

Therefore, there is a function $f^{\natural}$ of compact support on N satisfying $\varkappa f^{\natural}=f^*$. Let f be a function of compact support on N whose support is contained in L. Let $Pu=f$, where u is also a function of compact support. Define inductively $u_0=u$ and $u_{n+1}=u_n^{\natural}$. We have $\pi\operatorname{supp}Pu_{n+1}=\pi\operatorname{supp}Pu_n^{\natural}=\pi\operatorname{supp}\varkappa Pu_n^{\natural}=\pi\operatorname{supp}P\varkappa u_n^{\natural}=\pi\operatorname{supp}Pu_n^*\subset\pi\operatorname{supp}Pu_n\cup\operatorname{supp}\tilde{u}_n$. If $\pi\operatorname{supp}Pu_n\subset K$ then $\operatorname{supp}\tilde{u}_n\subset K$ since then $K\supset\operatorname{supp}\widetilde{Pu_n}=\operatorname{supp}\tilde{P}\tilde{u}_n$ and K is $\tilde{P}$-full. Therefore if $\pi\operatorname{supp}Pu_n\subset K$ then $\pi\operatorname{supp}Pu_{n+1}\subset K$ and also $\operatorname{supp}\tilde{u}_n\subset K$. But $\pi\operatorname{supp}Pu_0\subset K$ so by induction on n we have $\pi\operatorname{supp}Pu_n\subset K$ and $\operatorname{supp}\tilde{u}_n\subset K$, for all n.

Furthermore $\pi\operatorname{supp}(u_n^*-u_n)\subset\operatorname{supp}\tilde{u}_n\subset K$ and $\pi\operatorname{supp}u_{n+1}=\pi\operatorname{supp}\varkappa u_{n+1}=\pi\operatorname{supp}u_n^*$.

Suppose now that $x\notin L$. On the set xZ we have

$$u_n^*(xz)=u_n(xz) \quad \text{and} \quad \varkappa u_{n+1}(xz)=u_n^*(xz)$$

since xZ is disjoint from $L=\pi^{-1}(K)$. So on xZ we have $\varkappa u_{n+1}=u_n$ and $\varkappa^n u_n=u_0=u$. Therefore, if $\phi_n(z)=u_n(xz)$, then ϕ_0 is a function of compact support on Z such that for arbitrary n there exists a function ϕ_n of compact support on Z such that $\varkappa^n\phi_n=\phi_0$. Applying the Fourier transform to ϕ_0, we see that $\hat{\phi}_0$ is a real analytic function on the dual $\hat{Z}$ of Z with a zero of arbitrary high order at $0\in\hat{Z}$. Therefore $\hat{\phi}_0\equiv 0$ and $\phi_0\equiv 0$. Therefore $u(xz)=0$ for all z and $x\notin\operatorname{supp}u$. QED.

Theorem 1. *Let P be a non-zero biinvariant differential operator on a simply connected nilpotent Lie group N. Then any compact set of N is contained in a compact C-convex P-full subset of N.*

Proof. The proof is by double induction on the dimension of N and the degree of P, the assertion being trivial if the dimension of N or the degree of P is $\leqq 1$. We, therefore, suppose the theorem true whenever the dimension of the nilpotent group is $<n=\dim N$ or the degree of the operator is $<p=\text{degree } P$.

If Z is a one parameter central subgroup of N, the action of P on Z-invariant functions gives rise to a differential operator $\tilde{P}$ on N/Z satisfying $Pf(x)=\tilde{P}\check{f}(\pi(x))$ whenever $f(x)=\check{f}(\pi(x))$ where $\check{f}$ is a function on N/Z and π: $N\to N/Z$ is the natural projection. If $\tilde{P}\equiv 0$ it follows from lemma 1 that $P=\varkappa\circ P_1$ where $\varkappa$ is a generator of the Lie algebra of Z and P_1 is a biinvariant operator on N. Since degree $P_1=p-1$ any compact set of N is contained in a P_1-full compact C-convex subset K of N which is also $\varkappa$-full since this is the case for any C-convex subset of N. Now if $Pf=u$ where f and u are compactly supported functions on N with $\operatorname{supp}u\subset K$, then $Pf=\varkappa\circ P_1 f=u$ so $P_1 f$ is supported in K since K is $\varkappa$-full and f is supported in K since K is P_1-full. Thus the induction is valid whenever P annihilates all Z-invariant functions. Thus we can assume that whenever Z is a one-parameter central subgroup of N, the differential operator $\tilde{P}$ on N/Z induced by the action of P on Z-invariant functions is non-zero and, therefore, by inductive hypothesis that any compact subset of N/Z is contained in a $\tilde{P}$-full compact C-convex subset of N/Z.

The remainder of the proof is divided into two cases, viz.

Case 1. The center of N has dimension 1.

Case 2. The center of N has dimension $\geqq 2$.

We deal with Case 1 first. Let Z be the center of N, and let z be a generator of the Lie algebra of Z. Since the center of N/Z is non-trivial, we can find a vector $y \in \mathfrak{N}$, the Lie algebra of N such that for all $x \in \mathfrak{N}$, we have $[y, x] = \phi(x) z$ where ϕ is a non-zero linear functional on $\mathfrak{N}$. Also $[y[x_1, x_2]] = [[y, x_1] x_2] + [x_1, [y, x_2]] = [\phi(x_1) z, x_2] + [x_1, \phi(x_2) z] = 0$ so $\phi([x_1, x_2]) = 0$ and ϕ vanishes on the derived algebra of $\mathfrak{N}$. The kernel $\mathfrak{M}$ of ϕ is, therefore, a codimension one ideal of $\mathfrak{N}$ and we let $M = \exp \mathfrak{M}$ which is a simply connected nilpotent Lie subgroup of N with Lie algebra $\mathfrak{M}$. We pick $w \in \mathfrak{N}$ with $\phi(w) = 1$. Let i: $U(\mathfrak{M}) \to U(\mathfrak{N})$ be the inclusion of envelopping algebras induced by the inclusion of $\mathfrak{M}$ in $\mathfrak{N}$.

By the Poincaré-Birkhoff-Witt theorem we can write P uniquely as $P = w^k \circ i(p_0) + w^{k-1} \circ i(p_1) + \cdots + w \circ i(p_{k-1}) + i(p_k)$ where the p_i's are elements of $U(\mathfrak{M})$ then

$$0 = [y, P] = [k w^{k-1} \circ i(p_0) + (k-1) w^{k-2} \circ i(p_1) + \cdots + i(p_{k-1})] \circ z.$$

This implies, again by the Poincaré-Birkhoff-Witt theorem that $0 = p_0 = p_1 = \cdots = p_{k-1}$ and, therefore, that $P = i(p_k)$. It follows that any subset S of N such that $x^{-1}(S \cap xM)$ is a p_k-full subset of M for all x is a P-full subset of N. Furthermore, since the center of N is contained in M, if $x^{-1}(S \cap xM)$ is a C-convex subset of M for all $x \in N$, then S is a C-convex subset of N.

We pick a continuous M-equivariant projection ψ: $N \to M$ for instance

$$\psi(x) = x[\exp \phi(-\log x) w].$$

Now let K be a compact subset of N. By inductive hypothesis $\psi(K)$ is contained in a compact C-convex p_k-full subset L of M. Also $\phi(\log K)$ is contained in a compact connected interval I of $\mathbb{R}$. Now $\exp \phi^{-1}(I) \cap \psi^{-1}(L)$ is a compact C-convex P-full subset of N. This completes the proof of case 1.

Case 2. The center of N has dimension greater than 1. Let z_1 and z_2 be vectors in the center of $\mathfrak{N}$ which are orthonormal for a Euclidean metric ρ on $\mathfrak{N}$. Let $Z_1 = \exp \mathbb{R} z_1$ respectively $Z_2 = \exp \mathbb{R} z_2$, and let π_1 respectively π_2 be the projections of N on N/Z_1, respectively N/Z_2. Also let P_1 respectively P_2 be the differential operators on N/Z_1 respectively N/Z_2 induced by the action of P on Z_1-invariant respectively Z_2-invariant functions on N. We can assume that neither P_1 nor P_2 is the zero operator. Let K be a compact subset of N. Then $\pi_1(K)$ and $\pi_2(K)$ are compact subsets of N/Z_1 and N/Z_2 and by inductive hypothesis we can choose $F_1 \supset \pi_1(K)$ and $F_2 \supset \pi_2(K)$ such that F_i is a P_i-full compact C-convex subset of N/Z_i. Then $\pi_i^{-1}(F_i)$ is a C-convex P-full subset of N for $i = 1, 2$ by Proposition 1 and, therefore, $Q = \pi_1^{-1}(F_1) \cap \pi_2^{-1}(F_2)$ is a C-convex closed P-full subset of N containing K.

We assert Q is compact, or equivalently that $\log Q$ is compact. Let ρ_1 and ρ_2 be the Euclidean metrics induced by ρ on $\mathfrak{N}_1 \cong z_1^\perp$ and $\mathfrak{N}_2 \cong z_2^\perp$, the Lie algebras of N/Z_1 and N/Z_2. We can find a real number r such that the ρ_i distance of $\log F_i$ from the origin of $\mathfrak{N}_i$ is $\leqq r$ for $i = 1, 2$. Then if $v \in \log(\pi_1^{-1}(F_1) \cap \pi_2^{-1}(F_2))$ we have $\rho(v, \mathbb{R} z_1) \leqq r$ and $\rho(v, \mathbb{R} z_2) \leqq r$ so we can choose t_1 and t_2 such that $\rho(v, t_1 z_1) \leqq r$ and $\rho(v, t_2 z_2) \leqq r$. Then $\rho(t_1 z_1, t_2 z_2) \leqq 2r$ so $\sqrt{t_1^2 + t_2^2} \leqq 2r$ so

$t_1^2 \leqq 4r^2$ and $\rho(t_1 x_1, 0) = |t_1| \leqq 2r$. It follows that $\rho(v, 0) \leqq 3r$ for all $v \in \log(\pi_1^{-1}(F_1) \cap \pi_2^{-1}(F_2))$. Thus $\log(\pi_1^{-1}(F_1) \cap \pi_2^{-1}(F_2))$ is a closed bounded subset of $\mathfrak{N}$ and, therefore, compact. Therefore, $\pi_1^{-1}(F_1) \cap \pi_2^{-1}(F_2)$ is a compact C-convex P-full subset of N containing K. This completes the inductive step in Case 2 and concludes the proof of the theorem.

Corollary. *If K is any compact set in N, then there is a compact set L such that whenever $Pu = f$ is a distribution supported in K and u is a distribution of compact support on N, then the support of u is contained in L.*

Proof. This follows immediately from the theorem upon convoluting with a smooth approximate identity of N. Here L can be any compact P-full set containing a compact neighborhood of K.

Theorem 2. *Any non-zero biinvariant differential operator on a connected simply connected nilpotent Lie group is globally solvable.*

Proof. Semi-global solvability of such operators is contained in results of Rais [8] or Rouvière [9]. But by theorem 1.9 in the book of Trèves [11], global solvability follows from semi-global solvability and the P-convexity result proved above.

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