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Stability of vector bundles and extremal metrics

D. Burns* and P. de Bartolomeis

Department of Mathematics, The University of Michigan, Ann Arbor, MI 48109, USA
Istituto di Matematica Applicata, Università degli Studi di Firenze, I-50139 Firenze, Italia

It has been known for some time now that not every compact kähler manifold of positive first Chern class admits a kähler-einstein metric, or even a kähler metric of constant scalar curvature. This is due to structure theorems of Matsushima and Lichnerowicz on the algebra of holomorphic vector fields on M. For a summary, cf. [1]. Such metrics are special examples of the so-called extremal metrics of Calabi, obtained by fixing the fundamental class $[\omega]\in H^2(M, \mathbb{R})$, and looking for critical points g of the functional

$$I(g)=\int_M R^2\,\mathrm{dvol}$$

where g runs over kähler metrics with the given fundamental class and the scalar curvature and volume element are computed with respect to g. The Euler-Lagrange equations for $I(g)$ can be expressed as

$$\bar{\partial}(\mathrm{grad}^{(1,0)}(R))=0,$$

that is, the (1, 0)-component of the gradient of the scalar curvature is a holomorphic vector field. The problem of finding extremal metrics is quite natural but quite difficult. Extremal metrics should be easier to find than kähler-einstein metrics or metrics of constant scalar curvature. Nevertheless, Calabi has proved some (weaker) structure theorems for the algebra of holomorphic vector fields on an M with an extremal kähler metric, and M. Levine [8] has shown that these conditions are sufficient to obstruct the existence of an extremal metric on some M with the "wrong kinds" of algebras. In a different direction, Futaki has studied the very interesting interrelationship between the algebra of holomorphic vector fields and the given kähler class $[\omega]$ which was fixed in the definition above.

In this note, we give examples of ruled surfaces M which have no non-trivial holomorphic vector fields, and yet which admit no extremal kähler metric in a specifically given kähler class. For such an example, an extremal metric would

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necessarily be a metric of constant scalar curvature, and the obstruction found here in new in that context as well. The obstruction involves the borderline semi-stability properties of hermitian vector bundles with hermite-einstein connections (cf., e.g., [7, 9]). We came across these examples as an empirical off-shoot of our work on the integrability of twistor spaces over four-manifolds (cf. [2]). We have not been able to digest a simple general principle from the calculations, but it is clear that the borderline stability properties play the key role.

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To construct the examples, let C be a compact Riemann surface of genus $g \geqq 2$. Consider the complex surface $S_0 = C \times \mathbb{P}^1$, and give S_0 the kähler metric g_0, the product of the metric of constant curvature -1 on C and that of constant curvature $+1$ on $\mathbb{P}^1$. It is easy to see that this metric has scalar curvature $R \equiv 0$.

We write S_0 in terms of vector bundles over C in the obvious way, namely, $S_0 = \mathbb{P}(E_0)$, where $E_0 = C \times \mathbb{C}^2$. We will deform E_0 in order to construct new ruled surfaces over C. Write E_0 as an extension of two trivial line bundles over C:

$$0 \to L_0 \to E_0 \to L_0 \to 0, \qquad L_0 = C \times \mathbb{C}.$$

Since g is non-zero, one can deform L_0 slightly to a line bundle L over C such that $L^{\otimes 2}$ is non-trivial. Simultaneously, one can deform the trivial extension above to an extension

$$0 \to L \to E \to L^* \to 0 \qquad (*)$$

over C, where L^* denotes the dual bundle of L. Since $g \geqq 2$, $H^1(C, \mathcal{O}(L^{\otimes 2}))$ is non-zero, and we can assume that (*) doesn't split. Let S be the ruled surface $\mathbb{P}(E)$ over C.

Since S is a small, continuous perturbation of S_0, we can identify the topological cohomology groups $H^2(S_0, \mathbb{Z})$ and $H^2(S, \mathbb{Z})$, and under this identification, $c_1(S_0) = c_1(S)$. We let ω_0 denote the kähler form of g_0 on S_0, and note that by the stability of kähler metrics, if L is close enough to L_0 in Pic(C) and (*) is close enough to the trivial extension $0 \in H^1(C, \mathcal{O}(L^{\otimes 2}))$, then the class $[\omega_0]$ in $H^2(S_0, \mathbb{R}) = H^2(S, \mathbb{R})$ is again a kähler class. We are finally in a position to state our theorem.

Theorem. *If $S = \mathbb{P}(E)$ is a sufficiently small perturbation of S_0 such that (*) doesn't split and $L^{\otimes 2}$ is non-trivial, then*

(i) S does not admit an extremal kähler metric g whose kähler class $= [\omega_0]$ in $H^2(S, \mathbb{R})$;

(ii) there are no non-trivial holomorphic vector fields on S.

Proof. The proof is by contradiction. The proof proceeds by a succession of simple observations. We first note that it suffices to prove the theorem with statement (i) replaced by:

(i)′ S does not admit a kähler metric of constant scalar curvature R with kähler class $[\omega_0]$ in $H^2(S, \mathbb{R})$.

Indeed, the Euler-Lagrange equation for an extremal metric is that

$$\bar{\partial}(\mathrm{grad}^{(1,0)}(R))=0,$$

and thus $\mathrm{grad}^{(1,0)}(R)$ is a holomorphic vector field, and by statement (ii) of the theorem, must be zero. Hence R must be constant.

Lemma 1. *Let g be a kähler metric on S with kähler form ω and scalar curvature R. If $[\omega]=[\omega_0]$, and R is constant, then $R\equiv 0$.*

Proof. For any compact kähler manifold M of constant scalar curvature, one can calculate R cohomologically:

$$\int_M c_1(M)\wedge\omega^{n-1}=\frac{(n-1)!}{\pi}\int_M R\,\mathrm{dvol}$$
$$=\frac{R}{\pi n}\int_M \omega^n,$$

where $n=\dim_{\mathbb{C}} M$. For our S, since $[\omega]=[\omega_0]$, $c_1(S)=c_1(S_0)$, we get that $R=R_0=0$.

Lemma 2. *Let g be a kähler metric on S with $R\equiv 0$ and $[\omega]=[\omega_0]$. Then g is conformally flat, and the universal cover $\tilde{S}$ of S, with the induced metric $\tilde{g}$, is holomorphically isometric to $\tilde{S}_0=\Delta\times\mathbb{P}^1$, equipped with the induced product metric. Here $\Delta=$ the unit disk.*

Proof. Most of this was proved in [2], but we recall briefly the argument. One denotes by W_+, W_- the self-dual and anti-self-dual components of the Weyl conformal curvature tensor of g. For a kähler surface, $R\equiv 0$ if and only if $W_+\equiv 0$. Furthermore, the signature $\sigma(S)$ is S is given by

$$\sigma(S)=\frac{1}{48\pi^2}\int_S \{|W_+|^2-|W_-|^2\}\,\mathrm{dvol},$$

and since $\sigma(S)=\sigma(S_0)=0$, $W_-\equiv 0$. Thus g is conformally flat, and more precisely, due to Theorem 1 of Derdzinski [5], g is locally Hermitian symmetric. A quick glance at the (topological) possibilities shows that $\tilde{S}$ must be $\Delta\times\mathbb{P}^1$, as claimed. The volume of S and $R\equiv 0$ fix the two constants in the Hermitian symmetric metric.

At this point we conclude that S is a unitary, flat $\mathbb{P}^1$-bundle over C. That is, one has a homomorphism $\rho\colon \Gamma\to PSU(2)$, where $\Gamma=\pi_1(C)=\pi_1(S)$, and $PSU(2)$ is the isometry group of $\mathbb{P}^1$. On the other hand, $S\cong\mathbb{P}(E)$, where E is uniquely determined up to tensoring with a holomorphic line bundle. One thus concludes that

(a) ρ lifts to a homomorphism $\tilde{\rho}\colon\Gamma\to SU(2)$;

(b) the lifting $\tilde{\rho}$ can be chosen so that E is isomorphic to the associated flat, unitary bundle $E(\tilde{\rho})$ over C.

(These are because $\Lambda^2 E\cong L\otimes L^*$ is trivial). Thus our E admits a hermitian metric with a compatible flat connection.

Finally, we return to (*). Since $\Lambda^2 E \cong L \otimes L^*$, one has $\deg E = 0$. Since $\deg L = 0$ as well, by the borderline case of the theorem of Kobayashi-Lübke (cf. [7, 9]), E must split holomorphically and metrically as a direct sum $L \oplus L^*$ over C. This contradicts the assumption that (*) doesn't split, thereby proving part (i)′ of the theorem.

Part (ii) of the theorem is a standard cohomological calculation, which we include for the convenience of the reader. Let $\pi: S \to C$ be the projection, TS, TC the holomorphic tangent bundles of S, C respectively, and TF the line bundle over S of (holomorphic) tangents along the fibers of π. One has the usual exact sequence of vector bundles over S:

$$0 \to TF \to TS \to \pi^*(TC) \to 0.$$

We wish to show $H^0(S, \mathcal{O}(\pi^* TS)) = 0$.

(A) $$\begin{aligned} H^0(S, \mathcal{O}(\pi^* TC)) &\cong H^0(S, \pi_*(\mathcal{O}(\pi^*(TC))) \\ &\cong H^0(C, \mathcal{O}(TC)) \\ &= 0, \text{ since } g \geqq 2. \end{aligned}$$

(B) As above, $H^0(S, \mathcal{O}(TF)) = H^0(S, \pi_* \mathcal{O}(TF))$. It is clear that $\pi_* \mathcal{O}(TF)) \cong \mathcal{O}(sl(E))$ on C, where $sl(E)$ is the bundle of traceless endomorphisms of E. For any $\varphi \in H^0(C, \mathcal{O}(sl(E)))$, let χ be the composition

$$L \longrightarrow E \xrightarrow{\varphi} E \longrightarrow L^*.$$

Since χ is a section of $(L^*)^{\otimes 2}$, $\chi = 0$, since $\deg L^* = 0$, and $(L^*)^{\otimes 2}$ is non-trivial. Thus, every $\varphi \in H^0(S, \mathcal{O}(sl(E)))$ takes L to itself. The restriction of φ to L must be identically zero, since otherwise the sequence (*) would split according to the eigenspaces of φ. Thus, φ must induce the zero map on L^* as well, since trace$(\varphi) = 0$, and φ therefore factors through $E \to L^*$ and has its image in L. But by the same argument as above, the induced homomorphism from L to L^* is trivial, since $L^{\otimes 2}$ is non-trivial and of degree 0. Thus, $\varphi = 0$, proving part (ii) of the theorem.

We conclude this note with two remarks. First, if the curve C has no non-trivial automorphisms, then S has no non-trivial automorphisms. Secondly, the phenomenon above is sometimes generic, in the sense that the surfaces above form an open set in moduli, e.g., if the genus g of the base curve is 2.

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