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PROJECTIVE CONNECTIONS IN CR GEOMETRY

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Holomorphic invariants of an analytic real hypersurface in $\mathbb{C}^{n+1}$ can be computed by several methods, coefficients of the Moser normal form [4], pseudo-conformal curvature and its covariant derivatives [4], and projective curvature and its covariant derivatives [3]. The relation between these constructions is given in terms of reduction of the complex projective structure to a real form and exponentiation of complex vectorfields to give complex coordinate systems and corresponding Moser normal forms. Although the results hold for hypersurfaces with non-degenerate Levi-form, explicit formulas will be given only for the positive definite case.

Introduction

Let M be a real hypersurface in $\mathbb{C}^{n+1}$ with non-degenerate Levi form. Chern [4] and Tanaka [9] have shown how to associate to M a principal bundle Y_M with Cartan connection such that a local diffeomorphism $f: M \to M'$ is the boundary value of a holomorphic mapping if and only if f lifts to a connection preserving map $Y_M \to Y_{M'}$. The bundle Y_M is called the pseudoconformal or CR structure bundle. For a real analytic hypersurface (see remark at the end of the paper concerning C^∞ hypotheses) Chern [3] has also associated another structure bundle, with Cartan connection the projective

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structure bundle R_M. Let H_zM be the maximal complex subspace of the real tangent space $T_zM \subset T_z\mathbb{C}^{n+1}$ and let $HM = \cup_{z\in M} H_zM$. Each H_zM determines a point in the complex projective space $\mathbb{P}(T_z\mathbb{C}^{n+1})$, which is naturally identified with $\mathbb{P}(\mathbb{C}^{n+1}) = \mathbb{P}_n$. In this way we consider M as a submanifold of $\mathbb{C}^{n+1} \times \mathbb{P}_n$ and R_M is a principal bundle over a neighborhood of M in $\mathbb{C}^{n+1} \times \mathbb{P}_n$. As in the case of Y_M a local real-analytic diffeomorphism $f: M \to M'$ is the restriction of a holomorphic mapping if and only if it lifts to a connection preserving mapping $R_M \to R_{M'}$. The construction of R_M uses the complexification of M and it is reasonable to expect that R_M is the complexification of Y_M. We will show that this is indeed true and one can use R_M to define holomorphic coordinate systems in a neighborhood of any point of M, in which coordinates the defining function of M reduces to a normal form differing only slightly from Moser's normal form [4]. One can readily adjust the normalization of the Cartan connection so that the associated normal form is exactly Moser's. This new proof of Moser's theorem establishes the exact relation of the curvature functions on Y_M to the coordinates of the normal form. This may be used in studying the relation of the linear representation of H, the structure group of Y_M, on the curvature functions and the complicated nonlinear representation of H on the coefficients of the normal form. The result also completes the set of relations between biholomorphic invariants of real hypersurfaces computed from different constructions-pseudoconformal, projective, Kähler (using Monge-Ampère), Lorentz, and normal forms. See [2], [6], [10]. Using other methods from the theory of projective connections, Faran [5] has established the same result on normal forms, as well as several other results on the existence of projectively equivalent but pseudoconformally inequivalent hypersurfaces.

1.

We will quickly review the construction of the bundles R_M and Y_M, for details see [3], [4]. Let $T_z^*\mathbb{C}^{n+1}$ be the real linear functionals on $T_z\mathbb{C}^{n+1}$. The complex vector space of complex valued functionals linear over the real field can be identified with $T_z^*\mathbb{C}^{n+1} \otimes_{\mathbb{R}} \mathbb{C}$, which decomposes as a direct sum of $T^{(1,0)}$ and $T^{(0,1)}$, the complex linear and conjugate linear functionals respectively. Assume M is defined by

$$r(z,\bar{z}) = 0, \; z \in \mathbb{C}^{n+1}$$

then decomposing dr by type, on T_zM

(1) $$dr = \partial r + \bar{\partial} r = 0,$$

thus the complex linear functional ∂r is purely imaginary on T_zM. The null space of ∂r, denoted H_zM, is a complex subspace of real codimension one of T_zM, where the complex structure arises from the natural identification $T_z\mathbb{C}^{n+1} = \mathbb{C}^{n+1}$. The annihilator of H_zM in T_z^*M is a real line generated by the form $i\partial r$

which is real valued on T_zM. The set of all these lines defines a line bundle E, which is subbundle of T_z^*M. The restriction of the canonical one form on T^*M to E will be denoted by ω and for $p \in E$ with $\pi(p) = z \in M$, π is the projection $\pi : E \to M$,

$$(2) \qquad \omega_p = t\pi^*(i\partial r)_p$$

for some real number t. To define Y_M consider the real coframes on E $\{\omega, \mathrm{Re}\,\omega^\alpha, \mathrm{Im}\,\omega^\alpha, \varphi\}$ which satisfy the equation

$$(3) \qquad d\omega = i\,\Sigma_{\alpha=1\ldots n}\,\omega^\alpha \wedge \bar{\omega}^\alpha + \varphi \wedge \omega$$

and such that

$$(4) \qquad \omega_p^\alpha = \mathrm{Re}\,\omega_p^\alpha + i\,\mathrm{Im}\,\omega_p^\alpha = \pi^*(\theta_z^\alpha)$$

for some complex one form $\theta_z^\alpha \in T_z^*M \otimes \mathbb{C}$ which is complex linear on H_zM. This defines Y as a reduction of the frame bundle of E.

Next we define R_M. Let $\mathbb{P}(T_z\mathbb{C}^{n+1})$ be the complex projective space of complex hyperplanes in $T_z\mathbb{C}^{n+1}$, or dually the complex projective space of complex lines in the complex dual space $T^{(1,0)}$. The hyperplane H_zM determines a unique point $\tau(z) \in \mathbb{P}(T_z\mathbb{C}^{n+1})$. In the dual construction this point is represented by the complexification of the fiber of E over z, which fiber is a real line in $T^{(1,0)}$. Let U be a neighborhood of $z_0 \in M$ where $\frac{\partial r}{\partial z^{n+1}} \neq 0$ and set $w = z^{n+1}$. Further restrict U so that on U the equations

$$(5) \qquad r(z,a) = 0$$

$$r_{z^\alpha}(z,a) + p_\alpha r_w(z,a) = 0 \qquad \alpha = 1,\ldots,n ,$$

define $a = (a^j)$ $j = 1,\ldots,n+1$, uniquely as functions of (z^α, w, p_α) near $(z_0^\alpha, w_0, p_{\alpha 0})$ where

$$p_{\alpha 0} = -r_{z^\alpha}(z_0,\overline{z_0}) \Big/ r_w(z_0,\overline{z_0})$$

$$a^j(z_0,p_{\alpha 0}) = \overline{z_0^j} .$$

The existence of U and of functions a^j is guaranteed by the implicit function theorem and the non-degeneracy of the Levi-form of M at Z_0. If the p_α are interpreted as affine fiber coordinates in $\mathbb{P}(T_z\mathbb{C}^{n+1})$ the equations a^j = constant define a foliation of an open neighborhood of $\tau(U \cap M)$ in $\mathbb{P}(T\mathbb{C}^{n+1})$. The bundle R_M is a reduction of the coframe bundle of $T^{(1,0)}$ which is compatible with this foliation.

The foliation is also defined by the differential system

$$(6) \qquad dw - p_\alpha dz^\alpha = 0 \text{ (summation convention)}$$

$$dp_\alpha - r_{\alpha\beta} dz^\beta = 0$$

where

$$r_{\alpha\beta} = \frac{1}{r_w^3}(r_{z^\alpha z^\beta} r_w^2 - r_{wz^\alpha} r_{z^\beta} + r_{wz^\beta} r_{z^\alpha}) + r_{ww} r_{z^\alpha} r_{z^\beta})$$

as a function of (z^α, w, p_α), near $\tau(U \cap M)$.

Define

$$\theta = u(dw - p_\alpha dz^\alpha) \tag{7}$$

$$\theta^\alpha = u^\alpha_\beta dz^\beta + u^\alpha(dw - p_\beta dz^\beta)$$

$$\theta_\alpha = v^\beta_\alpha(dp_\beta - r_{\beta\gamma} dz^\gamma) + v_\alpha(dw - p_\beta dz^\beta) \ .$$

The distribution $\{\theta, \theta_\alpha\}$ defines the foliation given by (6), the distribution $\{\theta, \theta^\alpha\}$ defines the foliation by fibers of $\mathbb{P}(T\mathbb{C}^{n+1})$. We will use π for the projection $T^{(1,0)} \to \mathbb{P}(T\mathbb{C}^{n+1})$ unless the context requires a distinction from $\pi: E \to M$. Then $\pi^*\theta$ is a multiple of the canonical complex linear one form ω on $T^{(1,0)}$. We consider coframes of the form $\{\omega, \omega^\alpha, \omega_\alpha, \varphi\}$ where

$$\omega^\alpha = \pi^*\theta^\alpha, \quad \omega_\alpha = \pi^*\theta_\alpha \quad \text{and}$$

$$d\omega = i \textstyle\sum_{\alpha=1,\ldots n} \omega^\alpha \wedge \omega_\alpha + \varphi \wedge \omega \ . \tag{8}$$

What are the structure groups of $Y \to E$ and $R \to T^{(1,0)}$? Consider $\mathbb{P}Sl(n+2, \mathbb{C})$ acting on $\mathbb{P}_{n+1}$. Let L be the isotropy group of the point with homogeneous coordinates $[0,0,\ldots,1]$. Let K be the intersection of L with the subgroup preserving the hyperplane at infinity, $z^1 = 0$. Let H be the intersection of L with the subgroup preserving the real quadric

$$i/_2 (z^1 \overline{z^{n+2}} - \overline{z^1} z^{n+2}) + \sum_{j=2,..n+1} |z^j|^2 = 0 .$$

The bundle R has structure group K as a principal bundle over $T^{(1,0)}$ and Y has structure group H as a principal bundle over E. There are exact sequences of groups

(9)
$$e \to \tilde{N} \to K \rightleftarrows Gl(n,C) \to e$$

$$e \to N \to H \rightleftarrows U(n) \to e$$

where $\tilde{N}$ is the complex Heisenberg group of complex dimension $2n+1$ and N is the real Heisenberg group of real dimension $2n+1$, thus we can write K and H as semi-direct products

(10)
$$K \cong Gl(n,\mathbb{C}) \cdot \tilde{N}$$

$$H \cong U(n) \cdot N \quad .$$

On Y there is a Cartan connection ω_Y with values in $su(n+1,1)$ satisfying

$$R_h^* \, \omega_Y = \mathrm{Ad}(h^{-1})\omega_Y, \quad h \in H. \qquad *$$

On R there is a Cartan connection ω_R with values in $s\ell(n+2,\mathbb{C})$, satisfying

$$R_k^* \omega_R = \mathrm{Ad}(k^{-1})\omega_R \quad k \in K.$$

These connections are uniquely determined if their curvatures are required to satisfy certain trace conditions, see [3], [4]. We note here that R is a holomorphic principal bundle and that ω_R is a holomorphic Cartan connection.

<u>Theorem 1</u>. Let j be the imbedding of E into $T^{(1,0)}$, then there is an imbedding $\tilde{j}: Y \to R$ as a totally real submanifold such that

$$\tilde{j}^* \omega_R = \omega_Y$$

where ω_R and ω_Y are the respective Cartan connections and

$$\begin{tikzcd} Y \arrow[r, "\tilde{j}"] \arrow[d] & R \arrow[d] \\ E \arrow[r, "j"] & T^{(1,0)} \end{tikzcd} \quad \text{commutes.}$$

* R_n denotes right translation along the fiber

Proof. We will show that Y is a reduction of the structure group of j^*R, the pull-back of R over E. A direct calculation shows that for any coframe $\{\omega, \omega^\alpha, \omega_\alpha, \varphi\}$ in R

(10) $$j^*\omega_\alpha = \Sigma\, a_{\alpha\bar\beta}\, \overline{j^*\omega^\beta} \mod j^*\omega$$

for some hermitian matrix $(a_{\alpha\bar\beta})$. Considering those frames for which

(11) $$j^*\omega_\alpha = \overline{j^*\omega^\alpha} \mod j^*\omega$$

that is

(12) $$a_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$$

reduces that $Gl(n,\mathbb{C}) \subset K$ to $U(n)$. That is the set of frames satisfying (11) is a principal bundle with structure group $U(n)\cdot\tilde N$. For such frames we have

(13) $$j^*\omega_\beta = \overline{j^*\omega^\beta} + a_\beta j^*\omega$$

where (a_β) is a $\mathbb{C}^n$-valued function whose restriction to the fiber has complex rank n Set.

(14) $$a_\beta = 0.$$

Then from the structure equation (8) we conclude

$$(15) \qquad j^*\varphi = j^*\bar{\varphi} + sj^*\omega$$

where s is a complex function whose restriction to the fiber has $ds \neq 0$.

(16) Set $$s = 0.$$

The three equations (12), (14), and (16) define a submanifold Y of j^*R which is a principal H bundle, consisting of $\{\omega, \omega^\alpha, \omega_\alpha, \varphi\}$ such that the $2n+2$-tuple $\{j^*\omega,\ \mathrm{Re}\ j^*\omega^\alpha,\ \mathrm{Im}\ j^*\omega^\alpha,\ j^*\varphi\}$ is a real coframe for $T(E)$, and

$$(17) \qquad dj^*\omega = i \sum_\alpha j^*\omega^\alpha \wedge j^*\overline{\omega^\alpha} + j^*\varphi \wedge j^*\omega.$$

Represent the inclusion of Y in R by $\tilde{j}$, and where no confusion will result, identify Y with its image $\tilde{j}Y$. Since $E \subset T^{(1,0)}$ is totally real of maximal dimension, $Y \subset R$ is totally real of maximal dimension.

We want to show that the projective connection ω_R, a Cartan connection of type $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$, restricts to the pseudo conformal connection ω_Y, a Cartan connection of type $(\mathfrak{g}, \mathfrak{h})$, where $\tilde{\mathfrak{g}} = s\ell(n+2,C)$ and $\mathfrak{g} = (su(n+1,1)$. Represent $\tilde{\mathfrak{g}}$ as a real direct sum $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus i\mathfrak{g}$ and for a real linear form ω to a complex linear form on $T_pY + J(T_pY) = T_pR$ by the rule

$$\tilde{\omega}(\xi+J\eta) = \omega(\xi) + i\omega(\eta) \quad . \tag{18}$$

Applying the standard complexification procedure for extending real analytic functions defined on a maximal totally real submanifold to the coefficients of ω_Y we can extend ω_Y to a connected neighborhood U of Y in R. Let $\tilde{\omega}_R$ be the connection on U defined this way. We claim $\tilde{\omega}_R = \omega_R$. This follows from 1) the uniqueness of the normalized connection on R; 2) the fact that the complexification of the curvature of ω_Y is the curvature of the complexification; and 3) the traces of the curvature functions which vanish identically on Y must complexify to holomorphic functions vanishing identically on U. The details can be checked using these observations and comparing the normalization formulas in [3], [4], concluding the proof of the theorem.

Represent the Cartan connection ω_R in matrix form

$$\omega_R = \begin{pmatrix} \omega_0^0 & \omega^\alpha & 2\omega \\ -i\varphi_\beta & \omega_\beta^\alpha & 2i\omega_\beta \\ \psi/4 & \frac{\varphi^\alpha}{2} & \omega_{n+1}^{n+1} \end{pmatrix} \in s\ell(n+2,C) \quad .$$

Then the condition that ω_R restricted to Y takes values in su(n+1,1) implies that on Y:

$$
\begin{aligned}
\omega &= \bar{\omega} \\
\omega_\alpha &= \overline{\omega^\alpha} \\
\omega^\alpha_\beta &= -\overline{\omega^\beta_\alpha} \\
\omega^0_0 &= -\overline{\omega^{n+1}_{n+1}} \\
\varphi_\alpha &= \overline{\varphi^\alpha} \\
\psi &= \bar{\psi} \, .
\end{aligned}
\tag{20}
$$

Let

(21) $\{X_{n+1}, X_\alpha, X^\alpha, X^0, A^\alpha_\beta, B^\alpha, B_\alpha, C\}$ be the dual basis to $\{\omega, \omega^\alpha, \omega_\alpha, \omega^0_0, \omega^\beta_\alpha, \varphi_\alpha, \varphi^\alpha, \psi\}$.

Y is an integral manifold of the not everywhere integrable system (10), or equivalently the distribution

$$
\begin{gathered}
\{X_{n+1}, X_\alpha + X^\alpha, \ J(X_\alpha - X^\alpha), \ X^0, \\
A^\alpha_\beta - A^\beta_\alpha, \ J(A^\alpha_\beta + A^\beta_\alpha), \ B_\alpha + B^\alpha, \\
J(B_\alpha - B^\alpha), \ C\} \, .
\end{gathered}
\tag{22}
$$

In the next section we will use this relation between the fundamental vectorfields on R given by (21) and the vectorfields given by (22), which are tangent to Y, to describe the reduction of the defining function of M to normal form in certain distinguished coordinate systems.

2. In this section we will show that using the exponentials of certain fundamental vectorfields one can define a holomorphic coordinate system on $\mathbb{C}^{n+1}$ in which the defining function of M reduces to a normal form equivalent to Moser's.

Before defining the coordinates and computing the associated defining function we give some definitions. Although R is a complex manifold, the tangent space TR is taken in the sense of a real manifold. Let X be a real vectorfield on R and let $\varphi(X,t)$ be the flow of X at time t; assuming this is defined for time 1, let $\varphi(X,1) = \varphi(X)$. Let J be the almost complex structure tensor. If X is a (real) holomorphic vectorfield, that is $\mathcal{L}_x J = 0$, then

$$s+it \mapsto \varphi(sX+tJX)p$$

defines a holomorphic curve through p. If $X_1,\ldots,X_n$ are (real) holomorphic vectorfields which are linearly independent near p then

$$(23) \qquad (z^1,\ldots z^N) \to \varphi(x^1X_1+y^1JX_1 +\ldots+ x^NX_N+y^NJX_N)p$$

defines a holomorphic imbedding of a neighborhood of 0 in $\mathbb{C}^N$ into R. Since TR consists of tangent vectors with real coefficients, use the following convention, for $z = x + iy$ and $X \in TR$, let $zX = (x+iy)X = xX+yJX$ and write (23) as

(24) $$(z^1,\dots z^N) \to \varphi(z^1X_1 + \dots + z^NX_N)p \ .$$

Let π_1 be the projection of R on $\mathbb{P}(T^{(1,0)})$ and π_2 the projection of $\mathbb{P}(T^{(1,0)})$ on $\mathbb{C}^{n+1}$ and $\pi_3 = \pi_2 \circ \pi_1$. For X_α, X_{n+1} defined at the end of 1.

$$\Phi\colon\ (z^\alpha, w) \to \ \varphi(z^\alpha X_\alpha)\varphi(wX_{n+1})p$$

defines an imbedding of an open set U in $\mathbb{C}^{n+1}$ into R and $\pi_3 \circ \Phi$ defines a local parametrization of $\mathbb{C}^{n+1}$ near $\pi_3(p)$. Note that each submanifold $\pi_1 \circ \Phi$ (w = constant) is a leaf of the foliation used to define R. The coordinates (z^α, w) are called projective Fermi coordinates.

We will compute the defining function of M in projective Fermi coordinates and show where it differs from Moser normal form.

<u>Theorem 2.</u> The projective Fermi coordinates relative to R bases at $p \in Y$ provide holomorphic coordinates on an open set $U \subset \mathbb{C}^{n+1}$ containing $\pi_3(p) \in M$ with respect to which the defining function of M in U is in a modified normal form differing from Moser normal form only in that

$$\operatorname{tr}^3 F_{3\bar{3}} = -\ \left(\frac{2n+1}{n}\right)\| F_{2\bar{2}} \|^2 .$$

<u>Remark.</u> The notation $F_{k\bar{\ell}}$, etc. is the same as in [4] and is recalled below equations (38) and (39), as is the definition of the trace operator.

Proof. Since $M = \pi Y$ and the vectorfields $X_\alpha + X^\alpha$, $J(X_\alpha - X^\alpha)$, and X_{n+1} are tangent to Y and transverse to the fiber, we have

$$\pi_3\varphi(a^\alpha(X_\alpha+X^\alpha)+b^\alpha J(X_\alpha-X^\alpha))\varphi(cX_{n+1})p$$

covers an open set $M\cap V$ in M for (a^α,b^α,c) in some open neighborhood of 0 in $\mathbb{R}^{2n+1}$. The problem is to solve

$$\pi_3\varphi(f^\alpha X_\alpha+\overline{f^\alpha}X^\alpha)\varphi(hX_{n+1})p = \pi_3\Phi(z^\alpha,u+iF) \tag{25}$$

for real analytic functions f^α, h, F of $(z^\alpha,\overline{z^\alpha},u)$, all vanishing at 0. There is no problem of existence or analyticity, which follow from the inverse function theorem, what is being determined is the precise expression. The defining equation of M will be

$$v = F(z^\alpha,\overline{z^\alpha},u) \quad . \tag{26}$$

First look at the corresponding equation on R. For some vectorfield P tangent to the fibers of $\pi_3: R \to \mathbb{C}^{n+1}$, $P = p_\alpha X^\alpha + p_0 X^0 + p^\alpha_\beta A^\beta_\alpha + \tilde{p}^\alpha B_\alpha + \tilde{p}_\alpha B^\alpha + \tilde{p}C$,

$$\varphi(P)\varphi(f^\alpha X_\alpha + \overline{f^\alpha}X^\alpha)\varphi(hX_{n+1})p = \tag{27}$$

$$\varphi(z^\alpha X_\alpha)\varphi((u+iF)X_{n+1})p \ .$$

Letting $Z = z^\alpha X_\alpha$, $W = iFX_{n+1}$, $X = f^\alpha X_\alpha + \overline{f^\alpha X^\alpha}$, $H = (h-u)X_{n+1} = gX_{n+1}$ and $q = \varphi(uX_{n+1})p$, (27) becomes:

$$\varphi(P)\varphi(X)\varphi(H)q = \varphi(Z)\varphi(W)q. \tag{28}$$

The point q is on the integral curve γ of X_{n+1} through p; this curve projects onto a chain in M[4].

Recall that for a real analytic function f, a real analytic vectorfield X and t sufficiently small

$$f(\varphi(X,t)p) = (e^{tX}f)p \ .$$

From this fact we conclude that to satisfy equation (27) for all u sufficiently small, or equivalently (28) for all q near p on γ, we must solve the identity

$$e^W e^Z = e^H e^X e^P \tag{29}$$

where each side may be written as a single exponential, using the Baker-Campbell-Hausdorff (BCH) formula and computing commutators at q. the BCH formula as given in [8] says that there is an identity in the free associative algebra generated by two elements, a,b -

$$e^a e^b = e^c \tag{30}$$

where C is given by a formula involving only commutators of arbitrary order in a,b.

$$(31)\quad c = a + b + \frac{1}{2}[a,b] + \frac{1}{12}[a,[a,b]] + \frac{1}{12}[b,[b,a]] + \ldots\ .$$

To apply this formula to (29) with vectorfields W and Z or H, X and P, we compute the successive brackets at the point q using the structure equation (8) and the invariant equation for the deRham differential

$$2d\lambda(\xi,\eta) = \xi\lambda(\eta) - \eta\lambda(\xi) - \lambda([\xi,\eta]),$$

with λ a one-form and ξ and η vectorfields. The dependence on q is equivalent to dependance on u. Then equating the two exponents determines $F, g, f^\alpha, p_\alpha, \ldots \tilde{p}$ as power series in $z^\alpha, \bar{z}^\alpha$ with coefficients analytic functions of u.

More explicitly rewrite (29) as

$$(32)\qquad e^G e^Z = e^X e^P$$

where $G = -H + W = (-g+iF)X_{n+1} = \tilde{F}X_{n+1}$, then determine the formulas for coefficients in the power series expansion of $\tilde{F}$. The curvature Ω_R of the Cartan connection ω_R describes the difference between brackets of vectorfields computed on R and the brackets of the corresponding left invariant vectorfields on $Sl(n+2,\mathbb{C})$ where the correspondence is given by the Cartan connection on R. The only brackets which differ from those computed on the group are the following

$$[X_\alpha, X_{n+1}] = R^\beta_{\alpha\gamma} A^\gamma_\beta - \frac{i}{2} Q^\gamma_\alpha B_\gamma + P_{\alpha\gamma} B^\gamma + H_\alpha C$$

$$(34) \qquad [X^\alpha, X_{n+1}] = T^{\alpha\gamma}_\beta A^\beta_\gamma - \frac{i}{2} Q^\alpha_\gamma B^\gamma + L^{\alpha\gamma} B_\gamma + K^\alpha C$$

$$(35) \quad [X^\alpha, X_\beta] = i\delta^\alpha_\beta X_{n+1} + S^{\alpha\sigma}_{\beta\rho} A^\rho_\sigma + T^{\alpha\gamma}_\beta B_\gamma + R^\alpha_{\beta\gamma} B^\gamma + Q^\alpha_\beta C.$$

These are derived from the formulas for projective curvature in [3].

If we write $e^G e^Z = e^U$ and $e^X e^P = e^V$ then equating terms in U, V.

$$(36) \quad f^\alpha + \frac{1}{2} f^\alpha p^\gamma_\beta \delta^\beta_\gamma + \frac{1}{2} f^\gamma p^\alpha_\gamma - \frac{1}{4} f^\alpha_{p_0} = z^\alpha + \ldots$$

$$p^\alpha + \overline{f^\alpha} - \frac{1}{2} \overline{f^\alpha} p^\gamma_\beta \delta^\beta_\gamma - \frac{1}{2} \overline{f^\gamma} p^\gamma_\alpha - \frac{1}{2} \overline{f^\alpha} p_0 = 0 + \ldots$$

$$\tilde{F} = \frac{i}{2} f^\alpha p_\alpha + \ldots$$

$$p^\alpha_\beta + \frac{i}{2} f_\alpha \tilde{p}_\beta + \frac{i}{2} \overline{f^\beta} \tilde{p}^\alpha - \frac{1}{2} f^\lambda p_\mu \delta^\mu_\lambda \delta^\alpha_\beta = \frac{1}{2} z^\lambda \tilde{F} R^\alpha_{\beta\gamma} + \ldots$$

$$p_0 - \frac{i}{2} f^\alpha \tilde{p}_\beta \delta^\beta_\alpha + \frac{i}{2} \overline{f^\alpha} \tilde{p}^\beta \delta^\alpha_\beta = 0 + \ldots$$

$$\tilde{p}_\alpha - \frac{1}{4} \overline{f^\alpha} \tilde{p} - \frac{1}{2} f^\gamma p_\beta R^\beta_{\alpha\gamma} = \frac{1}{2} z^\gamma \tilde{F} P_{\alpha\gamma} + \ldots$$

$$\tilde{p}^\alpha - \frac{1}{4} f^\alpha \tilde{p} - \frac{1}{2} f^\gamma p_\beta T^{\alpha\beta}_\gamma = -\frac{i}{4} z^\gamma \tilde{F} Q^\alpha_\gamma + \ldots$$

$$\tilde{p} - \frac{1}{2} f^\beta p_\alpha Q^\alpha_\beta = \frac{1}{2} z^\alpha \tilde{F} H_\alpha + \ldots$$

where dots indicate an infinite series with monomials of cubic or higher order in $z^\alpha, \tilde{F}, p_\alpha$ etc., with coefficients universal polynomial expressions in $S^{\alpha\beta}_{\rho\sigma}$, $R^\alpha_{\beta\gamma}$, $T^{\alpha\gamma}_\beta$, Q^α_β, $P_{\alpha\beta}$, $L^{\alpha\beta}$, H_α, K^α and their X_λ, X^λ, X_{n+1} derivatives.

Clearly one can solve these equations recursively for f^α, p_α, etc., as power series in z^α, $\overline{z^\alpha}$ with coefficients given by expressions in the curvature functions and their successive X_λ, X^λ, X_{n+1} derivatives (generalized covariant derivatives) computed at the point $q = \varphi(uX_{n+1})p$.

We find

$$f^\alpha = z^\alpha + \ldots$$

$$p_\alpha = \overline{z^\alpha} + \ldots$$

$$\tilde{F} = -\frac{i}{2}\Sigma|z^\alpha|^2 + \ldots$$

(37) $$p_\theta = 0 + \ldots$$

$$p^\alpha_\beta = \frac{1}{2} z^\lambda \overline{z^\mu} S^{\mu\alpha}_\beta + \ldots$$

$$\tilde{p}_\alpha = \frac{1}{2} R^\beta_{\alpha\gamma} \overline{z^\beta} z^\gamma + \ldots$$

$$\tilde{p}^\alpha = \frac{1}{2} T^{\alpha\beta}_\gamma z^\gamma \overline{z^\beta} + \ldots$$

$$\tilde{p} = \frac{1}{2} Q^\alpha_\beta z^\beta \overline{z^\alpha} + \ldots \quad .$$

In order to solve for higher terms we use an argument involving homogeneity.

(i) Assign to the vectorfields X_{n+1}, X_α, etc., weights as follows:

X_{n+1}, weight -2; X_α, X^α, weight -1;

X^0, A^α_β, weight 0; B_α B^α, weight +1;

C, weight +2.

(ii) Assign to the curvature function weights:

$S^{\alpha\rho}_{\beta\sigma}$, weight -2; $R^\alpha_{\beta\gamma}$, $T^\alpha_{\beta\gamma}$, weight -3;

Q^α_β, $P_{\alpha\beta}$, $L^{\alpha\beta}$, weight -4; H^α, K_α, weight -5.

(iii) A covariant derivative by X_α or X^α lowers the weight by -1, a covariant derivative by X_{n+1} lowers the weight by -2.

(iv) Assign to the variables z^α, p_α, f^α etc., weights:

z^α, p_α, f^α weight +1; p_θ, p^α_β weight 0;

$\tilde{p}^\alpha$, $\tilde{p}_\alpha$ weight -1; $\tilde{p}$ weight -2.

Then one sees that the equations in (36) are homogeneous. This is clear for the leading terms (37) and, in general, it is because G,Z,X,P are of total degree 0, hence the same holds for U, V. Thus, when we separate components of the vector equation U = V, the coefficients of each vector X_{n+1}, X_α, etc., must have the opposite weight, e.g., the coefficient $\tilde{F}$ of X_{n+1} must have weight 2. Further there are no "stray" indices:

the coefficient $\tilde{F}$ of X_{n+1} is a sum of homogeneous polynomial terms, each one of which has all indices in certain possibly higher order curvature functions summed with respect to z^α or $\overline{z}^\beta$, where $\overline{z}^\beta$ is considered as having index lower β, the coefficient p^α of X_α is a sum of terms each of which has one free index upper α, and so on. If $a = (a_1 \ldots a_n)$, $b = (b_1 \ldots b_n)$ and $z^a \overline{z}^b = z_1^{a_1} \ldots \overline{z}_n^{b_n}$ then the coefficient of $z^a \overline{z}^b$ in the power series expansion of $F(z,\overline{z},u)$ polynomial is giver by a universal polynomial in tensors of weight $2 - |a| - |b|$ with lower indices $1, a_1$ - times, $2, a_2$ - times,... and upper indices $1, b_1$ - times, $2, b_2$ - times... .

Write $\operatorname{Im} \tilde{F}(z,\overline{z},u) = F(z,\overline{z},u) = \Sigma F_{k\overline{\ell}}(z,\overline{z},u)$ where $F_{k\overline{\ell}}(tz,s\overline{z},u) = t^k s^\ell F_{k\overline{\ell}}(z,\overline{z},u)$.

Let

$$F_{k\overline{\ell}}(z,\overline{z},u) = \sum_{\substack{\alpha_1 \ldots \alpha_\kappa \\ \beta_1 \quad \beta_\ell}} F^{\beta_1 \ldots \beta_\ell}_{\alpha_1 \ldots \alpha_\kappa}(u) z^{\alpha_1} \ldots z^{\alpha_\kappa} Z^{\beta_1} \ldots z^{\beta_2} \tag{38}$$

then $F^{\beta_1 \ldots \beta_\ell}_{\alpha_1 \ldots \alpha_\kappa}$ is expressible as a polynomial homogeneous of weight $2-k-\ell$ in the curvature tensors, $S^{\alpha\rho}_{\beta\sigma}$, $R^\alpha_{\beta\gamma}$ etc., their covariant derivatives, and δ^α_β. A simple argument shows that if $k \leq 1$ or $\ell \leq 1$, then $F_{k\overline{\ell}} = 0$ unless $k = \ell = 1$, since a curvature polynomial of weight $-m$ with $2+m$ indices must be either δ^α_β or have at least 2 upper and 2 lower indices. Define

(39) $$\operatorname{tr} F_{k\bar{\ell}} = \operatorname{trace} F_{k\bar{\ell}} = \sum_{\substack{\alpha_2 \ldots \alpha_\kappa \\ \beta_2 \ldots \beta_\ell \\ j}} F^{j\alpha_2 \ldots \alpha_\kappa}_{j\beta_2 \ldots \beta_\ell}(u) z^{\alpha_2} \ldots z^{\alpha_\kappa} a^{\bar{\beta}_1 \ldots \bar{\beta}_\ell} .$$

By inspection of the curvature polynomials of weights $-2, -3, -4$ we conclude that

$$\operatorname{tr} F_{2\bar{2}} = \operatorname{tr}^2 F_{3\bar{2}} = 0$$

(40) $$\operatorname{tr}^3 F_{3\bar{3}} = \text{const } \| S^{\alpha\rho}_{\beta\sigma} \|^2$$

$$= \text{const } \| F^{\alpha\rho}_{\beta\sigma} \|^2 .$$

That $F_{2\bar{2}}$, $F_{3\bar{2}}$ and $F_{3\bar{3}}$ are not identically zero can be seen by computing the connection in local coordinates given by projective Fermi coordinates. The result is (see [1])

$$S^{\alpha\rho}_{\beta\sigma} = -4 \; F^{\alpha\rho}_{\beta\sigma}$$

$$V^{\alpha}_{\beta\gamma} = R^{\alpha}_{\beta\gamma} = - \frac{12i}{n+2} F^{\lambda\alpha}_{\lambda\beta\gamma}$$

(41) $$Q^{\alpha}_{\beta} = \frac{n}{(n+1)(n+2)} F^{\lambda\mu\alpha}_{\lambda\mu\beta} + T^{\alpha}_{\beta}$$

$$\text{where} \quad T^{\alpha}_{\alpha} = \frac{2n+1}{(n+1)(n+2)} \| F^{\alpha\rho}_{\beta\sigma} \|^2 .$$

Therefore from (41)

$$F_{2\bar{2}} = 0 \Rightarrow S^{\alpha\beta}_{\rho\sigma} = 0$$

(42)
$$F_{3\bar{2}} = 0 \Rightarrow V^{\alpha}_{\beta\gamma} = 0$$

$$F_{3\bar{3}} = 0 \Rightarrow Q^{\alpha}_{\beta} = T^{\alpha}_{\beta} \Rightarrow \| F^{\alpha\beta}_{\rho\sigma} \|^2 =$$

$$= 0 \Rightarrow S^{\alpha\beta}_{\rho\sigma} = 0.$$

The last equation of (41) determines the constant $= -\frac{2n+1}{n}$, concluding the proof of the theorem.

By changing the definition of the Cartan connection on R (equivalently, on Y) we can get exactly the Moser normal form from projective Fermi coordinates. Define on R

$$\tilde{\psi} = \psi - a\omega .$$

The induced change in the dual basis is only in

(25)
$$\tilde{X}_{n+1} = X_{n+1} + aC.$$

Since the distribution on $R\{X_{n+1}, X^0, C\}$ is integrable, and its integral manifolds projective to complex curves in $\mathbb{C}^{n+1}$ which intersect M in chains, the proposed change will only affect the parametrization of chains and this is what yields $\operatorname{tr}^3 F_{3\bar{3}} = 0$, A direct calculation shows that this change in X_{n+1} alters the term $F_{3\bar{3}}$ by $-\frac{a}{4}\|z\|^6$. Choosing a properly it is possible to find a defining function F_a with $tr^3 F_{a\ 3\bar{3}} = 0$, the first two trace conditions remaining unchanged. This has the effect of changing only the curvature function

$$Q^\alpha_\beta \quad \text{to} \quad \tilde{Q}^\alpha_\beta = \frac{2n+1}{n(n+1)(n+2)}\|F^{\lambda\rho}_{\beta\mu}\|^2\ \delta^\alpha_\beta + Q^\alpha_\beta\ .$$

<u>Corollary.</u> The curvature functions and their covariant derivatives at a point q of the pseudoconformal bundle determine the coefficients of the normal form at $p = \pi(q)$ according to universal homogeneous, polynomial formulas derived from the Baker-Campbell-Hausdorff formula.

This corollary gives the relation between the CR curvature functions on Y and the coefficients of the normal form via the complexification procedure described in Theorem 1. The relation between these curvature functions and the invariants derived from Fefferman's asymptotic solution to the Monge-Ampère equation has been explained by Webster [10].

We conclude with the remark that if $M \subset \mathbb{C}^{n+1}$ is not real-analytic, but a C^∞ hypersurface, then the above theory carries over directly in the context of formal power series. Thus, the bundle R will be the almost-analytic extension, in the sense of A. Melin and J. Sjöstrand, of the totally-real bundle Y imbedded in R as in 1. If all exponentials of vectorfields in 2 are interpreted as formal power series in the appropriate variables, the (non-convergent) normal-form for M at P is derived by universal formulas involving the curvature of R and its covariant derivatives, computed to infinite order along Y. This curvature is, as in 1, the almost-analytic extension of the curvature of Y, so that the normal form coefficients are given by universal polynomial expressions in the curvature of Y and its covariant derivatives, the same polynomials as in the corollary above.

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