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Rigidity of Minimal Surfaces in $\mathbb{S}^3$

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Isometric deformations of compact minimal surfaces in the standard three-sphere are studied. It is shown that a given surface admits only finitely many noncongruent minimal immersions into $\mathbb{S}^3$ with the same first fundamental form.

0. Introduction

The purpose of this paper is to prove a rigidity result for compact surfaces minimally immersed in the standard three sphere. Let $\langle\ ,\ \rangle$ denote the standard inner product on $\mathbb{R}^4$ and let $\mathbb{S}^3 = \{ x \in \mathbb{R}^4 : \langle x,x\rangle = 1\}$ with the induced metric. The main result is stated below.

Theorem 1 *Let* M *be a compact surface and* $x:M \to \mathbb{S}^3$ *a branched minimal immersion into the three sphere. Then there are at most finitely many pairwise noncongruent, minimal immersions* $x^{(p)}:M \to \mathbb{S}^3$, $p = 1,\dots,N$ *such that*

$$\langle dx^{(p)},dx^{(p)}\rangle = \langle dx,dx\rangle \text{ for } p=1,\dots,N.$$

1. Preliminaries

Let $x:M \to \mathbb{S}^3$ be an immersion of an oriented surface M. Let $ds^2=\langle dx,dx\rangle$ be the induced metric on M. Since M is oriented, the unit normal to M is a globally defined function $\nu:M \to \mathbb{S}^3$. Locally the metric on M can be written as $ds^2= (\omega^1)^2+(\omega^2)^2$ where ω^1,ω^2 is an orthonormal coframe. The classical structure equations for such an immersion are

$$\begin{aligned} dx &= \qquad\quad e_1\omega^1 + e_2\omega^2 \\ de_1 &= -x\omega^1 + e_1\omega_1^1 + e_2\omega_1^2 + \nu\psi_1 \\ de_2 &= -x\omega^2 + e_1\omega_2^1 + e_2\omega_1^2 + \nu\psi_2 \\ d\nu &= \qquad\ - e_1\psi_1 - e_2\psi_2 \end{aligned} \tag{1}$$

where $(e_i)_{i=1,2}$ is the orthonormal frame dual to the coframe (ω^i), (ω_j^i) is the 2×2 matrix of connection one forms and $(\psi_i)_{i=1,2}$ are one forms that determine the second fundamental form of the immersion. The components of the second fundamental form with respect to the given coframe are given by the formulae

$$\psi_i = \sum_{i=1,2} h_{ij}\omega_j \quad i=1,2. \tag{2}$$

The mean curvature of the immersion is given by $H = h_{11} + h_{22}$.

Let $x:M \longrightarrow \mathbb{S}^3$ be a minimal immersion of an oriented surface M. In particular $H \equiv 0$. Choose a local frame as above and set $h = h_{11} - ih_{12}$. The Gauss equation in this context, can be written as

$$1 - K = |h|^2 \tag{3}$$

where K is the Gaussian curvature of M.

The following two results, due to Lawson [L1], are needed later.

<u>Proposition</u> <u>2</u> [L1] *Let* (M,ds_M^2) *be a surface* M *with a Riemannian metric* ds_M^2 *such that* $K \not\equiv 1$.

a) *If* $x,\tilde{x}$: $M \longrightarrow \mathbb{S}^3 \subseteq \mathbb{R}^4$ *are two minimal immersions both inducing the given metric on* M, *then*

$$\tilde{h} = \tilde{h}_{11} - i\tilde{h}_{12} = \exp(i\theta)\ (h_{11} - ih_{12}) = \exp(i\theta)\ h \tag{4}$$

where $\theta \in [0,2\pi)$. *Moreover,* x *and* $\tilde{x}$ *are congruent if and only if* $\theta = 0$ *or* $\theta = \pi$.

b) *Suppose* $x:M \longrightarrow \mathbb{S}^3$ *is a minimal immersion inducing the given metric on* M. *The for any simply connected domain* $U \subseteq M$ *and* $\theta \in [0,2\pi)$, *there is a minimal immersion* $\tilde{x} = x_\theta : U \longrightarrow \mathbb{S}^3$ *satisfying equation* 2.

<u>Lemma</u> <u>3</u> [L1] *Let* $x:M \longrightarrow \mathbb{S}^3$ *be a minimal immersion of an oriented surface* M *such that* $K \not\equiv 1$. *Then the normal map* $\nu:M \longrightarrow \mathbb{S}^3$ *is a branched*

minimal immersion with the induced metric given by $\langle d\nu, d\nu\rangle = (1-K)\ \langle dx, dx\rangle$.

Remark 4 *Let* $x, x' : M \to S^3$ *be two immersions of a surface* M *such that their normal maps* $\nu, \nu' : M \to S^3$ *are also immersions. For any* $T \in O(4)$, $x' = T\circ x$ *if and only if* $\nu' = T\circ\nu$. *(The proof is straight-forward.)*

2. Proof of Main Result and an Application

The proof of Theorem 1 depends on the following lemma.

Lemma 5 *Let* (M, ds_M^2) *be a surface with Gaussian curvature* $K \not\equiv 1$. *Let* $x^{(1)}, \ldots, x^{(N)}$ *be pairwise noncongruent minimal immersions of* M *inducing the metric* ds_M^2. *If*

5) $$\sum_{p=1}^{N} \langle v_p, x^{(p)}\rangle = 0 \qquad v_p \in \mathbb{R}^4$$

then $v_p = 0$ *for* $p = 1, \ldots, N$.

Proof Suppose such a nontrivial relation exists, with each $v_p \neq 0$ for $p=1,\ldots,N$. Applying the exterior differentiation operator to this relation gives the following

6) $$\sum_{p=1}^{N} \langle v_p, e_i^{(p)}\rangle = 0 \qquad i=1,2$$

where $(e_i^{(p)})$ are the images in $\mathbb{R}^4$ under $dx^{(p)}$ of the same local oriented orthonormal frame field on M. Applying the exterior differentiation operator to equation 6 and then using the structure equations, equations 5 and 6 yields the following relations

7) $$\sum_{p=1}^{N} \langle v_p, \nu^{(p)}\rangle \psi_i^{(p)} = 0 \qquad i=1,2.$$

Let $\psi_i^{(p)} = \sum_{j=1,2} h_{ij}^{(p)} \omega^j$ and $h^{(p)} = h_{11}^{(p)} - i h_{12}^{(p)}$. Note that equation 7 implies

8) $$0 = \sum_{p=1}^{N} \langle v_p, \nu^{(p)}\rangle h^{(p)}.$$

Proposition 2 implies that $h^{(q)} = \exp(i\theta^{(q)}) h^{(1)}$ where $\theta^{(q)} \in \mathbb{R}$ and $q=2,\ldots,N$. Since the immersions $x^{(p)}$ are pairwise noncongruent, Proposition 1 implies that $\theta^{(q)} \not\equiv 0 \pmod{\pi}$ and $\theta^{(p)} \not\equiv \theta^{(q)} \pmod{\pi}$ for distinct $p,q = 2,\ldots,N$. The assumption that $K \not\equiv 1$ and the Gauss equation 3 imply that $h^{(1)} \not\equiv 0$. This implies the relation

$$0 = \langle v_1, \nu^{(1)} \rangle + \sum_{p=2}^{N} \langle v_p, \nu^{(p)} \rangle \exp(i\theta^{(p)}). \tag{9}$$

The imaginary part of equation 9 is a nontrivial relation among the normal maps $\nu^{(2)}, \ldots, \nu^{(N)}$. By Lemma 3 and Remark 4, these normal maps are also conformal minimal immersions of M into S^3 that induce the same metric on M and are pairwise noncongruent. Therefore the preceding argument may be iterated until one finally has either

$$0 \equiv \langle v, x^{(N)} \rangle \text{ for some } v \in \mathbb{R}^4 \backslash \{0\}$$

or

$$0 \equiv \langle v, \nu^{(N)} \rangle \text{ for some } v \in \mathbb{R}^4 \backslash \{0\}.$$

The first possibility implies that $K \equiv 1$ since $x^{(N)}(M)$ is then forced to lie in a totally geodesic two sphere in S^3. This is impossible. The second conclusion implies that $\nu^{(N)}(M)$ must be contained in a totally geodesic two sphere in S^3. Since $\langle d\nu^{(N)}, d\nu^{(N)} \rangle = (1-K)\, ds_M^2$ and $K \not\equiv 1$, $\nu^{(N)}$ must be an immersion on some open neighborhood of M. It follows that $x^{(N)}$ must be degenerate on this neighborhood. This is also impossible. Therefore no nontrivial relation like equation 5 can hold.■

<u>Proof of Theorem 1</u> Suppose $x^{(1)}, \ldots, x^{(N)}, \ldots$ is an infinite sequence of pairwise noncongruent minimal immersions of a compact surface M into S^3. Then Lemma 5 implies that the coordinate functions of these immersions, $\{x_i^{(N)} : N \in \mathbb{Z}^+,\ i = 1, \ldots, 4\}$, are linearly independent. However, it is well known that all these functions satisfy the equation $\Delta u = -2u$, where Δ is the Laplace-Beltrami operator of M with the induced metric. It is well known that the space of solutions to this equation is finite dimensional if M is compact. Contradiction.■

The following example, due to R. Bryant, shows that one cannot expect Theorem 1 to hold in arbitrary codimensions. (See also the paper [B].)

<u>Remark 6</u> Consider the map $f_t : \mathbb{R}^2 \to S^7 \subseteq \mathbb{C}^4$ defined by setting $f_t(x,y)$

to be

$$(1/2)\ ((1-t)^{1/2}e^{i(2x+y)},(1-t)^{1/2}e^{i(x-2y)},(1+t)^{1/2}e^{i(2x-y)},(1+t)^{1/2}e^{i(x+2y)}).$$

It is easy to verify that $\langle df_t, df_t\rangle = (5/2)\ (dx^2 + dy^2)$ and that f_t factors to a minimal immersion of the torus $\mathbb{R}^2/\Lambda$, where $\Lambda=\{\ (2\pi a, 2\pi b): a,b \in \mathbb{Z}\}$, for every t such that $|t|<1$. Moreover, f_t and f_s are noncongruent for $s \neq t$.

<u>Theorem</u> <u>7</u> *Let* $x:M \rightarrow S^3$ *be a minimal immersion from a compact orientable surface. Suppose that* M *admits a one parameter group of isometries* $\phi_t:M \rightarrow M$ *with respect to the induced metric . Then there exists a one parameter family of orientation preserving isometries* $\Phi_t:S^3 \rightarrow S^3$ *such that* $x\circ\phi_t=\Phi_t\circ x$ *for all* $t \in \mathbb{R}$.

<u>Proof</u> Let $x^{(1)},\ldots,x^{(N)}:M \rightarrow S^3$ be a maximal family of isometic, pairwise noncongruent, minimal immersions of M into S^3. ($N < \infty$ by Theorem 1.) Continuity of the second fundamental form of $x\circ\phi_t$, with respect to the parameter t, implies that

a) $x\circ\phi_t$ is congruent to exactly one $x^{(i)}$ for all $t \in \mathbb{R}$ and

b) $h_t \equiv h^{(i)}$ or $h_t\equiv -h^{(i)}$ for any oriented frame e_1,e_2 of M and for all $t \in \mathbb{R}$ (h_t and $h^{(i)}$ are defined as in equation 4.

If $h_t \equiv h^{(i)}$ (resp. $h_t \equiv -h^{(i)}$) then $x\circ\phi_t$ is congruent to $x^{(i)}$ by an orientation preserving (resp. reversing) isometry of S^3. Since $x\circ\phi_0 = x$, $x\circ\phi_t$ is congruent to x for all $t \in \mathbb{R}$ by an orientation preserving isometry of S^3. Note that this orientation preserving isometry is uniquely determined by the constraints that it must take $x(p)$ to $x(\phi_t(p))$, and $dx(e_i)$ to $dx(d\phi_t(e_i))$, $i=1,2$, where $p \in M$ is some fixed point and e_1 and e_2 is an oriented orthonormal frame at p. Denote this congruence by $\Phi_t:S^3 \rightarrow S^3$. The smooth dependence of Φ_t on t and the fact that Φ_t is a one parameter group of SO(4) follow easily from the above discussion.■

<u>Remark</u> <u>8</u> Hsaing and Lawson [HL] have classified all minimal immersions of compact surfaces in S^3 admitting a continuous group of ambient

symmetries. The above result implies that their work also classifies minimally immersed, compact surfaces with a continuous group of intrinsic isometries.

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