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On the Introduction of an Agile, Temporary Workforce into a Tandem Queueing System

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Abstract. We consider a two-station tandem queueing system where customers arrive according to a Poisson process and must receive service at both stations before leaving the system. Neither queue is equipped with dedicated servers. Instead, we consider three scenarios for the fluctuations of workforce level. In the first, a decision-maker can increase and decrease the capacity as is deemed appropriate; the *unrestricted* case. In the other two cases, workers arrive randomly and can be rejected or allocated to either station. In one case the number of workers can then be reduced (the *controlled capacity reduction* case). In the other they leave randomly (the *uncontrolled capacity reduction* case). All servers are capable of working collaboratively on a single job and can work at either station as long as they remain in the system. We show in each scenario that all workers should be allocated to one queue or the other (never split between queues) and that they should serve exhaustively at one of the queues depending on the direction of an inequality. This extends previous studies on flexible systems to the case where the capacity varies over time. We then show in the unrestricted case that the optimal number of workers to have in the system is non-decreasing in the number of customers in either queue.

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1. Introduction

The use of an agile workforce is becoming more prevalent in the manufacturing and service sectors. Practitioners would like to understand the benefits of cross-training to know whether they exceed the costs. If workers or machines can perform a variety of tasks, they may be reallocated to alleviate areas of high congestion. Most models, however, assume that the number of workers available is predetermined. In many practical situations it is often the case that the number of workers available is dynamic. For example, consider a simple two cell factory where a company manufactures two products, A and B. Suppose some of the workers have the primary tasks of manufacture and inspection of product A, but are also trained to perform the same tasks for product B. A decision-maker monitors the workload of the entire job shop and can decide to reallocate some of the workers to either the manufacture or inspection portion of the product B line. This, of course, reduces the capacity to produce A and can be costly to the system. Three questions arise:

- Should cross-trained workers be assigned to work on product B?
- If they are to be assigned to product B, should they be allocated to the manufacture or inspection station?
- How long should they remain working on the product B line?

We address these questions in three contexts; each from the point of view of the product B line. The first is called the *unrestricted* case where the decision-maker can increase or decrease capacity to produce product B at any time. In the second model, workers become available randomly but may be released by the decision-maker; *controlled capacity reduction*. In the final problem workers randomly become available and randomly leave the system; *uncontrolled capacity reduction*. We note that the decision-maker has less and less control over the availability of cross-trained workers from the first to the third problem.

To illustrate each problem, consider the availability of health care professionals (doctors, nurses, etc.) in various hospital operations. It is standard practice for a nurse or doctor who is currently assigned to work in other care units, such as the intensive care unit (ICU) or the outpatient clinic, to be called to the emergency room (ER) or trauma center to handle higher priority patients and then returned to their primary assignment when the workload has subsided. This practice is called "jeopardy". From the perspective of the other care units, the time between calls that take workers away and the time they (or some other worker) become available again can be modeled as random events; the uncontrolled capacity reduction case. On the other hand, since the ER and trauma center are usually the highest priority (and often the most costly) operation performed in a hospital, they can add and subtract workers as needed; the unrestricted case. Finally, while arrivals of workers may be random (from say the ER), it is often the case that some of these nurses are released to work in the general rounds when not needed; the controlled capacity reduction case.

Each problem is considered under the infinite horizon discounted expected cost and average expected cost criteria. Since the average case is considered, we present sufficient conditions for the stability of each model. We then show the rather surprising result that the optimal allocation policy is "independent" of the decision to increase or decrease capacity in the sense that a rule analogous to the classic $c\mu$-rule applies (no matter how many workers are available). Once this assertion is made, Markov decision processes are employed to prove that optimal policies in the unrestricted case are monotone in the number of customers in each queue. This result is used to develop heuristics for each case.

Of particular relation to the present work is the work of Ahn et al. [1,2] where the clearing system and dynamic versions of this problem with a static number of servers are considered. In each case, the authors provide conditions under which the optimal policy is exhaustive in each station. Duenyas et al. [13] and Iravani et al. [20], in parallel, consider a tandem queueing system with one flexible server and characterize the optimal policy to be a monotone switching curve. The control of a flexible workforce in minimizing the cycle time of each job is discussed in Van Oyen et al. [26]. Andradottir et al. [3] consider the control of flexible servers to maximize throughput when the service rate is additive. Javidi et al. [21] consider two interconnected queues with identical machines and derive sufficient conditions under which the policy that prioritizes jobs in queue 1 minimizes the expected value of the first time that the system becomes empty (i.e., the dynamic version of makespan). Other related works include the work on "bucket brigades" by Bartholdi and Eisenstein [5] and Bartholdi et al. [4,6].

The scheduling of (temporary) workers in queueing systems is closely related to the scheduling of a removable server. Early work in this area considered the question of when to schedule the server to be offline and online (cf. [19,27]). The authors show the existence of control limits determining when the server should be turned on and off. More recently, Feinberg and Kella [16] show the optimality of "D-policies" that turn the server off when the system empties and turn it back on when the workload reaches some level D. There has also been a considerable amount of work on service rate control in queueing systems. Classic models include the work of Crabill [10], where it is shown that the optimal service rate is non-decreasing in the number of customers in the system. More recent studies of service rate control problems include the work of Stidham and Weber [25] and George and Harrison [17]. In each case, the authors consider models analogous to our unrestricted case but with the cost function non-decreasing in the one-dimensional state. No such representation holds here. Other related works include [14,15].

The rest of the paper is organized as follows. Section 2 contains formal definitions of the problems considered and Markov decision process formulations. In the same section, we state results leading to the stability of the network and the existence of a solution to the average cost optimality equations. The main results of the paper are contained in Sections 3 and 4; the optimality of an exhaustive policy in each case and the monotone structure of an optimal policy in the unrestricted case. Some examples displaying a significant difference between the models are provided in Section 5. Promising heuristics are also introduced and analyzed. We conclude the paper in Section 6.

2. Preliminaries and model formulation

Consider a two station, tandem queueing system where customer arrivals to station 1 follow a Poisson process of rate $\lambda > 0$. After receiving service at station 1, customers proceed immediately to station 2 and must receive service in order to leave the system. The service requirement for all customers is exponential with (finite) means $1/\mu_1$ and $1/\mu_2$ for stations 1 and 2, respectively. Neither station is equipped with a dedicated, permanent server. Instead, in the unrestricted case the number of workers may be increased or decreased, while in the controlled or uncontrolled capacity reduction cases workers arrive according to a Poisson process at rate $\alpha(k) > 0$ when there are k workers currently in the system. In the uncontrolled capacity reduction case, accepted workers are available for an exponential amount of time with rate γ.

Once the decision is made to increase the capacity, workers are immediately assigned to a station. Assume that all workers are identical and able to work at rate 1 regardless of the station to which they are assigned. Note that this along with the assumption that the service requirements are exponentially distributed implies that the time to complete a job at station i is exponential with rate μ_i. Assume that customers in service can be preempted to reallocate workers and that when more than one worker is working at a station their rates are additive; servers can **collaborate** on a single job. Let $\ell < \infty$ represent the maximum number of workers that can be in the system at any particular time or equivalently set $\alpha(k) = 0$ for $k \geq \ell$.

Define h_i, $i = 1, 2$, to be the rate at which holding costs are accrued for each customer at station i and $r(k)$ to be the (finite, real-valued) cost rate of having k workers in the system. Intuitively, this is the opportunity cost of having the workers allocated to this station instead of in some other part of the job shop described in Section 1. We assume that $r(0) = 0$. At each customer arrival and departure time, the decision-maker must decide how to allocate the flexible workers to each station. In the unrestricted case, the decision-maker must decide how many workers to keep (or add) for the coming period. In the controlled capacity reduction case, the decision-maker can reduce the number of workers at any time, while in the uncontrolled capacity reduction case it must wait for workers to leave. In the controlled and uncontrolled capacity reduction cases, workers can only be added when the opportunity arises. A decision must then be made if they should be rejected. In either case, workers that are rejected are lost forever.

For the remainder of the paper we differentiate the unrestricted, controlled and uncontrolled capacity reduction cases with superscripts U, C, and F, respectively. Unless otherwise specified, quantities without superscripts have a common definition for each problem. In each scenario a *decision rule*, say $d(x)$, is a function that maps the state space to the set of potential actions (the action space). A sequence of decision rules is called a *policy* and prescribes what action should be taken for each state at any particular time. Note that although we have defined policies to only cover those that are Markovian, this set is sufficient to guarantee the optimality over the larger set of non-anticipating policies. Let $\mathbb{X}^U \equiv \mathbb{Z}^+ \times \mathbb{Z}^+$ and $\mathbb{X}^C = \mathbb{X}^F \equiv \mathbb{X}^U \times \{0, 1, \ldots, \ell\}$ denote the respective state spaces of each problem, where $\mathbb{Z}^+$ is the set of non-negative integers. The first two elements of the state space represent the number of customers at each station and the last element (in $\mathbb{X}^C$ and $\mathbb{X}^F$) is the number of workers available to be allocated to either station. Let Π denote the set of non-anticipating, non-idling policies and suppose for a fixed $\pi \in \Pi$ and $t \geq 0$ that $Q_1^\pi(t)$, $Q_2^\pi(t)$ and $Z^\pi(t)$ represent the queue length processes for station 1 and 2 and number of workers available at time t, respectively. Define

$$v_{t,\theta}^\pi(x) \equiv \mathbb{E}_x \int_0^t e^{-\theta s}\big[h_1 Q_1^\pi(s) + h_2 Q_2^\pi(s) + r(Z^\pi(s))\big]\, ds, \tag{2.1}$$

$$v_\theta^\pi(x) \equiv \lim_{t\to\infty} v_{t,\theta}^\pi(x), \tag{2.2}$$

$$\rho^\pi(x) \equiv \limsup_{t\to\infty} \frac{v_{t,0}^\pi(x)}{t}, \tag{2.3}$$

where $\theta \geq 0$ is the discount factor. The equations (2.1)–(2.3) define the finite horizon expected discounted cost, the infinite horizon expected discounted cost, and the long-run average expected cost, respectively. In the finite horizon case, only the portion of the policy required for the time horizon is used. In each case we define the optimal values

$$v_{t,\theta}(x) = \inf_{\pi\in\Pi} v_{t,\theta}^\pi(x), \tag{2.4}$$

$$v_\theta(x) = \inf_{\pi\in\Pi} v_\theta^\pi(x), \tag{2.5}$$

$$\rho(x) = \inf_{\pi\in\Pi} \rho^\pi(x), \tag{2.6}$$

where a policy that achieves the infimum of any of the respective criteria is deemed optimal.

We will find it instructive to consider both the continuous-time problem presented and its discrete-time analogue. In the discrete-time case assume that *uniformization* in the spirit of Lippman [22] has been applied so that the optimal discounted and average costs are scaled by a constant. The optimal decisions are unaffected. Let $\Psi = \lambda + \max_{k\in\{0,1,\ldots,\ell\}}\{\alpha(k)\} + \ell(\max\{\mu_1, \mu_2\} + \gamma)$ be the uniformization constant and $\beta = \frac{\Psi}{\theta+\Psi}$ be the scaled discount factor. Without loss of generality assume that $\Psi = 1$.

In order to simplify notation, for each decision-making scenario, let g be any real-valued function on the respective state space. Define the following mappings for the unrestricted model

$$\begin{aligned}H_\beta^U g(i,j) = \min_{k\in\{0,1,\ldots,\ell\}} \{r(k) + \beta(\lambda g(i+1,j) + \min_{0\leq x\leq k}\{x\mu_1 g((i-1)^+, j+1) \\ +(k-x)\mu_2 g(i,(j-1)^+) + [1-(\lambda + x\mu_1 + (k-x)\mu_2)]g(i,j)\})\},\end{aligned}$$

for the uncontrolled capacity reduction model

$$\begin{aligned} H_\beta^F g(i,j,k) = r(k) + \beta(&\lambda g(i+1,j,k) + \gamma k g(i,j,k-1) \\ &+ \alpha(k)\min\{g(i,j,k), g(i,j,(k+1)\wedge \ell)\} \\ &+ \min_{0\le x\le k}\{x\mu_1 g((i-1)^+, j+1, k) + (k-x)\mu_2 g(i,(j-1)^+,k) \\ &+ [1-(\lambda + k\gamma + \alpha(k) + x\mu_1 + (k-x)\mu_2)]g(i,j,k)\}), \end{aligned}$$

and for the controlled capacity reduction model

$$\begin{aligned} H_\beta^C g(i,j,k) = \min_{0\le m\le k}\{&r(m) + \beta(\lambda g(i+1,j,m) \\ &+ \alpha(m)\min\{g(i,j,m), g(i,j,(m+1)\wedge \ell)\} \\ &+ \min_{0\le x\le m}\{x\mu_1 g((i-1)^+, j+1, m) + (m-x)\mu_2 g(i,(j-1)^+,m) \\ &+ [1-(\lambda + \alpha(m) + x\mu_1 + (m-x)\mu_2)]g(i,j,m)\})\}, \end{aligned}$$

where in each case a^+ represents the positive part of a. These mappings represent the one-step cost associated with current workers, capacity increase/decrease decisions, and a terminal cost g. In order to construct the optimal infinite horizon value functions let $v_{0,\beta} = 0$ and define the following sets of equations,

$$v_{n,\beta}(i,j,k) = ih_1 + jh_2 + H_\beta^P v_{n-1,\beta}(i,j,k), \tag{2.7}$$

$$v_\beta(i,j,k) = ih_1 + jh_2 + H_\beta^P v_\beta(i,j,k), \tag{2.8}$$

$$\rho + w(i,j,k) = ih_1 + jh_2 + H_1^P w(i,j,k), \tag{2.9}$$

where $P = U, F,$ or C.

The system of equations (2.7) are referred to as the finite horizon optimality equations (FHOE), (2.8) the discounted cost optimality equations (DCOE) and (2.9) the average cost optimality equations (ACOE). It is well-known that a solution to (2.7) is such that $v_{n,\beta}$ is the optimal, β-discounted, n-stage cost for whichever problem is currently under consideration. Since the action set is finite, Proposition 1.7 of [7] implies that $v_{n,\beta} \to v_\beta$, a solution to the DCOE, as $n \to \infty$. Moreover, a policy that achieves the minimum in (2.7) ((2.8)) is n-stage cost (infinite horizon, discounted cost) optimal for the appropriate model. The average cost case is a bit more subtle since the recurrence structure of the Markov process induced by each policy must be considered. For the problems considered, when a solution (w, ρ) of the ACOE (2.9) exists it is such that ρ is the optimal cost (independent of the initial state) and $w(x)$, called a relative value function, is unique up to an additive constant.

The first result states that the finite and infinite horizon discounted cost value functions are non-decreasing in i and j. In order to ease notation, let $\Delta_i g(i,j) = g(i+1,j) - g(i,j)$ and $\Delta_j g(i,j) = g(i,j+1) - g(i,j)$.

Lemma 2.1. $v^P_{n,\beta}$ and v^P_β are non-decreasing in the number of customers in either queue for $P = U, C, F$.

Proof. We show the result for the unrestricted case. The other cases follow similarly. The $n = 0$ case is trivial. Assume that it holds at $n - 1$. Let $d_n(i + 1, j) = (k', x')$ be the optimal number of workers and the number of workers to place at station 1 when in state $(i + 1, j)$ at time n. If we use the same decision in state (i, j) we have for $i \geq 1$

$$\Delta_i v^U_{n,\beta}(i, j) \geq h_1 + \beta\big(\lambda\Delta_i v^U_{n-1,\beta}(i + 1, j) + x'\mu_1\Delta_i v^U_{n-1,\beta}(i - 1, j + 1) + (k'-x')\mu_2\Delta_i v^U_{n,\beta}(i, j-1)+[1-(\lambda + x'\mu_1 + (k'-x')\mu_2)]\Delta_i v^U_{n,\beta}(i, j)\big).$$

The inductive hypothesis and the non-negativity of h_1 implies $\Delta_i v^U_{n,\beta}(i, j) \geq 0$. A similar argument holds for $i = 0$ except that we idle the workers assigned to station 1 in the process starting in state $(0, j)$. Taking limits as $n \to \infty$ yields the result in the infinite horizon case. □

The next result yields that the ACOE holds in states that are positive recurrent for the average optimal policy. In Section 3 we provide results that allow us to identify the set of positive recurrent states.

Proposition 2.2. Suppose that in the unrestricted or controlled capacity reduction models there exists $k^* \in \{0, 1, \dots, \ell\}$ such that $\frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2} < k^*$ or in the uncontrolled capacity reduction case $\frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2} < z \equiv \sum_{k=1}^{\ell}(kp_k)$, where

$$p_0 = \left(1 + \sum_{k=1}^{\ell}\frac{\Pi_{n=0}^{k-1}\alpha(n)}{k!\gamma^k}\right)^{-1},$$
$$p_k = \frac{\Pi_{n=0}^{k-1}\alpha(n)}{k!\gamma^k}p_0, \quad k = 1, 2, \dots, \ell.$$

In each case, the following hold

1. The optimal average cost may be computed by $\rho = \lim_{\beta\uparrow 1}(1 - \beta)v_\beta(x)$ for any $x \in \mathbb{X}$.
2. There exists a (bounded) limit point $w(x) = \lim_{n\to\infty} w_{\beta_n}(x) = \lim_{n\to\infty}[v_{\beta_n}(x) - v_{\beta_n}(\mathbf{0})]$ such that (ρ, w) satisfy the ACOE with the equality replaced with "$\geq$", where $v_\beta(\mathbf{0}) = v^U_\beta(0, 0)$ in the unrestricted case and $v_\beta(\mathbf{0}) = \min_{k\in\{0,1,\dots,\ell\}}\{v^P_\beta(0, 0, k)\}$ for $P = C, F$.
3. Any policy, say f, achieving the minimum on the right hand side of the above inequality is average cost optimal with average cost ρ. Moreover, the inequality is an equality at any state that is positive recurrent under the Markov chain induced by f.

Proof. Lemma 2.1 yields that v_β is non-decreasing in the number of customers in each queue. Moreover, under the hypotheses of the proposition we have the existence of a policy with finite average cost (see the Appendix) so that the assumptions of Sennott [24, Proposition 7.2.4] hold. The result now follows by applying Theorems 7.2.3 and 7.4.3 of [24]. □

The remainder of the paper is dedicated to proving Theorem 2.3 below. The results on the allocation decision hold for all three models, while the capacity decisions are for the unrestricted model. In the latter, the monotonicity results serve as a baseline for heuristics in the controlled and uncontrolled capacity decrease models.

Theorem 2.3. In each scenario and under the infinite horizon discounted cost and average cost criteria, the following hold

1. there exists an optimal policy that allocates all workers to one station or the other (workers are not split between the two)
2. if $\mu_2 h_2 \leq (\geq) \mu_1(h_1 - h_2)$, there exists an optimal policy that is exhaustive in station 1 (2).

Moreover, in the unrestricted case, if $\mu_1 = \mu_2$,

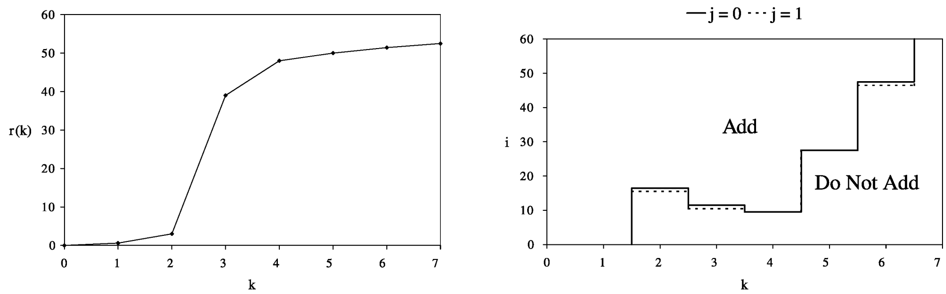
3. for each fixed $(i, j) \in \mathbb{X}^U$ there exists an optimal number of workers, $L(i, j)$, such that $L(i, j)$ is non-decreasing in i and j.

The third result above requires more comment. In all three cases, **all** of the examples we considered yielded that the optimal capacity increase/decrease policy was non-decreasing in i and j. However, we were only able to prove it in the unrestricted case. The difficulty is that the optimal capacity increase/decrease decisions in the other cases may vary in three directions (i, j, k). This significantly complicates the problem. Indeed, given the results for the unrestricted case, we originally conjectured that an optimal policy in the controlled and uncontrolled cases would continue to increase workers until reaching some level, say $M(i, j)$, after which it would stop increasing; monotone in k. The next example disproves this conjecture.

Example 2.4. Suppose we have the following inputs for the uncontrolled capacity reduction model: $h_1 = 2$; $h_2 = 3$; $\ell = 7$; $\lambda = 3$; $\mu_1 = 2$; $\mu_2 = 2$; $\alpha(k) = 2$; $\gamma = .001$. Figure 1(a) displays the cost function $r(k)$ and 1(b) the average optimal capacity increase actions for $j = 0, 1$.

The second result of Theorem 2.3 implies that the optimal allocation is exhaustive in queue 2 so that the set of recurrent states may only have $j = 0$ or 1. Note that despite the fact that the cost function is non-decreasing in k, the optimal capacity increase policy is not. We note that the cost function depicted in Figure 1(a) may be interpreted as a system where the first few workers allocated are not costly while allocating several workers to this station significantly hampers productivity of the whole system.

We also mention that the inequality $\mu_2 h_2 \geq (\leq) \mu_1(h_1 - h_2)$ is analogous to the $c\mu$-rule in parallel systems. In a parallel system, the choice of which job to serve next



(a) Worker cost function.

(b) Uncontrolled reduction optimal capacity increase policy.

Figure 1. Counter-example, Example 2.4, showing an optimal policy need not be monotone (in k).

is made based on the one that can reduce the cost the fastest; with the highest $c\mu$. The fact that the system has flexible workers makes the decision of where to serve a direct analogue. The right (left) hand side of the inequality is the rate at which costs can be drained if the workers are allocated to queue 1 (2). A further explanation is provided in [1].

3. Optimal worker allocation

We divide the control of the system into two parts: the first discusses how workers should be allocated to each station and the second discusses optimal capacity increase/decrease decisions. The next result shows that we need only consider policies that never divide the workers between queues. Although given the existence of a solution to the DCOE and ACOE, the proof is now simple, this result stands to simplify the other results of this section.

Proposition 3.1. In any of the models considered and under either the discounted or average cost criterion, there exists an optimal policy that always allocates all workers to either station 1 or station 2.

Proof. Note that the portion of $H_\beta^U g$, $H_\beta^F g$ and $H_\beta^C g$ that corresponds to the allocation decision has the form

$$\begin{aligned} G(i, j, k, x) \equiv\ & x\mu_1 g(i-1, j+1, k) + (k-x)\mu_2 g(i, j-1, k) \\ & + [1 - (\lambda + k\gamma + \alpha(k) + x\mu_1 + (k-x)\mu_2)]g(i, j, k), \end{aligned}$$

with perhaps γ and $\alpha(k)$ being zero. Since $G(i, j, k, x)$ is linear in x and a linear function achieves its maximum and minimum at the extreme points, the fact that the optimality equations under each criterion have a solution imply that it is optimal to set x equal to 0 or k and the result is proven. □

Proposition 3.1 implies that we may restrict attention to policies that always allocate all workers to one station or the other. Define $\Pi^R \subseteq \Pi$ to be this set of policies. Next we show the existence of optimal policies that are exhaustive in either queue 1 or 2. The proof is analogous to that in Nain [23] for scheduling in parallel queues. First we make the assumption that the cost rate of having workers in the system is zero ($r(\cdot) = 0$). This will be relaxed later. For a policy f let

$$\phi^f(s) \equiv \int_0^s Z^f(x-)\big[\mu_1(h_1 - h_2)a_1^f(x) + \mu_2 h_2 a_2^f(x)\big]\,dx$$

where $Z^f(x)$ is the controlled Markov process representing the number of workers currently available, $a_1^f(x) \equiv 1_{\{Q_1^f(x-)>0, d^f(x)=1\}}$, $a_2^f(x) \equiv 1_{\{Q_2^f(x-)>0, d^f(x)=2\}}$, and $d^f(x) = i$ represents the action that all workers are currently assigned to station i, for $i = 1, 2$.

Lemma 3.2. For a fixed policy $f \in \Pi^R$ and fixed $\theta \geq 0$,

$$v_{t,\theta}^f = \frac{1}{\theta}\big(1-e^{-\theta t}\big)\mathbb{E}\big[h_1 Q_1^f(0) + h_2 Q_2^f(0)\big] + \mathbb{E}\int_0^t e^{-\theta s} h_1 A_1(s)\,ds - \mathbb{E}\int_0^t e^{-\theta s}\phi_f(s)\,ds, \tag{3.1}$$

where $A_1(s)$ is a Poisson process of rate λ.

Proof. Denote the history of possible arrivals, services and the Markov process generated by the capacity increase/decrease decisions under policy f by

$$H(t) = \big\{\big(A_1(s), Y_k^m(s), Z^f(s)\big), k = 1, 2, \ldots, \ell,\ 0 \leq s \leq t\big\},$$

where $\{Y_k^m, k = 1, 2, \ldots, \ell\}$ are independent Poisson processes of rate μ_m for $m = 1, 2$. Let $\mathcal{F}_t$ be the σ-field generated by $H(t)$. That is to say that $\mathcal{F}_t$ is the smallest filtration to which $H(t)$ is adapted. The queue length processes can be written

$$Q_1^f(s) = Q_1^f(0) + A_1(s) - \sum_{k=1}^{\ell}\int_0^s 1_{\{Z^f(x-)\geq k\}} a_1^f(x)\,dY_k^1(x) \tag{3.2}$$

$$Q_2^f(s) = Q_2^f(0) + \sum_{k=1}^{\ell}\int_0^s 1_{\{Z^f(x-)\geq k\}} a_1^f(x)dY_k^1(x) - \sum_{k=1}^{\ell}\int_0^s 1_{\{Z^f(x-)\geq k\}} a_2^f(x)\,dY_k^2(x). \tag{3.3}$$

Note that since f is non-anticipating (and thus, left-continuous) the integrands in (3.2) and (3.3) are *predictable* with the respect to $\mathcal{F}_t$. Moreover, each $Y_k^m(t)$ is adapted to

$\mathcal{F}_t$. Applying a result from Bremaud [8] (see the "Partial result" on page 24 of [8]) we have

$$\mathbb{E}\sum_{k=1}^{\ell}\int_0^s 1_{\{Z^f(x-)\geq k\}}a_m^f(x)\,dY_k^m(x) = \mathbb{E}\sum_{k=1}^{\ell}\int_0^s 1_{\{Z^f(x-)\geq k\}}a_m^f(x)\mu_m dx$$
$$= \mathbb{E}\int_0^s Z^f(x-)a_m^f(x)\mu_m dx,$$

for $m = 1, 2$. Thus,

$$v_{t,\theta}^f = \mathbb{E}\int_0^t e^{-\theta s}h_1\left[Q_1^f(0) + A_1(s) - \int_0^s Z^f(x-)a_1^f(x)\mu_1 dx\right]ds$$
$$+\mathbb{E}\int_0^t e^{-\theta s}h_2\left[Q_2^f(0) + \int_0^s Z^f(x-)\,a_1^f(x)\,\mu_1 dx - \int_0^s Z^f(x-)\,a_2^f(x)\mu_2 dx\right]ds.$$

A little algebra yields

$$v_{t,\theta}^f = \frac{1}{\theta}(1-e^{-\theta t})\mathbb{E}\big[h_1 Q_1^f(0) + h_2 Q_2^f(0)\big] + \mathbb{E}\int_0^t e^{-\theta s}h_1 A_1(s)ds - \mathbb{E}\int_0^t e^{-\theta s}\phi_f(s)\,ds,$$

and the result is proven. □

Note that the above proof does not require the Poisson arrival or worker availability process assumptions; any counting processes will suffice. Moreover, since only the last term in (3.1) is dependent on the policy, showing that π^* minimizes $v_{t,\theta}^{\pi}$ is reduced to showing $\mathbb{E}\phi_{\pi^*}(s) \geq \mathbb{E}\phi_{\pi}(s)$ for all $\pi \in \Pi^R$ and all $0 \leq s \leq t$.

Theorem 3.3. Fix $t \geq 0$. Suppose $\mu_1(h_1 - h_2) \leq (\geq)\,\mu_2 h_2$. Then there exists a policy that minimizes $v_{s,\theta}^{\pi}$ over Π^R for all $0 \leq s \leq t$ that allocates all available workers to queue 2 (1) at time x if $Q_2^{\pi}(x) > 0$ ($Q_1^{\pi}(x) > 0$).

Proof. Consider the case with $\mu_1(h_1 - h_2) \leq \mu_2 h_2$. The other case is proved similarly and is omitted for brevity. The result is proved via an interchange argument. Consider 2 processes defined on the same probability space. Suppose Process 1 follows a policy $f \in \Pi^R$ while Process 2 follows a policy $\pi \in \Pi^R$ defined shortly. Since they are defined on the same space, the arrivals and (potential) services are the same. Moreover, assume that π is chosen so that the Markov chains $Z^f(s)$ and $Z^{\pi}(s)$ are such that $Z^f(s) = Z^{\pi}(s)$ almost surely. Suppose Process 1 allocates all workers to station 1 at some time $\sigma < t$ when $Q_2^f(\sigma) > 0$. Let τ be the first time after σ that Process 1 begins serving at station 2 and η_1 be the time after σ that it completes service of the first customer at station 2. Note that since customers may be pre-empted, there is no reason to believe that τ corresponds to a service completion.

The policy π follows the same policy as f until time σ. At this time, it serves at station 2 until either it completes service of the first customer in queue, or Process 1 stops serving at station 1; whichever occurs first. Denote this time by σ^*. Process 2 then follows the policy f until such time that the first customer in queue 2 has been served (in Process 2). Denote this time by η_2. On $[\eta_2, \eta_1)$ Process 2 serves at station 1. After η_1 the processes follow the same policy, f. Since the processes have the same number of workers, the amount of work that can be done before η_1 is the same. Moreover, since each process does precisely enough work at station 2 to serve the first customer (the rest of the capacity is spent serving station 1), after η_1 the two processes coincide. Thus, $Q_k^f(x) = Q_k^\pi(x)$ for $x \in [0, \sigma) \cup [\eta_1, \infty)$. Note that

$$a_1^\pi(x) - a_1^f(x) = \begin{cases} 0 & \text{if } x \in [0, \infty)\backslash([\sigma, \sigma^*) \cup [\eta_2, \eta_1)), \\ -1 & \text{if } x \in [\sigma, \sigma^*), \\ 1 & \text{if } x \in [\eta_2, \eta_1), \end{cases}$$

$$a_2^\pi(x) - a_2^f(x) = \begin{cases} 0 & \text{if } x \in [0, \infty)\backslash([\sigma, \sigma^*) \cup [\eta_2, \eta_1)), \\ 1 & \text{if } x \in [\sigma, \sigma^*), \\ -1 & \text{if } x \in [\eta_2, \eta_1). \end{cases}$$

Consider

$$\begin{aligned} \phi_\pi(s) - \phi_f(s) &= \int_0^s Z^f(x)\mu_1(h_1 - h_2)\big(a_1^\pi(x) - a_1^f(x)\big) + \mu_2 h_2\big(a_2^\pi(x) - a_2^f(x)\big)\, dx \\ &= [\mu_2 h_2 - \mu_1(h_1 - h_2)] \int_0^s Z^f(x)\big(1_{\{x\in[\sigma,\sigma^*)\}} - 1_{\{x\in[\eta_2,\eta_1)\}}\big)\, dx. \end{aligned}$$

Note that since the total amount of work done by each process must coincide, for $s \geq \eta_1$ the above integral is zero. On the other hand, since $[\sigma, \sigma^*) \cap [\eta_2, \eta_1) = \emptyset$, and $\sigma^* \leq \eta_2$, for $s < \eta_1$ the above integral is non-negative. Now, the assumption $\mu_2 h_2 \geq \mu_1(h_1 - h_2)$ implies $\phi_\pi(s) \geq \phi_f(s)$ for each $0 \leq s \leq t$.

It should be clear that the above argument can be repeated to create a finite sequence of policies $\{\pi_i, 0 \leq i \leq M\}$ such that π_M is exhaustive in station 2 and $\phi_{\pi_{i+1}}(s) \geq \phi_{\pi_i}(s)$ for each $0 \leq s \leq t$ where $\pi_0 = f$.

As in [23], we note that although the policies π_i, for $1 \leq i \leq M - 1$, are anticipating and thus not in Π, π_M is non-anticipating (since it is exhaustive). Moreover, since π_M only depends on f through the capacity increase/decrease decisions, the increase/decrease process does not violate the non-anticipation. The optimal policy's capacity increase/decrease policy may improve on this by using the queue length information currently available. That is, there exists an exhaustive policy that has expected cost lower than or equal to that of π_M and the result is proven. □

Note that since the policy π constructed in the proof of Theorem 3.3 has the same capacity increase/decrease process as that of f, the restriction that $r = 0$ is not necessary.

Moreover, we note that since the results of Theorem 3.3 hold for each t, they hold under the infinite horizon discounted and average cost cases (assuming stability) as well. In light of Proposition 3.1 they hold over the larger class of policies Π.

The results of this section suggest that the optimal capacity increase/decrease decisions are in some sense decoupled from the allocation decision. Originally we expected the two decisions to be intertwined, however, since the $c\mu$-rule is basically a "greedy" algorithm (drain the cost as fast as possible), this interpretation continues to hold no matter how much capacity is available. In the next section we analyze how capacity increase/decrease decisions are made.

4. Optimal capacity increase/decrease

In this section we consider the problem of when to increase and decrease workers. In the unrestricted model we show that the number of workers that should be made available is non-decreasing in both queue lengths. Although this result has also held true in all of our numerical studies in the controlled and uncontrolled capacity reduction cases, we have been unable to prove it to date. Recall from Example 2.4 that this monotonicity does not extend to the optimal number of workers.

Theorem 3.3 implies that when $\mu_2 h_2 \geq (\leq)\, \mu_1(h_1 - h_2)$ we may restrict attention to policies that are exhaustive in queue 2 (1). Consider the unrestricted case and suppose $\mu_2 h_2 \geq \mu_1(h_1 - h_2)$. For a (finite) function g on $\mathbb{X}^U$, H_β^U can now be simplified

$$H_\beta^U g(i,j) = \begin{cases} \min\limits_{k\in\{0,1,\ldots,\ell\}} \{r(k) + \beta(\lambda g(i+1,j) + k\mu_2 g(i,j-1) \\ \quad +[1-(\lambda + k\mu_2)]g(i,j))\} & \text{for } j \geq 1, \\ \min\limits_{k\in\{0,1,\ldots,\ell\}} \{r(k) + \beta(\lambda g(i+1,j) + k\mu_1 g(i-1,j+1) \\ \quad +[1-(\lambda + k\mu_1)]g(i,j))\} & \text{for } j = 0, \end{cases}$$

where for $i = j = 0$ set $k = 0$ in the minimum above. There is an analogous mapping with $k\mu_2 g(i, j-1)$ replaced with $k\mu_1 g(i-1, j+1)$ for the case with $\mu_2 h_2 < \mu_1(h_1 - h_2)$.

Inspecting $H_\beta^U g(i,j)$, we note that if $r(k)$ is concave, the minimum is that of a concave function. Thus, its minimum is achieved on the boundary (i.e. at $k = 0$ or ℓ). This is stated in the following proposition.

Proposition 4.1. If $r(k)$ is concave, it is optimal to have either zero workers or ℓ workers in the system.

When $r(k)$ is a general non-decreasing function but perhaps not concave, the optimal policy depends on the number of jobs in each queue as well as $r(k)$. Therefore, we investigate the structure of the optimal policy with respect to the number of jobs in the system. For the remainder of this section assume that $\mu \equiv \mu_1 = \mu_2$ and that $r(k)$ is a general non-decreasing function. Since β will be fixed throughout most of the section, we write v_n for $v_{n,\beta}$. The main theorem of this section is stated next.

Theorem 4.2. In the n-stage discounted cost problem, for each fixed (i, j), there exists an optimal number of workers to be made available, say $L_{n,\beta}(i, j)$. Moreover, these levels are such that $L_{n,\beta}(i, j)$ is non-decreasing in both i and j. Similarly for the infinite horizon discounted cost and, under the assumptions of Proposition 2.2, for average cost models.

Lemma 4.3. Consider the following cases.

1. If $2h_2 \geq h_1$, then for all $n \geq 0$,
   (a) $\Delta_i v_n^U(i, j-1) \leq \Delta_i v_n^U(i, j)$ for $j \geq 1$,
   (b) $\Delta_j v_n^U(i, j-1) \leq \Delta_j v_n^U(i, j)$, for $j \geq 1$,
   (c) $\Delta_i v_n^U(i-1, 1) \leq \Delta_i v_n^U(i, 0)$, for $i \geq 1$
   (d) $\Delta_j v_n^U(i, 0) \geq v_n^U(i, 0) - v_n^U(i-1, 1) \geq \Delta_j v_n^U(i-1, 0)$.
2. If $h_1 > 2h_2$, then for all $n \geq 0$,
   (a) $\Delta_j v_n^U(i, j) \leq \Delta_j v_n^U(i, j+1)$ for $i, j \geq 0$,
   (b) $\Delta_j v_n^U(i-1, j+1) \leq \Delta_j v_n^U(i, j)$ for $i \geq 1$ and $j \geq 0$.

Proof. See Appendix. □

Proof of Theorem 4.2 (and 2.3 (3)). We consider the cases when the optimal allocation policy is exhaustive at station 1 and 2 separately.

Case 1. $h_1 \leq 2h_2$. Recall that this implies the optimal allocation is for all available workers to serve exhaustively at station 2. Suppose instead of the optimal policy at stage n we use $d_n(i, j)$ in state $(i+1, j)$ and $d_n(i+1, j)$ in state (i, j). Then for $j \geq 1$

$$\begin{aligned} v_n^U(i, j) &= ih_1 + jh_2 + r(d_n(i, j)) + \lambda v_{n-1}^U(i+1, j) + d_n(i, j)\mu v_{n-1}^U(i, j-1) \\ &\quad + [1 - (\lambda + d_n(i, j)\mu)]v_{n-1}^U(i, j) \\ &\leq ih_1 + jh_2 + r(d_n(i+1, j)) + \lambda v_{n-1}^U(i+1, j) \\ &\quad + d_n(i+1, j)\mu v_{n-1}^U(i, j-1) + [1 - (\lambda + d_n(i+1, j)\mu)]v_{n-1}^U(i, j), \end{aligned}$$

and

$$\begin{aligned} v_n^U(i+1, j) &= (i+1)h_1 + jh_2 + r(d_n(i+1, j)) + \lambda v_{n-1}^U(i+2, j) \\ &\quad + d_n(i+1, j)\mu v_{n-1}^U(i+1, j-1) \\ &\quad + [1 - (\lambda + d_n(i+1, j)\mu)]v_{n-1}^U(i+1, j) \\ &\leq (i+1)h_1 + jh_2 + r(d_n(i, j)) + \lambda v_{n-1}^U(i+2, j) \\ &\quad + d_n(i, j)\mu v_{n-1}^U(i+1, j-1) + [1 - (\lambda + d_n(i, j)\mu)]v_{n-1}^U(i+1, j). \end{aligned}$$

Thus,

$$\begin{aligned}\Delta_i v_n^U(i,j) \geq h_1 + \lambda\Delta_i v_{n-1}^U(i+1,j) + d_n(i+1,j)\mu\Delta_i v_{n-1}^U(i,j-1)\\ + [1-(\lambda + d_n(i+1,j)\mu)]\Delta_i v_{n-1}^U(i,j),\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\Delta_i v_n^U(i,j) \leq h_1 + \lambda\Delta_i v_{n-1}^U(i+1,j) + d_n(i,j)\mu\Delta_i v_{n-1}^U(i,j-1)\\ + [1-(\lambda + d_n(i,j)\mu)]\Delta_i v_{n-1}^U(i,j).\end{aligned} \tag{4.2}$$

Subtracting (4.1) from (4.2) (and dividing by μ) yields

$$[d_n(i,j) - d_n(i+1,j)]\big(\Delta_i v_{n-1}^U(i,j-1) - \Delta_i v_{n-1}^U(i,j)\big) \geq 0. \tag{4.3}$$

When $j = 0$, and $i \geq 1$ a similar argument yields

$$[d_n(i,0) - d_n(i+1,0)]\big(\Delta_i v_{n-1}^U(i-1,1) - \Delta_i v_{n-1}^U(i,0)\big) \geq 0. \tag{4.4}$$

By symmetry we have the following inequalities also hold for $j \geq 1$,

$$[d_n(i,j) - d_n(i,j+1)]\big(\Delta_j v_{n-1}^U(i,j-1) - \Delta_j v_{n-1}^U(i,j)\big) \geq 0, \tag{4.5}$$

and for $j = 0$ and $i \geq 1$,

$$[d_n(i,0) - d_n(i,1)]\big(v_{n-1}^U(i,0) - v_{n-1}^U(i-1,1) - \Delta_j v_{n-1}^U(i,0)\big) \geq 0. \tag{4.6}$$

As noted previously, when $i = j = 0$ it should be clear that it is optimal to have zero workers and the monotonicity holds trivially. Lemma 4.3 implies that in (4.3)–(4.6) we have the difference in the actions between two states multiplied by a non-positive quantity. Consider for example (4.3). If $\Delta_i v_{n-1}^U(i,j-1) - \Delta_i v_{n-1}^U(i,j) < 0$, $d_n(i,j) - d_n(i+1,j) \leq 0$ and the result is proven. We note that the minimum in $H_\beta^U v_{n-1}^U(i,j)$ is realized when $r(k) + k\mu[v_{n-1}^U(i,j-1) - v_{n-1}^U(i,j)]$ is minimized. Similarly, for $H_\beta^U v_{n-1}^U(i+1,j)$ and $r(k)+k\mu[v_{n-1}^U(i+1,j-1)-v_{n-1}^U(i+1,j)]$. Thus, if $\Delta_i v_{n-1}^U(i,j-1) - \Delta_i v_{n-1}^U(i,j) = 0$ we may choose the number of workers in each state to be the same and the result is proven. The infinite horizon cases are precisely the same upon recalling that both v_β^U and w^U, solutions to the DCOE and ACOE, respectively, can be obtained by taking limits. That is, the inequalities of Lemma 4.3 hold with v_n^U replaced with either v_β^U or w^U. This completes the proof of Case 1.

Case 2. $h_1 > 2h_2$. It is optimal to serve exhaustively at station 1. We first show that $L_{n,\beta}(i,j)$ is non-decreasing in j. In an analogous manner to the case when $h_1 \leq 2h_2$, for $i \geq 1$

$$[d_n(i,j) - d_n(i,j+1)]\big(\Delta_j v_{n-1}^U(i-1,j+1) - \Delta_j v_{n-1}^U(i,j)\big) \geq 0, \tag{4.7}$$

and for $i = 0$

$$[d_n(0, j) - d_n(0, j+1)]\big(\Delta_j v_{n-1}^U(0, j-1) - \Delta_j v_{n-1}^U(0, j)\big) \geq 0. \tag{4.8}$$

The proof now follows in the same manner as Case 1 by applying the second set of inequalities in Lemma 4.3.

To prove the result for i, note that for $i \geq 1$

$$[d_n(i, j) - d_n(i+1, j)]\big(\Delta_i v_{n-1}^U(i-1, j+1) - \Delta_i v_{n-1}^U(i, j)\big) \geq 0, \tag{4.9}$$

and for $i = 0$

$$[d_n(0, j) - d_n(1, j)]\big(v_{n-1}^U(0, j+1) - v_{n-1}^U(0, j-1) - \Delta_i v_{n-1}^U(0, j)\big) \geq 0. \tag{4.10}$$

Thus, we must show that the following inequalities hold for all $n \geq 0$:

1. $\Delta_i v_n^U(i-1, j+1) - \Delta_i v_n^U(i, j) \leq 0$ for $i \geq 0$,
2. $v_n^U(i, j+1) - v_n^U(i+1, j) - v_n^U(i, j-1) + v_n^U(i, j) \leq 0$ for $i \geq 0$, $j \geq 1$,
3. $v_n^U(i-1, j) - v_n^U(i, j-1) \leq 0$ for $i, j \geq 1$,

where the last inequality, required for the proof of the other two, is proven by choosing $d_n(i-1, j)$ in state $(i, j-1)$ and is omitted for brevity. For the first inequality, note that for $i \geq 2$

$$\begin{aligned}
&\Delta_i v_n^U(i-1, j+1) - \Delta_i v_n^U(i, j) \\
&\quad \leq \lambda\big[\Delta_i v_{n-1}^U(i, j+1) - \Delta_i v_{n-1}^U(i+1, j)\big] \\
&\qquad + d_n(i-1, j+1)\mu\big[\Delta_i v_n^U(i-2, j+2) - \Delta_i v_{n-1}^U(i-1, j+1)\big] \\
&\qquad + [1 - (\lambda + d_n(i+1, j)\mu)]\big[\Delta_i v_{n-1}^U(i-1, j+1) - \Delta_i v_{n-1}^U(i, j)\big],
\end{aligned}$$

and for $i = 1$

$$\begin{aligned}
&\Delta_i v_n^U(0, j+1) - \Delta_i v_n^U(1, j) \\
&\quad \leq \lambda\big[\Delta_i v_{n-1}^U(1, j+1) - \Delta_i v_{n-1}^U(2, j)\big] \\
&\qquad + d_n(0, j+1)\mu\big[v_{n-1}^U(0, j+2) - v_{n-1}^U(0, j) - \Delta_i v_{n-1}^U(0, j+1)\big] \\
&\qquad + [1 - (\lambda + d_n(2, j)\mu)]\big[\Delta_i v_{n-1}^U(0, j+1) - \Delta_i v_{n-1}^U(1, j)\big].
\end{aligned}$$

The inductive hypothesis yields the result. For the second assertion with $i \geq 1$

$$\begin{aligned}
&v_n^U(i, j+1) - v_n^U(i+1, j) - v_n^U(i, j-1) + v_n^U(i, j) \\
&\quad \leq 2h_2 - h_1 + \lambda\big[v_{n-1}^U(i+1, j+1) - v_{n-1}^U(i+2, j) - v_{n-1}^U(i+1, j-1) \\
&\qquad + v_{n-1}^U(i+1, j)\big] + [1 - (\lambda + d_n(i+1, j)\mu)]\big[v_{n-1}^U(i, j+1) - v_{n-1}^U(i+1, j) \\
&\qquad - v_{n-1}^U(i, j-1) + v_{n-1}^U(i, j)\big] + d_n(i+1, j)\mu\big[v_{n-1}^U(i-1, j+2) - v_{n-1}^U(i, j+1) \\
&\qquad - v_{n-1}^U(i-1, j) + v_{n-1}^U(i-1, j+1)\big] + [d_n(i+1, j) - d_n(i, j-1)]\mu \\
&\qquad \times \big[v_{n-1}^U(i-1, j+2) - v_{n-1}^U(i, j+1)\big],
\end{aligned}$$

and for $i = 0$ and $j \geq 1$

$$\begin{aligned}
&v_n^U(0, j+1) - v_n^U(1, j) - v_n^U(0, j-1) + v_n^U(0, j) \\
&\quad \leq 2h_2 - h_1 + \lambda\big[v_{n-1}^U(1, j+1) - v_{n-1}^U(2, j) - v_{n-1}^U(1, j-1) + v_{n-1}^U(1, j)\big] \\
&\qquad + [1 - (\lambda + d_n(1, j)\mu)]\big[v_{n-1}^U(0, j+1) - v_{n-1}^U(1, j) - v_{n-1}^U(0, j-1) + v_{n-1}^U(0, j)\big] \\
&\qquad + d_n(0, j-1)\mu\big[\Delta_j v_{n-1}^U(0, j-2) - \Delta_j v_{n-1}^U(0, j)\big].
\end{aligned}$$

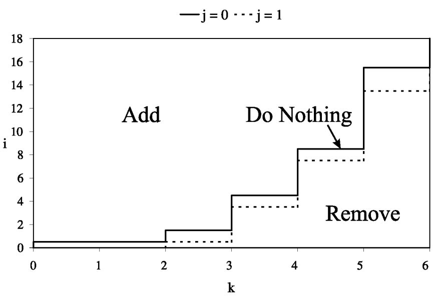
With the exception of the last term, in each case, the inductive hypothesis suffices for the non-positivity of each term. Furthermore, note that the inductive hypothesis implies, via (4.9) and (4.10), that $d_n(i+1, j) \geq d_n(i, j)$. Thus, since $d_n(i, j)$ is also non-decreasing in j, $d_n(i+1, j) \geq d_n(i, j-1)$. This completes the proof for $i \geq 1$. As for the last term for $i = 0$, recall that $\Delta_j v_n^U(i+1, j) \geq \Delta_j v_n^U(i, j)$ so the result is proven.

As in the proof for Case 1, noting that solutions to the DCOE and ACOE can be obtained by taking limits, the results hold for the infinite horizon discounted and average cost cases. □

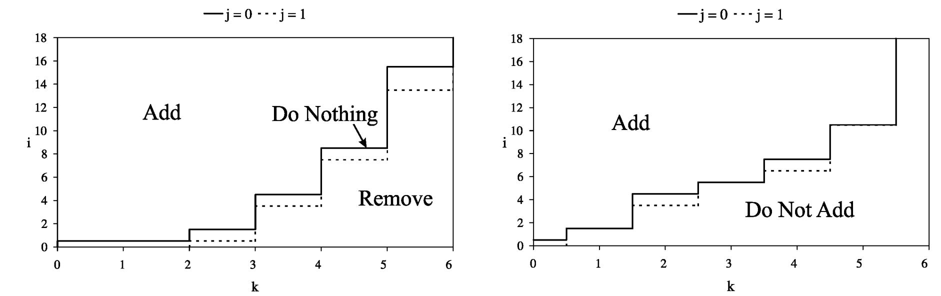
5. Numerical examples

In this section we include examples under the average cost criterion and propose promising heuristics based on the results in Section 4. A truncated state space is used to facilitate computation. In the next two examples the optimal allocation is exhaustive in queue 2, and hence only states with $j = 0$ or $j = 1$ will be occupied in the long-run. Since $\mu_1 = \mu_2$, the results of Theorem 2.3 part 3 also apply.

Example 5.1. Suppose we have the following parameter settings: $h_1 = 1$; $h_2 = 2$; $r(k) = k^2$; $\ell = 6$; $\lambda = .5$; $\mu_1 = \mu_2 = .5$; $\alpha(k) = .5$; $\gamma = .1$. Figure 2(a) displays the optimal capacity levels given the number of customers for the unrestricted model. Figure 2(b) displays the optimal capacity target levels for the controlled capacity reduction model; if k is less than the target level, then it is optimal to add workers when the opportunities arise up to the target, and if k is greater than the target level, then it is optimal to decrease the number of workers down to the target. For example, when $i = 3$,



(a) Unrestricted optimal capacity levels.



(b) Controlled reduction optimal capacity target levels: add (remove) workers when k is less (more) than target.

(c) Uncontrolled reduction optimal capacity increase policy.

Figure 2. Optimal policies for Example 5.1 with convex worker cost function $r(k) = k^2$.

$j = 0$, it is optimal add workers if $k < 3$, and remove workers when $k > 3$. Figure 2(c) displays the optimal capacity increase policy for the uncontrolled reduction model. For example, when $i = 3$, $j = 0$, it is optimal to add workers if $k < 2$.

The average cost in the unrestricted, controlled capacity reduction and uncontrolled capacity reduction cases is 7.6024, 8.2007 and 9.9530, respectively. As is expected, the unrestricted case dominates the controlled and uncontrolled capacity reduction cases (approximately 8% and 31%, respectively). In the class of policies with a fixed number of servers, the optimal average cost is 11.0000 (3 workers is optimal in this class). This is approximately 45% more than the unrestricted case. This indicates (at least for this example) that a company can derive significant benefits from having dynamically flexible capacity.

We know from Theorem 2.3 part 3 that the optimal number of workers for the unrestricted case is monotone in the number of jobs. This is evident in Figure 2(a); the optimal capacity levels are increasing in both i and j. The optimal policy for the controlled capacity reduction case, Figure 2(b), has a form close to the unrestricted policy and is also monotone in i and j. In addition, it is monotone in k. Likewise, the optimal policy for the uncontrolled capacity reduction case, Figure 2(c), is monotone

Table 1
Optimal average costs for Example 5.1 in the controlled reduction case for various $\alpha(k) = \alpha$. The second row compares this to the unrestricted optimal cost 7.6024.

α	0.1	0.2	0.5	1	2	5	10	20	50	100
Optimal cost	8.8802	8.6651	8.2007	7.9933	7.9069	7.7243	7.6632	7.6328	7.6145	7.6085
% from unrestricted	16.81%	13.98%	7.87%	5.14%	4.01%	1.60%	0.80%	0.40%	0.16%	0.08%

Table 2
Uncontrolled reduction optimal average costs for Example 5.1 with various $\alpha(k) = \alpha$ and γ. NA indicates infeasibility. The minimum cost in each column is in bold.

	α									
γ	0.1	0.2	0.5	1	2	5	10	20	50	100
0.01	**10.9166**	**9.9866**	9.6393	9.5561	9.5159	9.4960	9.4874	9.4830	9.4804	9.4796
0.05	NA	12.2387	**9.3982**	8.9979	8.8438	8.7609	8.7337	8.7207	8.7133	8.7108
0.1	NA	NA	9.9530	**8.9535**	**8.6594**	8.5054	8.4637	8.4449	8.4343	8.4309
0.5	NA	NA	NA	NA	9.2114	**8.2980**	**8.0951**	8.0198	7.9797	7.9676
1	NA	NA	NA	NA	NA	8.5664	8.1405	**7.9765**	**7.8836**	7.8555
5	NA	NA	NA	NA	NA	NA	NA	8.5887	7.8991	**7.7737**

in all three state variables. As stated previously, while we were not able to prove it, we conjecture that the optimal policies for the controlled and uncontrolled reduction cases are monotone in i, j, and k when $r(k)$ is non-decreasing and convex.

Tables 1 and 2 show the effect on the optimal average cost as $\alpha(k)$ is varied in the controlled reduction case, and $\alpha(k)$ and γ are varied in the uncontrolled reduction case, respectively. In the controlled reduction case, as $\alpha(k) = \alpha$ becomes large, that is, when excess capacity becomes available more frequently, the average cost decreases towards that of the unrestricted case. For a fixed γ, the average cost also decreases in the uncontrolled reduction case as α increases. For a fixed α, however, the average cost is not monotone in γ. For each α there is an "optimal" γ. If γ is too low, excess workers stay too long, while if γ is too high, the workers do not stay long enough. This suggests that heuristics based on the unrestricted case may do well in some cases and not well in others.

Example 5.2. Suppose we have the same parameters as in Example 5.1 except $r(k) = 15\sqrt{k}$. Figure 3 displays the optimal capacity increase policy for the uncontrolled reduction model.

We have seen in Examples 2.4 and 5.2 that optimal policies need not be monotone in k even when $r(k)$ is increasing. Our numerical studies suggest that such monotonicity is guaranteed only when $r(k)$ is convex. Example 5.2 has a worker cost function that is increasing and concave. We see in Figure 3 that the optimal policy for the uncontrolled

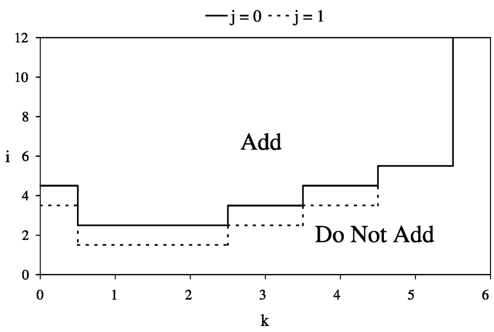


Figure 3. Uncontrolled reduction optimal capacity increase policy of Example 5.2 with concave worker cost function $r(k) = 15\sqrt{k}$.

reduction case is not monotone in k. For example, when $i = 3$, it is optimal to not add workers when $k = 0$, optimal to add workers when $k = 1$, and then optimal to not add workers again for larger k.

5.1. Heuristics

In this section, we consider some heuristics that have shown promise. An easily characterized heuristic is to try to maintain a constant workforce level independent of the number of jobs in the system. As noted for Example 5.1, this does not work well in the unrestricted case (costs 45% more than the optimal policy). Instead consider the policy that increases capacity to a constant level, say L, when there are jobs in the system and then removes all workers when there are no jobs. We refer to such a policy as a *0-L* heuristic. For the unrestricted case of Example 5.1, $L = 3$ is optimal in this class of policies with an average cost of 8.0000, only 5.23% above the optimal policy cost. For this system we exploited the fact that under a constant level of 3 workers, the probability of having zero customers in the system is significant: .3333. Thus, in this case, the 0-L heuristic significantly reduces the worker idle time.

For the controlled and uncontrolled reduction cases, a 0-L heuristic adds workers (when the opportunities arise) if and only if there are a positive number of jobs in the system and fewer than L workers on hand. In addition in the controlled capacity reduction case, the heuristic policy removes all workers when there are no jobs in the system. The results for the controlled reduction case of Example 5.1 are given in Table 3 for various values of $\alpha(k) = \alpha$. Again $L = 3$ performs the best. We see that for some parameters this heuristic performs well, while for other parameters it does not. In the uncontrolled capacity reduction case, the heuristic in general does not perform well. For example, for the parameters of Example 5.1 with $\alpha(k) = .5$ and $\gamma = .1$, the best such heuristic is only within 22% of optimal (average cost of 12.1440 compared to optimal 9.9530).

For the uncontrolled reduction case, consider the following policy: when there are less than T jobs in the system (in total), add workers only when there are less than L_1 servers on hand; when there are T or more jobs, add workers up to a total of

Table 3
Average costs of a 0-L heuristic with $L = 3$ applied to the controlled reduction model of Example 5.1 with various $\alpha(k) = \alpha$.

α	0.1	0.2	0.5	1	2	5	10	20	50	100
Heuristic cost	14.3166	11.1361	9.1993	8.5555	8.0838	8.2476	8.0380	8.0179	8.0069	8.0034
% from optimal	61.22%	28.52%	12.18%	7.03%	4.31%	4.65%	4.89%	5.05%	5.15%	5.19%

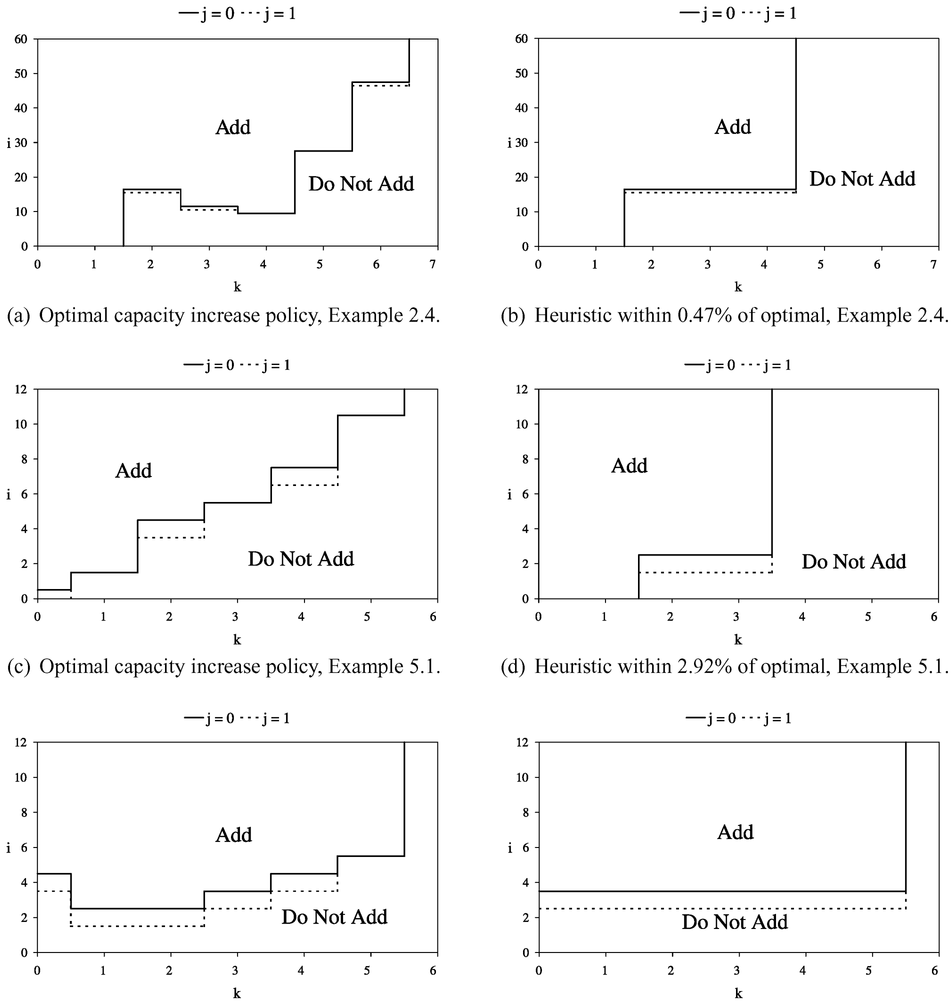
L_2. Thus, there are two target worker levels, L_1 and L_2, and a threshold total number of jobs T that indicates when to switch between the two. We refer to such a policy as a *2-level* heuristic. A 2-level heuristic may perform well and is fairly simple to implement.

For Example 5.1, the best 2-level heuristic has $L_1 = 2$, $L_2 = 4$, and $T = 3$. This is depicted in Figure 4(d). The average cost is 10.2438, within 2.92% of optimal. Figure 4(b) shows a 2-level heuristic for Example 2.4 that performs within 0.47% of optimal. The best 2-level heuristic for Example 5.2 is depicted in Figure 4(f). Here, $L_1 = 0$ and $L_2 = 6$ and the policy reduces to a simple job threshold policy that adds capacity if and only if there are $T = 4$ or more jobs in the system, independent of the number of workers (up to the capacity limit). This policy performs within 0.82% of optimal. It should be noted that a single job threshold policy ($L_1 = 0$ and $L_2 = \ell$) does not always perform well. For example, the best such policy for Example 5.1 has $T = 3$ and is only within 17.63% of optimal.

5.1.1. 2-Level heuristic numerical study

In this section we present the results of a numerical study of how well 2-level heuristics perform for a variety of parameters. For the study, the arrival rate and the total number of workers allowed are kept fixed at $\lambda = .5$ and $\ell = 6$. Three values for the ratio h_2/h_1 are considered: .5, 2, and 5; $h_1 = 2$ and $h_2 = 1$, $h_1 = 1$ and $h_2 = 2$, and $h_1 = 1$ and $h_2 = 5$, respectively. There are also three possible values for the ratio μ_1/μ_2: .5, 1, and 2. The specific values for μ_1 and μ_2 are such that the offered load, $\lambda(1/\mu_1 + 1/\mu_2)$, is equal to 2. The resulting values for μ_1 and μ_2 are $\mu_1 = .5$ and $\mu_2 = .5$, $\mu_1 = .75$ and $\mu_2 = .375$, and $\mu_1 = .375$ and $\mu_2 = .75$. Of the nine combinations of ratios h_2/h_1 and μ_1/μ_2, two combinations with $h_2/h_1 = .5$ result in optimal exhaustive allocation at queue 1 (with one of these two being indifferent between allocation to queue 1 or to queue 2); the rest are exhaustive at queue 2.

For the uncontrolled reduction case, we consider three values for the ratio α/γ. In order for the system to be stable, with an offered load of 2; it must be that $\alpha/\gamma >$ 2.0258. The ratios α/γ are chosen to be 3, 5, and 10. As a base case, consider the convex worker cost function $r(k) = k^2$. For this cost function, γ is chosen to be .05, .1, and .5. Hence, the base set of parameters have nine different α-γ combinations and 81 overall parameter combinations. In particular, the parameters of Example 5.1 are included.



(a) Optimal capacity increase policy, Example 2.4.

(b) Heuristic within 0.47% of optimal, Example 2.4.

(c) Optimal capacity increase policy, Example 5.1.

(d) Heuristic within 2.92% of optimal, Example 5.1.

(e) Optimal capacity increase policy, Example 5.2.

(f) Heuristic within 0.82% of optimal, Example 5.2.

Figure 4. Optimal uncontrolled reduction capacity increase policies and respective heuristics.

The results of the numerical study are given in Table 4. The average percent difference between the optimal average cost and the average cost of the best 2-level heuristic is 2.77%. We see that the percent difference gets smaller as α decreases (α/γ decreases for a fixed γ). In fact, the average percent difference for $\alpha \le \lambda = .5$ is only 1.77%. When α/γ is 3 (or less), it turns out that the best 2-level heuristics always have $L_2 = 6$; full capacity. In these cases both the optimal policy and the heuristic policy spend a lot of time in states where it is optimal to hire, and the percent differences between their average costs are very small. As α increases, the percent of arrivals that are hired decreases. For the parameters of Example 5.1 with $\alpha = .5$, the best 2-level heuristic has $L_1 = 2$, $L_2 = 4$,

Table 4
Results of the numerical study for the uncontrolled reduction base case with $r(k) = k^2$. Given are the optimal average cost, the average cost of the best 2-level heuristic, and the percent difference.

			$\frac{h_2}{h_1} = .5$			$\frac{h_2}{h_1} = 2$			$\frac{h_2}{h_1} = 5$		
			$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$
$\gamma = .05$	$\frac{\alpha}{\gamma} = 3$	Opt:	24.905	24.637	21.209	16.753	16.823	17.043	17.621	18.122	18.769
		Heur:	24.938	24.670	21.232	16.843	16.911	17.124	17.684	18.173	18.806
		% Diff:	0.14%	0.13%	0.11%	0.54%	0.52%	0.48%	0.36%	0.28%	0.20%
	$\frac{\alpha}{\gamma} = 5$	Opt:	13.805	13.594	12.452	10.786	10.881	11.074	11.632	12.138	12.750
		Heur:	13.953	13.737	12.606	11.064	11.143	11.333	11.871	12.355	12.947
		% Diff:	1.07%	1.05%	1.24%	2.58%	2.41%	2.34%	2.05%	1.79%	1.55%
	$\frac{\alpha}{\gamma} = 10$	Opt:	11.344	11.148	10.393	9.312	9.398	9.593	10.140	10.652	11.248
		Heur:	11.732	11.529	10.794	9.630	9.698	9.907	10.462	10.947	11.572
		% Diff:	3.42%	3.42%	3.86%	3.41%	3.19%	3.27%	3.17%	2.77%	2.88%
$\gamma = .1$	$\frac{\alpha}{\gamma} = 3$	Opt:	19.024	18.767	16.479	13.491	13.563	13.783	14.361	14.868	15.519
		Heur:	19.080	18.820	16.654	13.599	13.671	13.889	14.469	14.978	15.630
		% Diff:	0.30%	0.28%	1.06%	0.80%	0.80%	0.77%	0.76%	0.74%	0.72%
	$\frac{\alpha}{\gamma} = 5$	Opt:	12.310	12.102	11.187	9.861	9.953	10.152	10.708	11.231	11.836
		Heur:	12.648	12.442	11.578	10.159	10.244	10.438	10.995	11.499	12.110
		% Diff:	2.74%	2.81%	3.50%	3.03%	2.92%	2.82%	2.68%	2.39%	2.31%
	$\frac{\alpha}{\gamma} = 10$	Opt:	10.663	10.471	9.813	8.848	8.954	9.160	9.696	10.202	10.793
		Heur:	11.044	10.850	10.166	9.166	9.261	9.451	10.020	10.516	11.106
		% Diff:	3.57%	3.62%	3.60%	3.59%	3.44%	3.17%	3.34%	3.08%	2.91%
$\gamma = .5$	$\frac{\alpha}{\gamma} = 3$	Opt:	12.951	12.951	11.822	10.306	10.384	10.599	11.175	11.684	12.332
		Heur:	13.428	13.178	12.138	10.606	10.690	10.904	11.492	11.986	12.606
		% Diff:	3.69%	1.76%	2.67%	2.91%	2.94%	2.87%	2.84%	2.58%	2.23%
	$\frac{\alpha}{\gamma} = 5$	Opt:	10.325	10.140	9.578	8.712	8.825	9.007	9.548	10.045	10.619
		Heur:	10.870	10.666	10.081	9.083	9.168	9.366	9.932	10.443	11.064
		% Diff:	5.28%	5.19%	5.25%	4.26%	3.89%	3.98%	4.02%	3.96%	4.19%
	$\frac{\alpha}{\gamma} = 10$	Opt:	9.491	9.358	8.893	8.190	8.298	8.498	9.042	9.524	10.072
		Heur:	10.121	9.921	9.429	8.659	8.753	8.941	9.504	9.999	10.571
		% Diff:	6.64%	6.02%	6.02%	5.73%	5.49%	5.20%	5.11%	4.98%	4.96%

$T = 3$, and a 2.92% percent difference; when $\alpha = .3$, the best heuristic is $L_1 = 3$, $L_2 = 6, T = 4$, and the percent difference is only 0.80%; when $\alpha = 1$, the best heuristic is $L_1 = 2, L_2 = 4, T = 5$, and the percent difference is 3.44%. Increasing the opportunities to add workers makes the optimal policy in a sense more "dynamic" and the 2-level heuristic may not do as well in matching it. Below we will see results for other worker cost functions where sometimes the percent difference in costs may actually decrease as α increases.

So far we have only defined a 2-level heuristic for the uncontrolled reduction case. We can also define 2-level heuristics for the controlled reduction and unrestricted cases. In the unrestricted case, when there are less than T jobs in system, have L_1 workers on

Table 5
Results (optimal average cost, cost of the best 2-level heuristic, and percent difference) of the numerical study for the controlled reduction case of the base parameters with $r(k) = k^2$.

		$\frac{h_2}{h_1} = .5$			$\frac{h_2}{h_1} = 2$			$\frac{h_2}{h_1} = 5$		
		$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$
$\alpha = .15$	Opt:	10.439	10.265	9.544	8.647	8.735	8.919	9.461	9.958	10.547
	Heur:	10.620	10.418	9.672	8.737	8.819	9.012	9.560	10.054	10.651
	% Diff:	1.73%	1.50%	1.34%	1.04%	0.96%	1.04%	1.04%	0.97%	0.99%
$\alpha = .25$	Opt:	10.131	9.948	9.336	8.530	8.626	8.806	9.337	9.822	10.398
	Heur:	10.262	10.070	9.457	8.626	8.706	8.895	9.433	9.917	10.508
	% Diff:	1.30%	1.22%	1.30%	1.13%	0.92%	1.01%	1.03%	0.97%	1.06%
$\alpha = .3$	Opt:	10.058	9.872	9.286	8.399	8.506	8.685	9.244	9.761	10.360
	Heur:	10.172	9.983	9.399	8.601	8.680	8.868	9.402	9.883	10.470
	% Diff:	1.14%	1.13%	1.22%	2.41%	2.04%	2.11%	1.71%	1.24%	1.06%
$\alpha = .5$	Opt:	9.768	9.591	8.974	8.104	8.201	8.373	8.909	9.404	9.980
	Heur:	10.002	9.819	9.292	8.400	8.489	8.661	9.182	9.657	10.224
	% Diff:	2.40%	2.38%	3.53%	3.65%	3.51%	3.43%	3.06%	2.69%	2.44%
$\alpha = 1$	Opt:	9.266	9.099	8.606	7.903	7.993	8.160	8.670	9.140	9.691
	Heur:	9.866	9.689	9.018	8.226	8.308	8.470	8.958	9.401	9.933
	% Diff:	6.48%	6.49%	4.79%	4.09%	3.94%	3.80%	3.32%	2.86%	2.50%
$\alpha = 1.5$	Opt:	9.106	8.942	8.486	7.846	7.934	8.098	8.597	9.056	9.597
	Heur:	9.732	9.540	8.918	8.196	8.274	8.432	8.904	9.334	9.850
	% Diff:	6.87%	6.68%	5.09%	4.46%	4.30%	4.13%	3.58%	3.07%	2.63%
$\alpha = 2.5$	Opt:	8.985	8.822	8.392	7.761	7.847	8.005	8.486	8.931	9.452
	Heur:	9.626	9.437	8.854	8.114	8.190	8.342	8.800	9.217	9.714
	% Diff:	7.14%	6.97%	5.52%	4.54%	4.38%	4.22%	3.70%	3.20%	2.77%
$\alpha = 5$	Opt:	8.792	8.635	8.221	7.639	7.724	7.882	8.359	8.802	9.312
	Heur:	9.453	9.266	8.687	8.009	8.084	8.234	8.683	9.094	9.582
	% Diff:	7.51%	7.31%	5.67%	4.84%	4.65%	4.46%	3.88%	3.32%	2.89%

hand. When there are T or more jobs, have L_2 workers. In the uncontrolled reduction case, hire when there are less than T jobs and less than L_1 workers on hand. When there are less than T jobs and more than L_1 workers, reduce capacity down to L_1. When there are T or more jobs, hire up to or reduce down to L_2.

Results for 2-level heuristics in the controlled reduction and unrestricted cases for the base set of parameters are given in Tables 5 and 6, respectively. While there are nine different α/γ ratios in the uncontrolled reduction case, there are only eight different values for α. These are the values studied in the controlled reduction case. The average difference in costs for the controlled reduction case (over 8 different α's) is 3.16%, with an average of only 1.71% for $\alpha \leq \lambda$. The average difference in costs for the unrestricted case is 5.32%. For the parameters of Example 5.1, the best 2-level heuristic in the unrestricted case is $L_1 = 0$, $L_2 = 3$, and $T = 1$ with a cost difference of 5.23%. In other words, the best 2-level heuristic is a 0-L heuristic with $L = 3$, the best 0-L heuristic. All of the best 2-level heuristics in the unrestricted case are 0-L heuristics. In

Table 6
Results (optimal average cost, cost of the best 2-level heuristic, and percent difference) of the numerical study for the unrestricted case of the base parameters with $r(k) = k^2$.

	$\frac{h_2}{h_1} = .5$			$\frac{h_2}{h_1} = 2$			$\frac{h_2}{h_1} = 5$		
	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$
Opt:	8.544	8.387	8.007	7.517	7.602	7.760	8.233	8.671	9.170
Heur:	9.185	9.000	8.476	7.926	8.000	8.148	8.593	9.000	9.481
% Diff:	7.50%	7.31%	5.87%	5.43%	5.23%	5.00%	4.36%	3.79%	3.40%

the uncontrolled case, the parameters of Example 5.1 result in a best 2-level heuristic of $L_1 = 1$, $L_2 = 3$, and $T = 1$ with a cost difference of 3.51%. This is better than the 12.18% of the best 0-L heuristic found in Table 3. Note that in the controlled reduction case the cost difference is not always non-decreasing in α.

In addition to the convex worker cost function $r(k) = k^2$ (Function 1), consider the linear function $r(k) = 6k$ (Function 2), the concave function in Example 5.2, $r(k) = 15\sqrt{k}$ (Function 3), and the general function (neither concave nor convex) given by the vector $r = [0, 8, 13, 17, 22, 30, 36]$ (Function 4). Numerical results for these functions are given in Tables 7–9 for the uncontrolled reduction, controlled reduction, and unrestricted cases, respectively. We only give results for the case $\gamma = .1$ and the three resulting α's when $\alpha/\gamma = 3$, 5, and 10.

For Function 1 the average cost difference in the uncontrolled case restricted to $\gamma = .1$ is 2.29%. For Functions 2, 3, and 4 this average difference is 1.75%, 0.94%, and 1.43%, respectively. For Function 1 in the controlled firing case the overall average cost difference for $\alpha = .3$, .5, and 1 is 2.94%. For Functions 2, 3, and 4 this average difference is 0.37%, 0.90%, and 0.42%, respectively. For Function 1 in the unrestricted case the average cost difference is 5.32%. For Functions 2, 3, and 4 this average difference is 0.00%, 0.00%, and 0.74%. For Functions 2, 3, and 4 the 2-level heuristics do very well. In fact, for Functions 2 and 3, both of which are concave, the best 2-level heuristic in the unrestricted case is always $L_1 = 0$, $L_2 = 6$, and $T = 1$, which coincides with the overall optimal policy. For Function 3, the best 2-level heuristic in the unrestricted case is always $L_1 = 0$, $L_2 = 4$, and $T = 1$, a 0-L heuristic. Since it is not concave, it does not have $L_2 = 6$ and does not coincide exactly with the overall optimal policy. Note again that the percent difference in costs is not always non-decreasing in α. For example, for the parameters of Example 5.1 except with worker cost Function 4, the best 2-level heuristic ($L_1 = 1$, $L_2 = 4$, $T = 3$) is within 1.98% of the optimal cost while the best 2-level heuristics for α lowered to .3 ($L_1 = 3$, $L_2 = 6$, $T = 4$) and α raised to 1 ($L_1 = 0$, $L_2 = 4$, $T = 4$) are within 0.53% and 1.53%, respectively.

The 2-level heuristics displayed in Figures 4(b), (d), and (f) seem to work well because they are essentially smoothing the optimal policies, Figures 4(a), (c), and (e), respectively. While we found that 2-level heuristics work well in many settings and are

Table 7
Results (optimal average cost, cost of the best 2-level heuristic, and percent difference) of the numerical study for the uncontrolled reduction case of the base parameters restricted to $\gamma = .1$ with various worker cost functions.

			$\frac{h_2}{h_1} = .5$			$\frac{h_2}{h_1} = 2$			$\frac{h_2}{h_1} = 5$		
			$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$
Function 2	$\frac{\alpha}{\gamma} = 3$	Opt:	26.037	25.778	23.483	20.369	20.450	20.670	21.261	21.779	22.439
(linear)		Heur:	26.106	25.847	23.552	20.536	20.617	20.833	21.415	21.924	22.575
		% Diff:	0.26%	0.27%	0.29%	0.82%	0.81%	0.79%	0.72%	0.67%	0.61%
	$\frac{\alpha}{\gamma} = 5$	Opt:	19.661	19.479	18.539	17.079	17.191	17.381	17.957	18.491	19.112
		Heur:	20.053	19.849	18.933	17.482	17.575	17.765	18.331	18.851	19.464
		% Diff:	1.99%	1.90%	2.13%	2.36%	2.23%	2.21%	2.09%	1.95%	1.84%
	$\frac{\alpha}{\gamma} = 10$	Opt:	18.220	18.038	17.377	16.335	16.444	16.641	17.208	17.741	18.344
		Heur:	18.773	18.577	17.937	16.740	16.843	17.029	17.606	18.142	18.760
		% Diff:	3.03%	2.99%	3.22%	2.48%	2.42%	2.33%	2.31%	2.26%	2.27%
Function 3	$\frac{\alpha}{\gamma} = 3$	Opt:	33.459	33.211	30.984	27.864	27.953	28.167	28.753	29.277	29.926
(concave)		Heur:	33.467	33.219	31.024	27.989	28.075	28.280	28.862	29.384	30.025
		% Diff:	0.02%	0.02%	0.13%	0.45%	0.44%	0.40%	0.38%	0.36%	0.33%
	$\frac{\alpha}{\gamma} = 5$	Opt:	27.352	27.183	26.240	24.613	24.729	24.904	25.451	25.985	26.579
		Heur:	27.685	27.504	26.583	24.817	24.932	25.102	25.673	26.217	26.815
		% Diff:	1.22%	1.18%	1.31%	0.83%	0.82%	0.79%	0.87%	0.89%	0.89%
	$\frac{\alpha}{\gamma} = 10$	Opt:	25.870	25.702	25.058	23.546	23.662	23.831	24.349	24.871	25.452
		Heur:	26.222	26.052	25.360	23.926	24.045	24.211	24.780	25.323	25.898
		% Diff:	1.36%	1.36%	1.20%	1.61%	1.62%	1.59%	1.77%	1.82%	1.75%
Function 4	$\frac{\alpha}{\gamma} = 3$	Opt:	26.084	25.828	23.610	20.653	20.731	20.947	21.523	22.036	22.684
(general)		Heur:	26.097	25.842	23.689	20.766	20.841	21.056	21.633	22.136	22.773
		% Diff:	0.05%	0.05%	0.33%	0.55%	0.53%	0.52%	0.51%	0.45%	0.40%
	$\frac{\alpha}{\gamma} = 5$	Opt:	19.848	19.665	18.761	17.401	17.505	17.693	18.245	18.769	19.372
		Heur:	20.327	20.127	19.204	17.756	17.852	18.046	18.625	19.143	19.724
		% Diff:	2.41%	2.35%	2.36%	2.04%	1.98%	1.99%	2.08%	1.99%	1.82%
	$\frac{\alpha}{\gamma} = 10$	Opt:	18.399	18.218	17.599	16.607	16.723	16.905	17.439	17.948	18.523
		Heur:	18.810	18.620	18.043	16.873	16.979	17.155	17.721	18.225	18.787
		% Diff:	2.24%	2.21%	2.52%	1.60%	1.53%	1.48%	1.62%	1.55%	1.43%

definitely promising, if they do not work well in a particular instance then the idea of smoothing the optimal policy could be explored further.

In conclusion, our numerical study yields three major observations. First, the difference in the average costs between operating with a fixed number of servers and operating under the optimal policies for the unrestricted, controlled capacity reduction and uncontrolled capacity reduction cases can be significant. Second, our 2-level heuristic provides a simple, easily implementable alternative to optimal policies without (in most cases) a significant increase in the average costs. This observation seems to be robust in each of the model parameters. Finally, our numerical study supports our unproven conjecture that a sufficient condition for the optimal capacity increase/decrease policy to be monotone is that the worker cost function is increasing and convex. When the convexity assumption is relaxed, examples show that the optimal policy need not be monotone.

Table 8
Results (optimal average cost, cost of the best 2-level heuristic, and percent difference) of the numerical study for the controlled reduction case of the base parameters restricted to $\gamma = .1$ with various worker cost functions.

			$\frac{h_2}{h_1} = .5$			$\frac{h_2}{h_1} = 2$			$\frac{h_2}{h_1} = 5$		
			$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$
Function 2	$\alpha = .3$	Opt:	17.433	17.273	16.550	15.560	15.669	15.805	16.288	16.750	17.224
(linear)		Heur:	17.596	17.448	16.730	15.710	15.811	15.930	16.370	16.798	17.250
		% Diff:	0.93%	1.01%	1.09%	0.96%	0.90%	0.79%	0.50%	0.29%	0.15%
	$\alpha = .5$	Opt:	16.594	16.473	15.871	14.645	14.744	14.859	15.278	15.680	16.100
		Heur:	16.726	16.615	16.006	14.669	14.765	14.874	15.283	15.681	16.100
		% Diff:	0.80%	0.86%	0.85%	0.17%	0.14%	0.10%	0.03%	0.01%	0.00%
	$\alpha = 1$	Opt:	15.030	14.936	14.576	13.795	13.879	13.974	14.334	14.682	15.053
		Heur:	15.050	14.955	14.590	13.795	13.879	13.974	14.334	14.682	15.053
		% Diff:	0.13%	0.12%	0.10%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Function 3	$\alpha = .3$	Opt:	23.642	23.535	22.597	20.544	20.653	20.753	21.172	21.592	22.009
(concave)		Heur:	23.642	23.535	22.597	20.593	20.711	20.813	21.253	21.698	22.134
		% Diff:	0.00%	0.00%	0.00%	0.24%	0.28%	0.29%	0.39%	0.49%	0.57%
	$\alpha = .5$	Opt:	21.155	21.062	20.436	18.922	19.015	19.103	19.466	19.828	20.191
		Heur:	21.155	21.062	20.436	19.099	19.213	19.304	19.713	20.128	20.530
		% Diff:	0.00%	0.00%	0.00%	0.94%	1.04%	1.05%	1.27%	1.52%	1.68%
	$\alpha = 1$	Opt:	18.730	18.645	18.231	17.132	17.210	17.288	17.600	17.907	18.223
		Heur:	18.734	18.656	18.274	17.478	17.580	17.658	18.018	18.383	18.737
		% Diff:	0.02%	0.06%	0.23%	2.02%	2.15%	2.14%	2.37%	2.66%	2.82%
Function 4	$\alpha = .3$	Opt:	18.035	17.890	17.145	15.528	15.632	15.762	16.230	16.681	17.163
(general)		Heur:	18.192	18.031	17.254	15.595	15.697	15.826	16.289	16.736	17.215
		% Diff:	0.87%	0.79%	0.64%	0.43%	0.42%	0.41%	0.37%	0.33%	0.30%
	$\alpha = .5$	Opt:	16.562	16.433	15.786	14.493	14.592	14.712	15.147	15.567	16.016
		Heur:	16.722	16.589	15.873	14.518	14.615	14.737	15.177	15.600	16.056
		% Diff:	0.97%	0.95%	0.55%	0.17%	0.16%	0.17%	0.19%	0.21%	0.25%
	$\alpha = 1$	Opt:	14.885	14.775	14.360	13.550	13.640	13.751	14.150	14.533	14.945
		Heur:	15.021	14.897	14.416	13.570	13.658	13.772	14.177	14.564	14.987
		% Diff:	0.91%	0.83%	0.39%	0.15%	0.14%	0.16%	0.19%	0.22%	0.28%

Table 9
Results (optimal average cost, cost of the best 2-level heuristic, and percent difference) of the numerical study for the unrestricted case of the base parameters restricted to $\gamma = .1$ with various worker cost functions.

		$\frac{h_2}{h_1} = .5$			$\frac{h_2}{h_1} = 2$			$\frac{h_2}{h_1} = 5$		
		$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$	$\frac{\mu_1}{\mu_2} = .5$	$\frac{\mu_1}{\mu_2} = 1$	$\frac{\mu_1}{\mu_2} = 2$
Function 2	Opt:	12.815	12.750	12.646	12.574	12.625	12.685	12.907	13.125	13.352
(linear)	Heur:	12.815	12.750	12.646	12.574	12.625	12.685	12.907	13.125	13.352
	% Diff:	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Function 3	Opt:	13.062	12.997	12.893	12.822	12.872	12.933	13.155	13.372	13.599
(concave)	Heur:	13.062	12.997	12.893	12.822	12.872	12.933	13.155	13.372	13.599
	% Diff:	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Function 4	Opt:	12.431	12.341	12.176	12.026	12.096	12.188	12.478	12.774	13.091
(general)	Heur:	12.611	12.500	12.267	12.056	12.125	12.222	12.556	12.875	13.222
	% Diff:	1.45%	1.29%	0.74%	0.25%	0.24%	0.28%	0.62%	0.79%	1.00%

6. Conclusions

In this paper we considered several models for a potentially temporary, flexible workforce in a dynamic environment. We characterized the structure of the optimal allocation policy of flexible capacity in a tandem queueing system when the capacity is available only for a random amount of time. Furthermore, when the decision-maker has complete control over the capacity available in the system, we proved the intuitive result that the optimal number of workers is non-decreasing in the number of customers at each station. Each of these results stand to simplify both computation and analysis.

We remind the reader that we have assumed that workers can collaborate on a single job. When this does not hold, we believe that the allocation results of Section 3 no longer hold, at least not in both directions. Indeed, even when the capacity is not variable, only the direction leading to the optimality of a policy that is exhaustive in queue 1 holds (see the example in Ahn et al. [1]). We believe that a similar result holds here. Moreover, we do not believe that the assumption that $\mu_1 = \mu_2$ is required for Theorem 4.2, but have been unable to prove it without this assumption.

As of yet we have been unable to prove any analogous results for the capacity increase/decrease decisions in the controlled and uncontrolled capacity reduction cases. While some examples that we considered exhibit the monotonicity, the result does not extend to the case where the opportunity cost per unit time is increasing in the number of servers (as demonstrated by Example 2.4). However, our extensive numerical studies reinforced our conjecture that the optimal capacity increase/decrease policy is monotone in the number of workers when the worker cost function is increasing and convex; this also remains to be proven. Along with this study, we note that other future research directions include a similar analysis for a network of stations and those with capacity that is available for a fixed amount of time.

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Appendix

Structure of value functions for finite horizon problems

Proof of Lemma 4.3. Suppose $2h_2 \geq h_1$. By induction. The result holds trivially for $n = 0$. Assume that it holds for $n - 1$. Let $d_n(i, j)$ denote the optimal number of workers in state (i, j) at stage n. We show that each inequality holds for $H_\beta^U v_{n-1}^U$. The result then follows from the definition of v_n^U in (2.7). For the first inequality, choose the actions $d_n(i+1, j)$ and $d_n(i, j-1)$ in states (i, j) and $(i+1, j-1)$, respectively. For $j \geq 2$

$$\begin{aligned}\Delta_i H_\beta^U v_{n-1}^U(i, j) \geq \lambda \Delta_i v_{n-1}^U(i+1, j) + d_n(i+1, j)\mu \Delta_i v_{n-1}^U(i, j-1) \\ + [1 - (\lambda + d_n(i+1, j)\mu)]\Delta_i v_{n-1}^U(i, j),\end{aligned}$$

and

$$\begin{aligned}\Delta_i H_\beta^U v_{n-1}^U(i, j-1) \leq \lambda \Delta_i v_{n-1}^U(i+1, j-1) + d_n(i, j-1)\mu \Delta_i v_{n-1}^U(i, j-2) \\ + [1 - (\lambda + d_n(i, j-1)\mu)]\Delta_i v_{n-1}^U(i, j-1).\end{aligned}$$

Taking the difference we have

$$\begin{aligned}&\Delta_i H_\beta^U v_{n-1}^U(i, j-1) - \Delta_i H_\beta^U v_{n-1}^U(i, j) \\ &\leq \lambda\big[\Delta_i v_{n-1}^U(i+1, j-1) - \Delta_i v_{n-1}^U(i+1, j)\big] \\ &\quad + d_n(i, j-1)\mu\big[\Delta_i v_{n-1}^U(i, j-2) - \Delta_i v_{n-1}^U(i, j-1)\big] \\ &\quad + [1 - (\lambda + d_n(i+1, j)\mu)]\big[\Delta_i v_{n-1}^U(i, j-1) - \Delta_i v_{n-1}^U(i, j)\big].\end{aligned}$$

The inductive hypothesis yields the first assertion. In the case that $j = 1$ and $i \geq 1$, a similar argument yields

$$\begin{aligned}&\Delta_i H_\beta^U v_{n-1}^U(i, 0) - \Delta_i H_\beta^U v_{n-1}^U(i, 1) \\ &\leq \lambda\big[\Delta_i v_{n-1}^U(i+1, 0) - \Delta_i v_{n-1}^U(i+1, 1)\big] \\ &\quad + d_n(i, 0)\mu\big[\Delta_i v_{n-1}^U(i-1, 1) - \Delta_i v_{n-1}^U(i, 0)\big] \\ &\quad + [1 - (\lambda + d_n(i+1, 1)\mu)]\big[\Delta_i v_{n-1}^U(i, 0) - \Delta_i v_{n-1}^U(i, 1)\big].\end{aligned}$$

The result now follows directly from the inductive hypothesis. If $j = 1$ and $i = 0$, we first note that $d_n(0, 0) = 0$ is an optimal action (otherwise workers would be idle). Thus, the previous inequality holds with the second term equal to zero.

For the second inequality, let $\Delta_j^2 g(i, j) \equiv \Delta_j g(i, j+1) - \Delta_j g(i, j)$. Note for $j \geq 1$

$$\begin{aligned}\Delta_j^2 H_\beta^U v_{n-1}^U(i, j) \geq \lambda \Delta_j^2 v_{n-1}^U(i, j) + d(i, j)\mu \Delta_j^2 v_{n-1}^U(i, j-1) \\ + [1 - (\lambda + d(i, j+2)\mu)]\Delta_j^2 v_{n-1}^U(i, j),\end{aligned}$$

and for $j = 0$ and $i \geq 1$

$$\begin{aligned}\Delta_j^2 H_\beta^U v_{n-1}^U(i, 0) \geq \lambda \Delta_j^2 v_{n-1}^U(i, 0) + d(i, 0)\mu\big[\Delta_j v_{n-1}^U(i, 0) - \big(v_{n-1}^U(i, 0) - v_{n-1}^U(i-1, 1)\big)\big] \\ + [1 - (\lambda + d(i, j+2)\mu)]\Delta_j^2 v_{n-1}^U(i, j).\end{aligned}$$

The result follows by the inductive hypothesis. The case for $i = j = 0$ is analogous to the previous argument.

The third inequality yields

$$\begin{aligned}
&\Delta_i H_\beta^U v_{n-1}^U(i-1,1) - \Delta_i H_\beta^U v_{n-1}^U(i,0) \\
&\quad \le \lambda\big[\Delta_i v_{n-1}^U(i,1) - \Delta_i v_{n-1}^U(i+1,0)\big] \\
&\qquad + d_n(i,1)\mu\big[\Delta_i v_{n-1}^U(i-1,0) - \Delta_i v_{n-1}^U(i-1,1)\big] \\
&\qquad + [1-(\lambda + d_n(i,0)\mu)]\big[\Delta_i v_{n-1}^U(i-1,1) - \Delta_i v_{n-1}^U(i,0)\big].
\end{aligned}$$

Each term is non-positive by the inductive hypothesis by noting

$$\Delta_i v_{n-1}^U(i-1,0) - \Delta_i v_{n-1}^U(i-1,1) = \Delta_j v_{n-1}^U(i-1,0) - \Delta_j v_{n-1}^U(i,0).$$

To show the fourth inequality choose $d_n(i,1)$ and $d_n(i-1,1)$ for the two processes starting in $(i,0)$,

$$\begin{aligned}
&\Delta_j H_\beta^U v_{n-1}^U(i,0) - \big[H_\beta^U v_{n-1}^U(i,0) - H_\beta^U v_{n-1}^U(i-1,1)\big] \\
&\quad \ge \lambda[\Delta_j v_{n-1}^U(i+1,0) - \big[v_{n-1}^U(i+1,0) - v_{n-1}^U(i,1)\big]\big] \\
&\qquad + [1-(\lambda + d_n(i,1)\mu)]\big[\Delta_j v_{n-1}^U(i,0) - [v_{n-1}^U(i,0) - v_{n-1}^U(i-1,1)]\big] \\
&\qquad + d_n(i-1,1)\mu\big[\big[v_{n-1}^U(i,0) - v_{n-1}^U(i-1,1)\big] - \Delta_j v_{n-1}^U(i-1,0)\big].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&H_\beta^U v_{n-1}^U(i,0) - H_\beta^U v_{n-1}^U(i-1,1) - \Delta_j H_\beta^U v_{n-1}^U(i-1,0) \\
&\quad \ge \lambda\big[v_{n-1}^U(i+1,0) - v_{n-1}^U(i,1) - \Delta_j v_{n-1}^U(i,0)\big] \\
&\qquad + [1-(\lambda + d_n(i,0)\mu)]\big[v_{n-1}^U(i,0) - v_{n-1}^U(i-1,1) - \Delta_j v_{n-1}^U(i-1,0)\big] \\
&\qquad + d_n(i-1,0)\mu\big[\Delta_j v_{n-1}^U(i-1,0) - \big[v_{n-1}^U(i-1,0) - v_{n-1}^U(i-2,1)\big]\big].
\end{aligned}$$

In either case, the inductive hypothesis yields the result for $H_\beta^U v_{n-1}^U$. Recalling that $2h_2 \ge h_1$ yields the result for v_n^U.

Suppose now that $h_1 > 2h_2$. Again, the results hold for $n = 0$ trivially. Assume that it is true for $n-1$. To prove the first inequality, choose potentially sub-optimal actions to get for $i \ge 1$

$$\begin{aligned}
&\Delta_j v_n^U(i,j) - \Delta_j v_n^U(i,j+1) \\
&\quad \le \lambda\big[\Delta_j v_{n-1}^U(i+1,j) - \Delta_j v_{n-1}^U(i+1,j+1)\big] \\
&\qquad + d_n(i,j)\mu\big[\Delta_j v_{n-1}^U(i-1,j+1) - \Delta_j v_{n-1}^U(i-1,j+2)\big] \\
&\qquad + [1-(\lambda + d_n(i,j+2)\mu)]\big[\Delta_j v_{n-1}^U(i,j) - \Delta_j v_{n-1}^U(i,j+1)\big],
\end{aligned}$$

and for $i = 0$

$$\begin{aligned}
&\Delta_j v_n^U(0, j) - \Delta_j v_n^U(0, j+1) \\
&\quad \le \lambda\big[\Delta_j v_{n-1}^U(1, j) - \Delta_j v_{n-1}^U(1, j+1)\big] \\
&\qquad + d_n(0, j)\mu\big[\Delta_j v_{n-1}^U(0, j-1) - \Delta_j v_{n-1}^U(0, j)\big] \\
&\qquad + [1 - (\lambda + d_n(0, j+2)\mu)]\big[\Delta_j v_{n-1}^U(0, j) - \Delta_j v_{n-1}^U(0, j+1)\big].
\end{aligned}$$

Similarly, for the second inequality for $i \ge 2$

$$\begin{aligned}
&\Delta_j v_n^U(i-1, j+1) - \Delta_j v_n^U(i, j) \\
&\quad \le \lambda\big[\Delta_j v_{n-1}^U(i, j+1) - \Delta_j v_{n-1}^U(i+1, j)\big] \\
&\qquad + d_n(i-1, j+1)\mu\big[\Delta_j v_{n-1}^U(i-2, j+2) - \Delta_j v_{n-1}^U(i-1, j+1)\big] \\
&\qquad + [1 - (\lambda + d_n(i, j+1)\mu)]\big[\Delta_j v_{n-1}^U(i-1, j+1) - \Delta_j v_{n-1}^U(i, j)\big],
\end{aligned}$$

and for $i = 1$

$$\begin{aligned}
&\Delta_j v_n^U(0, j+1) - \Delta_j v_n^U(1, j) \\
&\quad \le \lambda\big[\Delta_j v_{n-1}^U(1, j+1) - \Delta_j v_{n-1}^U(2, j)\big] \\
&\qquad + d_n(0, j+1)\mu\big[\Delta_j v_{n-1}^U(0, j) - \Delta_j v_{n-1}^U(0, j+1)\big] \\
&\qquad + [1 - (\lambda + d_n(1, j+1)\mu)]\big[\Delta_j v_{n-1}^U(0, j+1) - \Delta_j v_{n-1}^U(1, j)\big].
\end{aligned}$$

In each case, the induction hypothesis proves the result. □

Stability results

We provide conditions under which a policy yields a stable Markov process. This leads to a guarantee that the process also lends finite average cost, and thus to the existence of a finite solution to the ACOE. The first result states conditions for stability for the unrestricted and controlled capacity reduction cases.

Proposition A.1. Suppose for some $k^* \in \{0, 1, \dots, \ell\}$

$$\frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2} < k^*.$$

Then in either the unrestricted or controlled capacity reduction models, there exists a policy π such that $\rho^\pi < \infty$.

Proof. In either case, one may choose the policy that has k^* workers available for all but a finite amount of time. The system is then the same system studied for 2 workers in Ahn et al. [1] where the average holding costs is shown to be finite. The k^* worker case is directly analogous. Adding $r(k^*)$ to this finite amount yields the result. □

When the capacity reduction is uncontrolled, the workforce cannot be made constant and the argument in Proposition A.1 does not carry over. For the remainder of this section we study this problem and thus, suppress the superscript F on all quantities. Note that this no longer means that quantities apply to all three models. Since the workers are flexible, the system can be viewed as a reentrant line with one common pool of resources that can be split amongst two classes—stage 1 and stage 2 customers. Stability and convergence of moments results for standard reentrant lines (with fixed service capacity) can be obtained using fluid limit techniques as in Dai [11] and Dai and Meyn [12]. We adapt these results to our setting with a varying service capacity. In general, the fluid limit results are very useful in analyzing network stability even when arrival and service times follow a general distribution. This is not taken advantage of here; only exponential times are addressed. What is made use of is the fact that the results go beyond stability in a positive recurrence sense and ensure finite average queue length.

Fix a Markovian head of the line (HL) policy, π. That is, π is such that workers can collaborate to work on only the first customers in each of the queues and worker allocations remain constant in between changes in the Markovian state $X(t) = (Q(t), K(t))$, where $Q(t) = (Q_1(t), Q_2(t))$ is the queue length process (including the customers in service) and $K(t)$ is the number of workers available at time t.

We explore the statistical regularity of this policy through *fluid scaling*, a law of large numbers type scaling of space and time. Define the norm

$$|X(t)| \equiv Q_1(t) + Q_2(t) + K(t)$$

and consider the scaled process

$$\bar{Q}^x(t) \equiv \frac{1}{|x|} Q^x(|x|t),$$

where the superscript denotes the dependence on the initial state $x = X(0)$. Letting $|x| \to \infty$, any limit point $\bar{Q}(t)$ is called a *fluid limit* of the queue length process. We will show that every fluid limit is a solution to a set of equations known as the *fluid model*. The fluid model is said to be *stable* if there exists a fixed time t_0 such that $\bar{Q}(t) = 0$ for all $t \geq t_0$. That is, the fluid model is stable if all queues eventually drain and once drained stay empty.

Let the customer inter-arrival times be $\xi(n),\ n = 1, 2, \ldots$, and the service requirements at stage m be $\eta_m(n),\ n = 1, 2, \ldots; m = 1, 2$. Recall that we have assumed that ξ and η_m are sequences of i.i.d. exponential random variables with means $1/\lambda$ and $1/\mu_m$ respectively. Let

$$E(t) = \max\{n \geq 0 : \xi(1) + \xi(2) + \cdots + \xi(n-1) \leq t\}, \quad t \geq 0,$$
$$S_m(t) = \max\{n \geq 0 : \eta_m(1) + \eta_m(2) + \cdots + \eta_m(n-1) \leq t\}, \quad t \geq 0,$$

where the maximum of an empty set is zero. $E(t)$ is the total number of arrivals by time t. $S_m(t)$ represents the total number of service completions from station m given t units of work. We make the following assumptions on π regarding capacity increases.

A1. (i) π does not reject any arriving workers up to capacity ℓ. (ii) Workers are never idle while customers are present.

Under **A1**, it is easy to see that the number of workers available follows a birth-death process with death rates $k\gamma$ and birth rates $\alpha(k)$, $k = 0, 1, \ldots, \ell$. Such a process has steady-state probabilities p_k given in Proposition 2.2. Hence, the average number of workers available in steady state is

$$z = \sum_{k=1}^{\ell}(kp_k). \tag{A.1}$$

Let $Y_k^x(t)$ be the cumulative time there are exactly k hired workers available in $[0, t)$ given the initial state x. Suppose $W_{mk}^x(t)$ is the cumulative amount of work done by t on stage m customers during the times there are exactly k workers available. Thus, $S_m(\sum_{k=0}^{\ell} W_{mk}^x(t))$ is the number of completions from stage m by t. Let $I_k^x(t)$ be the cumulative amount of idle time while exactly k workers are available so that $T_k^x(t) \equiv Y_k^x(t) - I_k^x(t)$ is the amount of time there are customers to serve in $[0, t)$ when there are exactly k workers available. These processes are all defined on the domain $[0, \infty)$ and are assumed to be right continuous with left limits. The definitions above imply the following *system equations*:

$$Q_1^x(t) = Q_1^x(0) + E^x(t) - S_1^x\left(\sum_{k=0}^{\ell} W_{1k}^x(t)\right), \tag{A.2}$$

$$Q_2^x(t) = Q_2^x(0) + S_1^x\left(\sum_{k=0}^{\ell} W_{1k}^x(t)\right) - S_2^x\left(\sum_{k=0}^{\ell} W_{2k}^x(t)\right), \tag{A.3}$$

$$Q^x(t) \geq 0, \tag{A.4}$$

$$\sum_{k=0}^{\ell} Y_k^x(t) = t, \tag{A.5}$$

$$T_k^x(t) + I_k^x(t) = Y_k^x(t), \quad k = 0, 1, \ldots, \ell, \tag{A.6}$$

$$W_{1k}^x(t) + W_{2k}^x(t) = kT_k^x(t), \quad k = 0, 1, \ldots, \ell, \tag{A.7}$$

$$Y_k^x(t), T_k^x(t), I_k^x(t), W_{1k}^x(t), \text{ and } W_{2k}^x(t) \text{ are nondecreasing and start from 0,} \tag{A.8}$$

$$\int_0^{\infty} \big(Q_1^x(t) + Q_2^x(t)\big) d\left(\sum_{k=0}^{\ell} I_k^x(t)\right) = 0. \tag{A.9}$$

Note that (A.9) is the non-idling constraint and guarantees that the idle time increases only if the total queue length is zero.

The next proposition, a variant of Theorem 4.1 in Dai [11], presents the fluid model and establishes convergence of the scaled processes. This convergence is *uniform on compact sets (u.o.c.)*.

Proposition A.2. Consider any Markovian HL policy under **A1**. The following holds with probability one. For any sequence of initial states $\{x_j\} \in \mathbb{X}$ with $|x_j| \to \infty$, there exists a subsequence $\{x_i\}$, $\{i\} \subseteq \{j\}$, with $|x_i| \to \infty$ such that

$$(\bar{Q}^{x_i}(0), \bar{K}^{x_i}(0)) \to (\bar{Q}(0), 0), \tag{A.10}$$

$$(\bar{Q}^{x_i}(t), \bar{T}^{x_i}(t), \bar{W}^{x_i}(t)) \to (\bar{Q}(t), \bar{T}(t), \bar{W}(t)) \ \text{ u.o.c.}, \tag{A.11}$$

where $(\bar{Q}, \bar{T}, \bar{W})$ satisfies the following set of equations:

$$\bar{Q}_1(t) = \bar{Q}_1(0) + \lambda t - \mu_1\left(\sum_{k=0}^{\ell} \bar{W}_{1k}(t)\right), \tag{A.12}$$

$$\bar{Q}_2(t) = \bar{Q}_2(0) + \mu_1\left(\sum_{k=0}^{\ell} \bar{W}_{1k}(t)\right) - \mu_2\left(\sum_{k=0}^{\ell} \bar{W}_{2k}(t)\right), \tag{A.13}$$

$$\bar{Q}(t) \geq 0, \tag{A.14}$$

$$\sum_{k=0}^{\ell} p_k = 1, \tag{A.15}$$

$$\bar{T}_k(t) + \bar{I}_k(t) = p_k t, \quad k = 0, 1, \ldots, \ell, \tag{A.16}$$

$$\bar{W}_{1k}(t) + \bar{W}_{2k}(t) = k\bar{T}_k(t), \quad k = 0, 1, \ldots, \ell, \tag{A.17}$$

$$\bar{Y}_k(t), \bar{T}_k(t), \bar{I}_k(t), \bar{W}_{1k}(t), \text{ and } \bar{W}_{2k}(t) \text{ are nondecreasing and start from } 0, \tag{A.18}$$

$$\int_0^{\infty} \big(\bar{Q}_1(t) + \bar{Q}_2(t)\big) d\left(\sum_{k=0}^{\ell} \bar{I}_k(t)\right) = 0. \tag{A.19}$$

Proof. Notice that for $m = 1, 2$,

$$\frac{1}{|x_j|} Q_m^{x_j}(0) \leq 1, \quad \frac{1}{|x_j|} K^{x_j}(0) \leq 1,$$

for all j. Also, $0 \leq K^{x_j}(0) \leq \ell$. Therefore, there exists a subsequence $|x_i| \to \infty$ such that (A.10) holds.

For any $0 \leq t_1 \leq t_2$, and each m and k, we have

$$0 \leq \bar{W}_{mk}^x(t_2) - \bar{W}_{mk}^x(t_1) \leq k\big(\bar{T}_{mk}^x(t_2) - \bar{T}_{mk}^x(t_1)\big) \leq \ell(t_2 - t_1).$$

That is, $\{\bar{T}_k^x(t), |x| \geq 1\}$ and $\{\bar{W}_{mk}^x(t), |x| \geq 1\}$ are uniformly Lipschitz, and hence equicontinuous. Setting $t_1 = 0$ and $t_2 = t$ yields uniform bounds $0 \leq \bar{T}_k^x(t) \leq t$ and $0 \leq \bar{W}_{mk}^x(t) \leq \ell t$. Therefore, by the Arzelà-Ascoli theorem, any subsequence of $\bar{T}_k^x(t)$ has a u.o.c. convergent subsequence. Similarly, for $\bar{W}_{mk}^x(t)$. The families $\{\bar{I}_k^x(t), |x| \geq 1\}$ and $\{\bar{Y}_k^x(t), |x| \geq 1\}$ can be shown to have a u.o.c. convergent subsequence in the same manner. Since $K^x(t)$ is ergodic, the (functional) Strong Law of Large Numbers (SLLN) (cf. Glynn and Whitt [18]) can be applied to get

$$\bar{Y}_k(t) = p_k t,$$
$$\bar{E}(t) = \lambda t,$$
$$\bar{S}_m(t) = \mu_m t.$$

Equations (A.12) and (A.13) then follow from (A.2) and (A.3) where the random time change is valid by Theorem 5.3 of Chen and Yao [9]. Equations (A.14)–(A.18) are a consequence of (A.4)–(A.8) and (A.19) follows from (A.9) and Lemma 4.4 of Dai [11]. □

The next proposition says that the desired stability results hold if on average the offered workload by the arriving customers is less than the capacity of the hired workers.

Proposition A.3. Suppose z is defined as in Proposition 2.2. If $\lambda(1/\mu_1 + 1/\mu_2) < z$, then for any Markovian HL policy π satisfying **A1**, the fluid model is stable and, thus, an invariant probability ψ exists for $X = \{X(t), t \geq 0\}$. Furthermore,

$$\lim_{t\to\infty} \mathbb{E}_x^\pi[Q_m(t)] = \mathbb{E}_\psi^\pi[Q_m(0)] < \infty, \quad m = 1, 2, \tag{A.20}$$

and $\rho^\pi < \infty$.

Proof. To show that the fluid model is stable, consider the immediate workload given by

$$\begin{aligned}\bar{L}(t) &\equiv \bar{Q}_1(t)/\mu_1 + (\bar{Q}_1(t) + \bar{Q}_2(t))/\mu_2 \\ &= \bar{Q}_1(0)/\mu_1 + (\bar{Q}_1(0) + \bar{Q}_2(0))/\mu_2 + \lambda(1/\mu_1 + 1/\mu_2)t - \sum_{k=0}^{\ell}(k\bar{T}_k(t)).\end{aligned}$$

$\bar{L}(t)$ is nonnegative and equals zero if and only if $\bar{Q}_1(t) = \bar{Q}_2(t) = 0$. Because $\bar{T}_k(t)$ and $\bar{W}_{mk}(t)$ are Lipschitz continuous, so are $\bar{Q}_m(t)$ and $\bar{L}(t)$. Lipschitz continuity implies absolute continuity, so $\bar{L}(t)$ is differentiable almost everywhere. When $(\bar{Q}_1(t) + \bar{Q}_2(t)) > 0$, (A.19) and (A.16) imply $\dot{\bar{I}}_k(t) = 0$ and $\dot{\bar{T}}_k(t) = p_k$, and thus

$$\dot{\bar{L}}(t) = \lambda(1/\mu_1 + 1/\mu_2) - z.$$

By Lemma 5.2 of Dai [11], $\bar{L}(t) \equiv 0$ for $t \geq (z - \lambda(1/\mu_1 + 1/\mu_2))^{-1}$. Therefore, the fluid model is stable.

The results showing that stability of the fluid model implies stability of the original system are proven in Dai [11] and Dai and Meyn [12]. Since $K(t)$ is uniformly bounded, it is easily incorporated into these proofs. The existence of an invariant probability for X follows from Theorem 4.2 of Dai [11]. From Theorem 4.1 of Dai and Meyn [12], we have (A.20). Since the holding cost rates are linear, this together with a bounded worker cost imply $\rho^\pi < \infty$. □

Note that the above arguments can be carried over without much difficulty to a network of any finite number of queues in tandem. Since this just amounts to more bookkeeping, the details are omitted.

References

[1] H. Ahn, I. Duenyas and M.E. Lewis, The optimal control of a two-stage tandem queueing system with flexible servers, Probability in the Engineering and Informational Sciences 16(4) (2002) 453–469.

[2] H. Ahn, I. Duenyas and R.Q. Zhang, Optimal stochastic scheduling of a two-stage tandem queue with parallel servers, Advances in Applied Probability 31 (1999) 1095–1117.

[3] S. Andradottir, H. Ayhan and D. Down, Server assignment policies for maximizing the steady-state throughput of finite queueing systems, Management Science 47(10) (October 2001) 1421–1439.

[4] J.J. Bartholdi III, L.A. Bunimovich and D.D. Eisenstein, Dynamics of 2- and 3-worker 'bucket brigade' production lines, Operations Research 47(3) (1999) 488–491.

[5] J.J. Bartholdi III and D.D. Eisenstein, A production line that balances itself, Operations Research 44(1) (1996) 21–34.

[6] J.J. Bartholdi III, D.D. Eisenstein and R.D. Foley, Performance of bucket brigades when work is stochastic, Operations Research 49(5) (2001) 710–719.

[7] D.P. Bertsekas, *Dynamic Programming and Optimal Control*, volume 2, (Athena Scientific, Belmont, MA, 1995).

[8] P. Brémaud, *Point Processes and Queues; Martingale Dynamics* (Springer-Verlag, New York, 1981).

[9] H. Chen and D. Yao, *Fundamentals of Queueing Networks* (Springer-Verlag, New York, 2001).

[10] T.B. Crabill, Optimal control of a service facility with variable exponential service times and constant arrival rate, Management Science 18(9) (May 1972) 560–566.

[11] J. Dai, On positive harris recurrence of multiclass queueing networks: a unified approach via fluid limit models, Annals of Applied Probability 5 (1995) 49–77.

[12] J. Dai and S. Meyn, Stability and convergence of moments for multiclass queueing networks via fluid limit models, IEEE Transactions on Automatic Control 40 (1995) 1889–1904.

[13] I. Duenyas, D. Gupta and T. Olsen, Control of a single server tandem queueing system with setups, Operations Research 46(2) (March-April 1998) 218–230.

[14] A. Federgruen and K. So, Optimal maintenance policies for single-server queuing-systems subject to breakdowns, Operations Research 38(2) (March-April 1990) 330–343.

[15] A. Federgruen and K. So, Optimality of threshold policies in single-server queuing-systems with server vacations, Advances in Applied Probability 23(2) (June 1991) 388–405.

[16] E.A. Feinberg and O. Kella, Optimality of D-policies for an M/G/1 queue with a removable server, Queueing Systems 42 (2002) 355–376.

[17] J.M. George and J.M. Harrison, Dynamic control of a queue with adjustable service rate, Operations Research 49(5) (September-October 2001) 720–731.

[18] P. Glynn and W. Whitt, Limit theorems for cumulative processes, Stochastic Processes and their Applications 47 (1993) 299–314.

[19] D. Heyman, Optimal operating policies for M/G/1 queuing systems, Operations Research 16(2) (March-April 1968) 362–382.
[20] S. Iravani, M. Posner and J. Buzacott. A two-stage tandem queue attended by a moving server with holding and switching costs, Queueing Systems 26(2/3) (1997) 203–228.
[21] T. Javidi, N. Song and D. Teneketzis, Expected makespan minimization of identical machines in two interconnected queues: Probability In The Engineering And Informational Sciences 15(4) (2001) 409–443.
[22] S.A. Lippman, Applying a new device in the optimization of exponential queueing system, Operations Research 23(4) (1975) 687–710.
[23] P. Nain, Interchange arguments for classical scheduling problems in queues, Systems and Control Letters 12 (1989) 177–184.
[24] L.I. Sennott, *Stochastic Dynamic Programming and the Control of Queueing Systems* (Wiley Series in Probability and Statistics. John Wiley and Sons, Inc., New York, 1999).
[25] S. Stidham, Jr and R. Weber, Monotonic and insensitive optimal policies for control of queues with undiscounted costs, Operations Research 37(4) (Jul-Aug 1989) 611–625.
[26] M. Van Oyen, E. Senturk-Gel and W. Hopp, Performance opportunity for workforce agility in collaborative and noncollaborative work systems, IIE Transactions 33(9) (2001) 761–777.
[27] M. Yadin and P. Naor, Queuing-systems with a removable service station, Operational Research Quarterly 14(1) (1963) 393–405.