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Probability Theory and Related Fields

Asymptotic Expansions for Renewal Measures in the Plane

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Summary. Let P be a distribution in the plane and define the renewal measure $R=\sum P^{*n}$ where $*$ denotes convolution. The main results of this paper are three term asymptotic expansions for R far from the origin. As an application, expansions are obtained for distributions in linear boundary crossing problems.

1. Introduction

The renewal measure associated with a distribution P is defined as $R=\sum_{n=0}^{\infty} P^{*n}$ where $*$ denotes convolution. When P is a distribution on $\mathbf{R}^1$ with positive mean $\mu=\int x\,dP$, the renewal theorem asserts that

$$\lim_{x\to\infty} R\{[x, x+K)\}-K/\mu=\lim_{x\to-\infty} R\{[x, x+K)\}=0 \tag{1.1}$$

for any K if P is nonlattice and for K any multiple of the span if P is arithmetic. Although applications of this result abound making it one of the most important tools in applied probability, to some extent the basic limit theory starts and ends here. As shown by Stone (1965ab), Carlsson (1983) and Grübel (1983, 1987), the rate of convergence in (1.1) is exceedingly fast – algebraic to any power provided P has sufficient moments, and exponential if P has a finite moment generating function. This should be contrasted with the central limit theorem where correction terms which depend on moments of P must be incorporated to improve the rate of convergence.

In higher dimensions, results analogous to (1.1) have been obtained by Bickel and Yahev (1965) and Stam (1968, 1969, 1971). Roughly speaking, Stam shows that near infinity along the direction of drift, R is like the product of Lesbegue measure over the length of μ in the direction of drift with a normal (or stable) distribution in the orthogonal direction. The covariance of the normal distribu-

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tion increases linearly in the distance from the origin. Unlike the case in one dimension, extra terms are now necessary to improve the rate of convergence in these theorems. The main results of this paper provide the first two correction terms for renewal theory in the the plane.

To describe these results, let $(X, Y) \sim P$, and define $v = EX$, $\gamma = EY/v$, $Z = Y - \gamma X$, $\sigma^2 = EZ^2$, $M_{jk} = EX^j Z^k/(v^j \sigma^k j!\,k!)$, $c_1 = -M_{11}$, $c_2 = M_{03} - M_{11}/2$, $c_3 = M_{20} - M_{12} + M_{11}^2$, $c_4 = M_{04} - M_{12}/2 + M_{20}/4 + M_{11}^2 - 2M_{11}M_{03}$, and $c_5 = M_{03}^2/2 + M_{11}^2/8 - M_{11}M_{03}/2$. For $x \leqq 0$ let $\hat{r}(x, y) = 0$, and for $x > 0$ let

$$\hat{r}(x, y) = \frac{e^{-\hat{y}^2/2}}{v\sqrt{2\pi x \sigma^2/v}} \cdot \left\{1 + \sqrt{\frac{v}{x}}\,[c_1 H_1(\hat{y}) + c_2 H_3(\hat{y})] + \frac{v}{x}\,[c_3 H_2(\hat{y}) + c_4 H_4(\hat{y}) + c_5 H_6(\hat{y})]\right\}$$

where $\hat{y} = (y - \gamma x)/\sqrt{x\sigma^2/v}$ and H_k is the k^{th} Hermite polynomial. Let $\tau = \tau(x, y) = \sqrt{|x| + |y - \gamma x|}$. Theorems 1.1 and 1.2 show that as $\tau \to \infty$, R is well approximated by the measure with Lesbegue or counting density $\hat{r}$.

Call P a *lattice* distribution on $\mathbf{Z}^2 = \{\ldots, -1, 0, 1, \ldots\}^2$ if $P(\mathbf{Z}^2) = 1$ and $P(A) < 1$ for A the translate of any proper subgroup of $\mathbf{Z}^2$.

Theorem 1.1. *If* $EX^2 \log^{3/2}(1 + X^-) < \infty$, $EZ^4 < \infty$, $v > 0$ *and* P *is a lattice distribution on* $\mathbf{Z}^2$, *then*

$$R(\{(x, y)\}) = \hat{r}(x, y) + o(1/\tau^3)$$

as $\tau \to \infty$ *uniformly for* $(x, y) \in \mathbf{Z}^2$.

The next result is an expansion in the "continuous case". To be specific, let χ denote the characteric function of P. Cramer's condition is

$$\limsup_{p^2 + q^2 \to \infty} \chi(p, q) < 1.$$

The theorem assumes Cramer's condition and approximates $\int f\,dR$ for non-negative functions that are zero in a large neighborhood of the origin. The distance of a function f from the origin is measured by

$$\tau_f = \inf\{\tau(x, y) : f(x, y) > 0\},$$

so $f(x, y) = 0$ if $\tau(x, y) < \tau_f$. Also, define the variation function

$$\omega_f(x, y, \varepsilon) = \sup\{f(x_1, y_1) - f(x_2, y_2) : (x_1, y_1), (x_2, y_2) \in B_\varepsilon + (x, y)\}$$

where $B_\varepsilon = \{(x, y) : x^2 + y^2 \leqq \varepsilon^2\}$ is the ε ball about the origin. Also let $\|f\| = \sup_{\mathbb{R}^2} |f|$ and define

$$m_f(x, y, \varepsilon) = \sup_{B_\varepsilon + (x, y)} |f|.$$

Theorem 1.2. *If* $EX^2\log^{3/2}(1+|X|)<\infty$, $EZ^4<\infty$, $v>0$, Cramer's condition holds, $0<\eta<1/6$, and $\varepsilon=\varepsilon_f=\exp\{-\tau_f^\eta\}$ then

$$\iint f\,dR=\iint f(x,y)\,\hat{r}(x,y)\,dx\,dy+O(1)\iint \omega_f(x,y,\varepsilon)\,\hat{r}(x,y)\,dx\,dy$$
$$+o(1)\iint m_f(x,y,1)\,\tau^{-3}(x,y)\,dx\,dy$$

uniformly in f *as* $\tau_f\to\infty$.

With f the indicator of a strip, we obtain the following corollary.

Corollary 1.3. *Under the conditions of Theorem 1.2,*

$$R\{(x-\delta,x+\delta)\times(-\infty,y)\}=\frac{2\delta}{v}\,\Phi(\hat{y})-\frac{2\delta}{\sqrt{vx}}\,\phi(\hat{y})\,[c_1+c_2H_2(\hat{y})]$$
$$-\frac{2\delta}{x}\,\phi(\hat{y})\left[\left(c_3+\frac{\delta^2\gamma^2}{6\sigma^2}\right)H_1(\hat{y})+c_4H_3(\hat{y})+c_5H_5(\hat{y})\right]+o\left(\frac{1+\delta^3}{x}\right)$$

as $x\to\infty$ *with* $\delta=o(\sqrt{x})$, *uniformly for* $y\in\mathbf{R}$.

Our final result gives expansions for densities. If $P(X\in\mathbf{Z})=1$, let λ_x be counting measure on $\mathbf{Z}$; otherwise let λ_x be Lesbegue measure. Define λ_y similarly. Let $R_n=\sum_{k=n}^{\infty}P^{*k}=P^{*n}*R$.

Theorem 1.4. *Suppose* $EX^2\log^{3/2}(1+X^-)<\infty$, $EZ^4<\infty$, P^{*k} *is absolutely continuous with respect to* $\lambda_x\times\lambda_y$ *with a bounded density, and* $(1+\tau)^3\,dP^{*n}/d(\lambda_x\times d_y)$ *is directly Riemann integrable where* $n=2k+4$. *Also, assume* $P(\mathbf{Z}^2)<1$ *and that* X *is lattice with span* 1 *if* $P(X\in\mathbf{Z})=1$ *and* Y *is lattice with span* 1 *if* $P(Y\in\mathbf{Z})=1$. *Then*

$$\frac{dR_n}{d(\lambda_x\times\lambda_y)}(x,y)=\hat{r}(x,y)+o(1/\tau^3)$$

as $\tau\to\infty$ *uniformly for* (x,y) *in the support of* $\lambda_x\times\lambda_y$.

Remark. In our proofs, the condition $EX^2\log^{3/2}(1+X^-)<\infty$ is used to deal with difficulties associated with $P(X<0)>0$. When $P(X\geqq 0)=1$, this condition can be relaxed to $EX^2<\infty$ (we conjecture $EX^2<\infty$ is always sufficient) and the integrability condition for $dP^{*n}/d(\lambda_x\times\lambda_y)$ can be dropped in Theorem 1.4. The condition $EX^2\log^{3/2}(1+X^-)<\infty$ can also be dropped when error rates of $o(|x|^{-3/2})$ are sufficient – see the proof of Theorem 2.5.

A unified treatment of the results just stated can be obtained by finding approximations for convolutions of test measures with symmetrized versions of R. These results are stated and proved in Sect. 2. Section 3 contains proofs of the results in this introduction and Sect. 4 has applications to boundary crossing problems.

2. Density Theorem for Smoothed Symmetrized Renewal Measures

The results of the introduction will all be derived from expansions for densities of $R*G$ with appropriate choices for the smoothing probability measure G.

If $X\in\mathbf{Z}$ a.s. P, let $a=\pi$; otherwise let $a=\infty$. Similarly let $b=\pi$ if $y\in\mathbf{Z}$ a.s. P, and let $b=\infty$ otherwise.

Aside from sorting out the discrete/continuous cases for X and Y, it is convenient to deal with the measure $Q=\mathscr{L}(X,Z)$. The characteristic function of Q will be denoted by ϕ. Define the measure λ by

$$\lambda(A)=\lambda_x\times\lambda_y(\{x,y):(x,y-\gamma x)\in A\})$$

for Borel sets in the plane. Let H be a probability measure which is absolutely continuous with respect to λ with density $h=dH/d\lambda$ and characteristic function $\hat{h}$. If $\iint_S|\hat{h}|<\infty$ where $S=(-a,a)\times(-b,b)$ then h can be found from the inversion formula

$$h(x,z)=\frac{1}{4\pi^2}\iint_S\hat{h}(p,q)\,e^{-ipx-iqz}\,dp\,dq.$$

Moments of H will be denoted $d_{jk}=\iint x^j z^k\,dH$. Let A be a fixed positive constant and let $\mathscr{H}$ be the set of all probability measures H absolutely continuous with respect to λ satisfying $d_{20}+d_{04}\leqq A$. The expansion of Theorem 2.1 holds uniformly for $H\in\mathscr{H}$. This uniformity is necessary for a smoothing argument to obtain Theorem 1.2.

The measure G mentioned earlier is related to H by

$$G(A)=H(\{(x,z):(x,z+\gamma x)\in A\})$$

for Borel sets A in the plane. Densities for G are related to densities for H by

$$h(x,z)=\frac{dG}{d(\lambda_x\times\lambda_y)}(x,z+\gamma x)$$

and consequently densities for $G*R$ are related to densities for $H*\sum_0^\infty Q^{*n}$ by

$$\frac{d\left(H*\sum_0^\infty Q^{*n}\right)}{d\lambda}(x,z)=\frac{d(G*R)}{d(\lambda_x\times\lambda_y)}(x,z+\gamma x).$$

To avoid integrability problems near the origin, results will be derived initially for symmetrized renewal measures V_n, $0\leqq n\leqq\infty$ defined by

$$V_n(A)=\sum_{j=0}^{n-1}\{Q^{*j}(A)+Q^{*j}(-A)\}.$$

Let $U_n=H*V_n$ and $u_n=dU_n/d\lambda$, and let $U=U_\infty$ and $u=u_\infty$.

Theorem 2.1. *Let* $\tau=\sqrt{|x|}+|z|$, *let* $0<\eta<1$ *and let* $n=n_\tau=o(\tau^{1/6})$ *as* $\tau\to\infty$. *If* $|\phi|$ *is bounded away from one on any open subset of* $S-\{(0,0)\}$ *then*

$$u=\hat{u}+u_n+o(\tau^{-3})\{1+\exp(-n^\eta)\iint_S[|\hat{h}|+|\hat{h}_p|+|\hat{h}_{pp}|+|\hat{h}_q|+|\hat{h}_{qq}|+|\hat{h}_{qqq}|+|\hat{h}_{qqqq}|]\}$$

as $\tau\to\infty$ *uniformly for* $H\in\mathscr{H}$, $x>0$ *and* $-\infty<z<\infty$. $\hat{u}$ *is given by*

$$\hat{u}(x,z)=\frac{1}{v}\frac{e^{-\hat{z}^2/2}}{\sqrt{2\pi x\sigma^2/v}}\left\{1+\sqrt{\frac{v}{x}}[C_1H_1(\hat{z})+C_2H_3(\hat{z})]+\frac{v}{x}[C_3H_2(\hat{z})+C_4H_4(\hat{z})+C_5H_6(\hat{z})]\right\},$$

where H_k *is the* k^{th} *Hermite polynomial,*

$$\hat{z}=z/\sqrt{x\sigma^2/v},\qquad C_1=d_{01}/\sigma-M_{11},\qquad C_2=M_{03}-M_{11}/2,$$

$$C_3=M_{20}-M_{12}+M_{11}^2-M_{11}d_{01}/\sigma+d_{02}/(2\sigma)-d_{10}/(2v),$$

$$C_4=M_{04}-M_{12}/2+M_{20}/4+M_{11}^2-2M_{11}M_{03}-M_{11}d_{01}(4\sigma),$$

and

$$C_5=M_{03}^2/2+M_{11}^2/8-M_{11}M_{03}/2.$$

A starting point for proving this theorem is the following inversion formula. Let $\Psi=\Re\{1/(1-\phi)\}$.

Lemma 2.2. *Under the conditions of Theorem 2.1,*

$$u(x,z)=\frac{1}{2\pi^2}\iint_S\Psi(p,q)\,\hat{h}(p,q)\,e^{-ipx-iqz}\,dp\,dq.$$

Proof. Let $W_r(A)=\sum_0^\infty r^n[Q^{*n}(A)+Q^{*n}(-A)]$ and $w_r=d(H*W_r)/d\lambda$. By monotone convergence, $u=\lim_{r\uparrow 1}w_r$, and by Fourier inversion,

$$\omega_r(x,z)=\frac{1}{2\pi^2}\iint_S\Re\left\{\frac{1}{1-r\phi(p,q)}\right\}\hat{h}(p,q)\,e^{-ipx-iqz}\,dp\,dq.$$

By Taylor expansion of ϕ near the origin,

$$\left|\Re\left\{\frac{1}{1-\phi(p,q)}\right\}\right|\sim\frac{\sigma^2q^2/2}{p^2v^2+\sigma^4q^4/4},$$

as $(p, q) \to (0, 0)$. Since this function is locally integrable at the origin, and since $\Re\{1/1-r\phi)\} \to \Psi$ as $r \uparrow 1$ uniformly away from the origin, it is sufficient to show that

$$\lim_{\varepsilon \downarrow 0} \lim_{r \uparrow 1} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \frac{dp\,dq}{|1-r\phi(p,q)|} = 0.$$

As $(p, q) \to (0, 0)$,

$$\phi(p,q) = 1 + ipv - q^2\sigma^2/2 + o(|p|+q^2).$$

Hence there is a constant K_1 such that $|1-\phi| \geqq K_1(|p|+q^2)$ in some neighborhood $\mathcal{N}_0$ of the origin. Since $\Re\{1-\phi\} \geqq 0$, in $\mathcal{N}_0$

$$\begin{aligned} |1-r\phi| &= r\{(1-r)^2/r^2 + |1-\phi|^2 + 2(1-r)\,\Re(1-\phi)/r\}^{1/2} \\ &\geqq r|1-\phi| \\ &\geqq rK_1(|p|+q^2). \end{aligned}$$

The lemma follows since

$$\int_{-\varepsilon}^{\varepsilon} dp \int dq \frac{1}{|p|+q^2} = 4\pi\sqrt{\varepsilon}.$$

Theorem 2.1 will be proved by careful analysis of the integral in Lemma 2.2, and the behavior of $\hat{h}\Psi$ near the origin is of crucial importance.

Lemma 2.3. As $(p, q) \to (0, 0)$, $\hat{h}\Psi = \hat{\Psi} + o(1)$ uniformly for $H \in \mathcal{H}$, where

$$\begin{aligned} \hat{\Psi} = {} & \frac{\frac{1}{2}q^2\sigma^2(1+iqd_{01}+ipd_{10}-q^2d_{02}/2)}{p^2v^2+q^4\sigma^4/4} \\ & + \frac{\left\{\begin{array}{c}[pq^5v\sigma^5\{M_{03}-M_{11}/4\}+p^3qv^3\sigma M_{11}](1+iqd_{01}) \\ +q^8\sigma^8M_{04}/4+p^2q^4v^2\sigma^4\{-M_{04}+M_{12}-M_{20}/4\}+p^4v^4M_{20}\end{array}\right\}}{(p^2v^2+q^4\sigma^4/4)^2} \\ & + \frac{\left\{\begin{array}{c}-q^{12}\sigma^{12}M_{03}^2/8+p^2q^8v^2\sigma^8\{M_{11}^2/8+3M_{03}^2/2-3M_{11}M_{03}/2\} \\ +p^4q^4v^4\sigma^4\{-3M_{11}^3/2+2M_{11}M_{03}\}\end{array}\right\}}{(p^2v^2+q^4\sigma^4/4)^3}. \end{aligned}$$

Also for $j=1$ or 2,

$$\frac{\partial^j}{\partial p^j}(\hat{h}\Psi) = \frac{\partial^j}{\partial p^j}\hat{\Psi} + o((|p|+q^2)^{-j})$$

and for $k=2$, 3, or 4,

$$\frac{\partial^k}{\partial q^k}(\hat{h}\Psi) = \frac{\partial^k}{\partial q^k}\hat{\Psi} + o((|p|+q^2)^{-k/2}).$$

Proof. Taylor expansion of ϕ about the origin gives

$$\phi(p,q)=1+ipv-q^2\sigma^2/2-pqv\sigma M_{11}-iq^3\sigma^3 M_{03}-p^2v^2M_{20}$$
$$-ipq^2v\sigma^2M_{12}+q^4\sigma_4M_{04}+o((|p|+q^2)^2)$$

as $(p,q)\to(0,0)$. Since

$$\frac{1}{x-\varepsilon}=\frac{1}{x}+\frac{\varepsilon}{x^2}+\frac{\varepsilon^2}{x^3}+O(\varepsilon^3/x^4)$$

as $\varepsilon/x\to 0$, with $x=q^2\sigma^2/2-ipv$ we obtain

$$\frac{1}{1-\phi}=\frac{1}{-ipv+q^2\sigma^2/2}$$
$$+\frac{-pqv\sigma M_{11}-iq^3\sigma^3M_{03}-p^2v^2M_{20}-ipq^2v\sigma^2M_{12}+q^4\sigma^4M_{04}}{(-ipv+q^2\sigma^2/2)^2}$$
$$+\frac{(pqv\sigma M_{11}+iq^3\sigma^3M_{03})^2}{(-ipv+q^2\sigma^2/2)^3}+o(1)$$

as $(p,q)\to(0,0)$. Taking real parts of this equation gives

$$\Psi=\tfrac{1}{2}q^2\frac{\sigma^2}{p^2v^2+q^4\sigma^4/4}$$
$$+\frac{\left\{\begin{matrix}pq^5v\sigma^5\{M_{03}-M_{11}/4\}+p^3qv^3\sigma M_{11}+q^8\sigma^8M_{04}/4\\ +p^2q^4v^2\sigma^4\{-M_{04}+M_{12}-M_{20}/4\}+p^4v^4M_{20}\end{matrix}\right\}}{(p^2v^2+q^4\sigma^4/4)^2}$$
$$+\frac{\left\{\begin{matrix}-q^{12}\sigma^{12}M_{03}^2/8+p^2q^8v^2\sigma^8\{M_{11}^2/8+3M_{03}^2/2-3M_{11}M_{03}/2\}\\ +p^4q^4v^4\sigma^4\{-3M_{11}^2/2+2M_{11}M_{03}\}\end{matrix}\right\}}{(p^2v^2+q^4\sigma^4/4)^3}+o(1)$$

as $(p,q)\to(0,0)$. Multiplying this by the Taylor expansion,

$$\hat{h}(p,q)=1+iqd_{01}+ipd_{10}-q^2d_{02}/2+o(|p|+q^2)$$

as $(p,q)\to(0,0)$ uniformly for $H\in\mathscr{H}$, proves the lemma.

Although $\hat{\Psi}$ is a good approximation for $\hat{h}\Psi$ near the origin, it has a few bad properties. It need not be integrable on S, and $\hat{\Psi}$ and its derivatives may not vanish at $\pm b$. To correct these deficiencies, let g: $(-b,b)\to[0,1]$ have a bounded continuous fourth derivative and satisfy $g(x)=1$ for $|x|<\pi/3$ and $g(x)=0$ for $|x|>2\pi/3$. Let $\tilde{\Psi}(p,q)=g(q)(\hat{\Psi}(p,q)-M_{20})$. As $(p,q)\to(0,0)$, the second and third assertions of Lemma 2.3 hold with $\tilde{\Psi}$ replacing $\hat{\Psi}$. An initial approximation for u is

$$\tilde{u}(x,z)=\frac{1}{2\pi^2}\int_{-b}^{b}dp\int_{-\infty}^{\infty}dp\,\tilde{\Psi}(p,q)\,e^{-ipx-iqz}.$$

Since $\tilde{\Psi}$ is not absolutely integrable over the region of integration the integral over p should be interpreted as $\lim_{\Delta\to\infty}\int_{-\Delta}^{\Delta} dp$. The order of integration can be interchanged as $\Psi(p,q)-iq^2\sigma^2 d_{10}/(2p)-qv^3\sigma M_{11}/p$ is absolutely integrable over $\{p\colon |p|>1\}\times(-b,b)$, and q^2/p and q/p both factor.

Lemma 2.4. Under the assumptions of Theorem 2.1,

$$u=\tilde{u}+u_n+o(\tau^{-3})\{1+\exp(-n^{\eta})\iint_S [|\hat{h}|+|\hat{h}_p|+|\hat{h}_{pp}|+|\hat{h}_q|+|\hat{h}_{qq}|+|\hat{h}_{qqq}| + |\hat{h}_{qqqq}|]\}$$

as $\tau\to\infty$ uniformly for $H\in\mathscr{H}$, $x>0$ and $-\infty<z<\infty$.

Proof. Let us first consider what happens if (x,z) vary with τ so that $\tau=O(\sqrt{x})$ as $\tau\to\infty$. Fix $\varepsilon>0$ and choose $\delta>0$ so that

$$\left|\frac{\partial^2}{\partial p^2}(\hat{h}\Psi-\tilde{\Psi})\right|\leqq\frac{\varepsilon}{(|p|+q^2)^2}$$

on $(-\delta,\delta)^2$. Let $S_1=(-1/x,1/x)\times(-1/\sqrt{x},1/\sqrt{x})$, $S_2=(-\delta,\delta)^2-S_1$, $S_3=S-S_1-S_2$, and $S_4=\{p\colon |p|>a\}\times(-b,b)$. For τ sufficiently large, x will be large and we will have

$$\left|\frac{\partial}{\partial p}(\hat{h}\Psi-\tilde{\Psi})\right|\leqq\frac{\varepsilon}{|p|+q^2}$$

on S_1 and $S_1\subset(-\delta,\delta)^2$. Integration by parts gives

$$\begin{aligned}
x[u-\tilde{u}]&=-\frac{i}{2\pi^2}\iint_S\frac{\partial}{\partial p}(\hat{h}\Psi-\tilde{\Psi})e^{-ipx-iqz}-\frac{i}{2\pi^2}\iint_{S_4}\frac{\partial\tilde{\Psi}}{\partial p}e^{-ipx-iqz}\\
&=-\frac{i}{2\pi^2}\iint_{S_1}\frac{\partial}{\partial p}(\hat{h}\Psi-\tilde{\Psi})e^{-ipx-iqz}-\frac{1}{2\pi^2 x}\iint_{S_2}\frac{\partial^2}{\partial p^2}(\hat{h}\Psi-\tilde{\Psi})e^{-ipx-iqz}\\
&\quad-\frac{1}{2\pi^2 x}\iint_{S_3}\frac{\partial^2}{\partial p^2}(\hat{h}\Psi-\tilde{\Psi})e^{-ipx-iqz}-\frac{i}{2\pi^2}\iint_{S_4}\frac{\partial\tilde{\Psi}}{\partial p}e^{-ipx-iqz}\\
&\quad-\frac{1}{2\pi^2 x}\int_{-1/\sqrt{x}}^{1/\sqrt{x}}\left[\frac{\partial}{\partial p}(\hat{h}\Psi-\tilde{\Psi})e^{-ipx}\right]_{p=-1/x}^{p=1/x}e^{-ipx}dq\\
&=I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

By the choice of δ,

$$|I_1|\leqq\frac{1}{2\pi^2}\int_{-1/x}^{1/x}dp\int dq\,\frac{\varepsilon}{|p|+q^2}=\frac{2\varepsilon}{\pi\sqrt{x}},$$

$$|I_2|\leqq\frac{1}{2\pi^2 x}\left\{\iint_{|p|>1/x}+\iint_{|q|>1/\sqrt{x}}\right\}\frac{\varepsilon\,dp\,dq}{(|p|+q^2)^2}=\frac{(1+\pi)\varepsilon}{\pi^2\sqrt{x}}$$

and

$$|I_5|\leqq\frac{1}{2\pi^2 x}\int\frac{2\varepsilon\,dq}{q^2+1/x}=\frac{\varepsilon}{\pi\sqrt{x}}.$$

Since $\partial^2\tilde{\Psi}/\partial p^2$ is integrable on S_4, after an integration by parts, $I_4=O(1/x)$.

I_3 is the troublesome contribution. Let $\alpha = \sup_{S_3} |\phi|$. Since $\partial^2(\hat{h}\phi^j)/\partial p^2$ is bounded by a constant plus a multiple of j^2 uniformly for $H \in \mathscr{H}$ and $j \geqq 0$, by the identity $1/(1-\phi) = \phi^n/(1-\phi) + \sum_{j=0}^{n-1} \phi^j$,

$$I_3 = \frac{1}{-2\pi^2 x} \iint_S \frac{\partial^2}{\partial p^2}\left(\hat{h}\Re\left\{\sum_{j=0}^{n-1} \phi^j\right\}\right) e^{-ipx-iqz} + O(1/x) + O(n^3/x)$$
$$+ 0(\alpha^n/x) \iint_S \{|\hat{h}| + |\hat{h}_p| + |\hat{h}_{pp}|\}$$

as $\tau \to \infty$ uniformly for $H \in \mathscr{H}$. The first term in this equation after two integration by parts is just $x u_n$. Since ε is arbitrary and $\alpha < 1$ this establishes the lemma for sequences where $\tau = O(\sqrt{x})$ as $\tau \to \infty$.

To complete the proof, consider now sequences for which $\tau = O(z)$ as $\tau \to \infty$. Fix $\varepsilon > 0$ and choose $\delta > 0$ so that on $(-\delta, \delta)^2$, $|\partial^j(\hat{h}\Psi - \tilde{\Psi})/\partial q^j| \leqq \varepsilon/(|p| + q^2)^{j/2}$, for $j = 2$, 3, and 4. Let $S_1 = (-1/z^2, 1/z^2) \times (-1/|z|, 1/|z|)$, $S_2 = (-\delta, \delta)^2 - S_1$, $S_3 = S - S_1 - S_2$ and $S_4 = \{p : |p| > a\} \times (-b, b)$. For τ sufficiently large $S_1 \subset (-\delta, \delta)^2$ and three integration by parts gives (boundary terms vanish in the lattice case because $\partial^j \phi(p, \pi)/\partial q^j = \partial^j \phi(p - 2\pi\gamma, -\pi)/\partial q^j$)

$$z^3[u - \tilde{u}] = \frac{i}{2\pi^2} \iint_S \frac{\partial^3}{\partial q^3} (\hat{h}\Psi - \tilde{\Psi}) e^{-ipx-iqz} + \frac{i}{2\pi^2} \iint_{S_4} \frac{\partial^3 \tilde{\Psi}}{\partial q^3} e^{-ipx-iqz}$$
$$= -\frac{z}{2\pi^2} \iint_{S_1} \frac{\partial^2}{\partial q^2} (\hat{h}\Psi - \tilde{\Psi}) e^{-ipx-iqz} + \frac{1}{2\pi^2 z} \iint_{S_2} \frac{\partial^4}{\partial q^4} (\hat{h}\Psi - \tilde{\Psi}) e^{-ipx-iqz}$$
$$+ \frac{1}{2\pi^2 z} \iint_{S_3} \frac{\partial^4}{\partial q^4} (\hat{h}\Psi - \tilde{\Psi}) e^{-ipx-iqz} + \frac{1}{2\pi^2} \iint_{S_4} \frac{\partial^3 \tilde{\Psi}}{\partial q^3} e^{-ipx-iqz}$$
$$+ \frac{1}{2\pi^2} \int_{-1/z^2}^{1/z^2} \left[\left\{i \frac{\partial^2}{\partial q^2} (\hat{h}\Psi - \tilde{\Psi}) + \frac{1}{z} \frac{\partial^3}{\partial q^3} (\hat{h}\psi - \tilde{\psi})\right\} e^{-iqz}\right]_{q=-1/|z|}^{q=1/|z|} e^{-ipx} dp$$
$$= I_1 + I_2 + I_3 + I_4 + I_5.$$

By the choice of δ,

$$|I_1| \leqq \frac{z}{2\pi^2} \int_{-1/z^2}^{1/z^2} dp \int dq \frac{\varepsilon}{|p| + q^2} = \frac{2\varepsilon}{\pi},$$

$$|I_2| \leqq \frac{1}{2\pi^2 z} \{ \iint_{|p| > 1/z^2} + \iint_{|q| > 1/|z|} \} \frac{\varepsilon\, dp\, dq}{(|p| + q^2)^2} = \frac{(1+\pi)\varepsilon}{\pi^2},$$

and

$$|I_5| \leqq \frac{1}{2\pi^2} \int_{-1/z^2}^{1/z^2} \frac{2\varepsilon\, dp}{|p| + 1/z^2} + \frac{1}{2\pi^2 z} \int \frac{2\varepsilon\, dp}{(|p| + 1/z^2)^3/2} = \frac{2\varepsilon \log(2)}{\pi^2} + \frac{4\varepsilon}{\pi^2}.$$

After an integration by parts, $I_4=O(1/z)$ as $\tau\to\infty$ uniformly for $H\in\mathscr{H}$. To approximate I_3, again let $\alpha=\sup_{S_3}|\phi|$. Using the identity $1/(1-\phi)=\phi^n/(1-\phi)+\sum_{j=0}^{n-1}\phi^j$,

$$I_3=\frac{1}{2\pi^2 z}\iint_S \frac{\partial^4}{\partial q^4}\left(\hat{h}\Re\left\{\sum_{j=0}^{n-1}\phi^j\right\}\right)e^{-ipx-iqz}+O(1/z)+O(n^5/z)$$
$$+O(\alpha^n/z)\iint_S\{|\hat{h}|+|\hat{h}_q|+|\hat{h}_{qq}|+|\hat{h}_{qqq}|+|\hat{h}_{qqqq}|\}$$

as $\tau\to\infty$ uniformly for $H\in\mathscr{H}$. The first term in this equation after four integration by parts is just z^3u_n. Since ε is arbitrary and $\alpha<1$ this completes the proof of the lemma.

Proof of Theorem 2.1. By Lemma 2.4, the theorem follows provided $\tilde{u}=\hat{u}+o(\tau^{-3})$. Integration over p in the equation defining $\tilde{u}$ gives

$$\tilde{u}(x,z)=\frac{1}{2\pi v}\int_{-\infty}^{\infty} g(q)\,e^{-iqz-\hat{q}^2/2}\left\{1+\sqrt{\frac{v}{x}}[i\hat{q}C_1+(i\hat{q})^3C_2]+\frac{v}{x}[(i\hat{q})^2C_3+(i\hat{q})^4C_4+(i\hat{q})^6C_5]\right\},$$

where $\hat{q}=\sigma q\sqrt{x/v}$. Although the algebra to obtain this equation is tedious, the necessary integrals can all be evaluated by residue calculus, or by repeated differentiation of the identity

$$\int\frac{e^{-ipx}}{p^2+\xi^2}\,dp=\frac{\pi}{\xi}e^{-\xi|x|}$$

with respect to ξ and x. At this stage there is no harm in setting $g\equiv 1$; errors due to this change are $o(\tau^{-3})$. Integration over q using the formula (derived from equations XVI.1.6 and XVI.1.7 of Feller 1971)

$$\frac{1}{2\pi}\int(i\hat{q})^k e^{-ipz-\hat{q}^2/2}\,dq=H_k(\hat{z})\frac{e^{-\hat{z}^2/2}}{\sqrt{2\pi x\sigma^2/v}}$$

completes the proof of Theorem 2.1.

The next result will be used to convert expansions of the symmetrized measure V to expansions for $W=\sum_0^\infty Q^{*n}$.

Theorem 2.5. *Let H be a finite measure with density $h=dH/d\lambda$, and let $g=(1+\tau)^3h$. If g is directly Riemann integrable, if $EX^2\log^{3/2}(1+X^-)<\infty$ and EZ^4*

$<\infty$, *and if* $|\phi|$ *is bounded away from one on any open subset of* $S-\{(0,0)\}$, *then*

$$\frac{d}{d\lambda} H*W=o(\tau^{-3})$$

as $\tau\to\infty$ *uniformly for* (x,z) *in the support of* λ *with* $x\leqq 0$.

Proof. Suppose initially that H is a probability measure with support the unit ball B_1, and that $\iint|\partial^j \hat{h}/\partial p^j|<\infty$ for $j=0,1,2$ and $\iint|\partial^j \hat{h}/\partial q^j|<\infty$ for $j=1,2,3,4$. By Theorem 2.1 with $n_\tau\equiv 0$,

$$\begin{aligned}\frac{d}{d\lambda} H*W(x,z) &\leqq u(x,z)\\ &=\hat{u}(x,z)+o(\tau^{-3})\\ &=O\left(\frac{1}{\sqrt{|x|}}\exp-\left\{\frac{z^2 v}{4|x|\sigma^2}\right\}\right)+o(\tau^{-3})\end{aligned} \tag{2.1}$$

as $\tau\to\infty$ uniformly in x and z. In particular, for some constant K_1,

$$\frac{d}{d\lambda} H*W(x,z)\leqq\frac{K_1}{\sqrt{2+|x|}}$$

for all x and z. Also, due to the rapid decay of the exponential function, it is sufficient to restrict attention to sequences where $z=O(\sqrt{|x|\log|x|})$ as $\tau\to\infty$, and for these sequences $\tau=O(\sqrt{|x|\log|x|})$. Consequently it is sufficient to show that

$$\frac{d}{d\lambda} H*W(x,z)=o(\{|x|\log|x|\}^{-3/2}) \tag{2.2}$$

as $x\to\infty$ uniformly in z.

Let (X_i,Z_i), $i\geqq 1$ be i.i.d. from Q, independent of (X,Z), let $\mathbf{S}_n=(S_n^{(1)},S_n^{(2)})=\left(\sum_1^n X_i,\sum_1^n Z_i\right)$, and let $\mathscr{F}_n=\sigma\{\mathbf{S}_1,\ldots,\mathbf{S}_n\}$. Then

$$\frac{d}{d\lambda} H*W(x,z)=E\sum_{n=0}^{\infty} h(\mathbf{S}_n-(x,z)).$$

Let $t=t_x=\inf\{n\geqq 1: S_n^{(1)}\leqq x/3\}$. Since $h=0$ off B_1, for x sufficiently negative, conditioning on $\mathscr{F}_t$,

$$\begin{aligned}\frac{d}{d\lambda} H*W(x,z)&=E\left[\frac{d}{d\lambda} H*W(\mathbf{S}_t-(x,z));\ t<\infty\right]\\ &=E\left[\frac{d}{d\lambda} H*W(\mathbf{S}_t-(x,z));\ t<\infty,\ S_t^{(1)}>x/2\right]\\ &\quad+E\left[\frac{d}{d\lambda} H*W(\mathbf{S}_t-(x,z));\ t<\infty,\ S_t^{(1)}\leqq x/2\right].\end{aligned}$$

To see that the first of these terms is small, let $\tilde{t}=\inf\{n>t\colon S_n^{(1)}\leqq S_t^{(1)}+x/3\}-t$ on $\{t<\infty\}$ and $\tilde{t}=\infty$ on $\{t=\infty\}$. Then $\tilde{t}|t<\infty\sim t$, so $P(\tilde{t}<\infty)=P(t<\infty)^2$. Conditioning on $\mathscr{F}_t$, for x sufficiently negative,

$$E\left[\frac{d}{d\lambda}H*W(\mathbf{S}_t-(x,z));\, t<\infty,\, S_t^{(1)}>x/2\right]=E\left[\frac{d}{d\lambda}H*W(\mathbf{S}_{\tilde{t}}-(x,z));\, \tilde{t}<\infty,\, S_t^{(1)}>x/2\right]$$

$$\leqq K_1\,P(t<\infty)^2/2. \tag{2.3}$$

This expression is $O(1/x^2)$ as $x\to-\infty$ by Markov's inequality; $EX^2<\infty$ and $EX>0$ imply $E[\inf_n S_n^{(1)}]>-\infty$. The second term is bounded by

$$E\left[\frac{K_1}{\sqrt{2+|S_t^{(1)}-x|}};\, t<\infty,\, S_t^{(1)}\leqq x/2\right]$$

$$\leqq\sum_{n=0}^{\infty}E\left[\frac{K_1}{\sqrt{2+|S_n^{(1)}+X-x|}};\, S_n^{(1)}>x/3,\, S_n^{(1)}+X\leqq x/2\right]$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}E\left[\frac{K_1}{\sqrt{2+|S_n^{(1)}+X-x|}};\, S_n^{(1)}\in(j+x/3, j+1+x/3],\, S_n^{(1)}+X\leqq x/2\right]$$

$$\leqq\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}E\left[\frac{K_1}{\sqrt{1+|j+X-2x/3|}};\, S_n^{(1)}\in(j+x/3, j+1+x/3],\, X\leqq -j+x/6\right]$$

$$\leqq K_2E\left[\sum_{0\leqq j\leqq -X+x/6}\frac{K_1}{\sqrt{1+|j+X-2x/3|}}\right]$$

where $K_2=\sup_x\sum_{n=0}^{\infty}P(S_n^{(1)}\in(x, x+1])$ which is finite by the renewal theorem (in one dimension). To continue, with x sufficiently negative, this last expression is bounded by

$$K_1K_2E\int_{-1}^{1-X+x/6}\frac{dy}{\sqrt{|y+X-2x/3|}}$$

$$\leqq 2K_1K_2E[\sqrt{|1+x/2|}+\sqrt{|X-1-2x/3|};\, X\leqq 2+x/6]$$

$$\leqq 8K_1K_2E[\sqrt{|X|};\, X<x/10]$$

$$\leqq 8K_1K_2E[X^2\log^{3/2}(1+X^-);\, X<x/10]/(x\log(1-x))^{3/2} \tag{2.4}$$

and (2.2) follows by dominated convergence finishing the proof in this case.

In the general case, first pick a function h that satisfies the conditions of the special case and exceeds one on $B_{1/2}$. Then

$$W(B_{1/2}+(x,z))\leqq\frac{d}{d\lambda}H*W(x,z)=o(\tau^{-3})$$

as $\tau\to\infty$ with $x<0$. Since the arguments leading to this result remain valid under rescaling, for any (fixed) bounded measurable set I,

$$W(I+(x,z))=o(\tau^{-3}) \tag{2.5}$$

as $\tau\to\infty$ uniformly for $x<0$. Using similar arguments and the bound (2.1), for any bounded measurable set I,

$$W(I+(x,z))=O(\tau^{-3})+O\left(\exp-\frac{z^2 v}{4\sigma^2|x|}\right) \tag{2.6}$$

as $\tau\to\infty$ uniformly for $x\geqq 0$.

To continue, let $\{I_n\}_{n\geqq 1}$ be the unit squares with integer corners and let $g_n=\sup_{I_n} g$. Since g is directly Riemann integrable, $\sum g_n<\infty$. Define $(x_n, z_n)\in I_n$ so that $\tau(x_n, z_n)=\inf_{I_n}\tau(x,z)$. Then

$$\begin{aligned}\tau^3(x,z)\frac{d}{d\lambda}H*W(x,z)&=\sum_n \iint_{(x,z)-I_n}\frac{\tau^3(x,z)\,g(x-x',z-z')}{(1+\tau(x-x',z-z'))^3}\,dW(x',z')\\&\leqq\sum_n g_n\frac{\tau^3(x,z)}{(1+\tau(x_n,z_n))^3}W((x,z)-I_n).\end{aligned}$$

This approaches zero by dominated convergence – pointwise convergence of the summands follows from (2.5) and domination follows from the next lemma using (2.6) and the observation $(x,z)-I_n\subset(x-x_n, z-z_n)+I$ where I is the square $[-1,1]^2$.

Lemma 2.6. *For all x, x', z and z',*

$$\frac{\tau(x,z)}{1+\tau(x-x',z-z')}\leqq 2+\tau(x',z')$$

and for all x, x', z and z' with $x'<x<0$,

$$\tau(x,z)\exp\left\{-\frac{(z-z')^2 v}{12\sigma^2|x-x'|}\right\}\leqq|z'|+\sqrt{|x'|}\{1+\sqrt{6\sigma^2/(ev)}\}.$$

Proof. If $x_1, x_2, x_3\geqq 0$ then $(x_1+x_2)/(x_1+x_3)\leqq(x_2+x_3)/x_3$ and $\sqrt{x_1+x_2}\leqq\sqrt{x_1}+\sqrt{x_2}$. Using these bounds,

$$\begin{aligned}\frac{\sqrt{|x|}+|z|}{1+\sqrt{|x-x'|}+|z-z'|}&\leqq\frac{\sqrt{|x|}+|z'|+|z-z'|}{1+\sqrt{|x-x'|}+|z-z'|}\\&\leqq 1+\frac{\sqrt{|x|}+|z'|}{1+\sqrt{|x-x'|}}\\&\leqq 1+\frac{\sqrt{|x'|}+\sqrt{|x-x'|}+|z'|}{1+\sqrt{|x-x'|}}\\&\leqq 2+\tau(x',z').\end{aligned}$$

Using the identity $x_2 \exp\{-x_1 x_2^2\} \leqq 1/\sqrt{2 e x_1}$ for all $x_1 > 0$,

$$\tau(x, z) \exp\left\{-\frac{(z-z')^2 v}{12\sigma^2 |x-x'|}\right\} \leqq \sqrt{|x|} + |z'| + |z-z'| \exp\left\{-\frac{(z-z')^2 v}{12\sigma^2 |x-x'|}\right\}$$

$$\leqq \sqrt{|x|} + |z'| + \left\{\frac{6\sigma^2 |x-x'|}{e v}\right\}^{1/2}$$

and the lemma follows.

The final lemma of this section will be used to show in applications of Theorem 2.1 that terms associated with u_n are negligible. The proof is similar to the initial arguments in the proof of Theorem 2.5.

Lemma 2.7. *Suppose* $n = n_\tau = o(\tau)$ *as* $\tau \to \infty$. *If* $EZ^4 < \infty$, $Ex^2 \log^{3/2}(1+|X|) < \infty$ *and* $|\phi|$ *is bounded away from one on open subsets of* $S - \{(0,0)\}$, *then for any bounded measurable set* I,

$$V_n(I + (x, y)) = o(\tau^{-3})$$

uniformly in (x, y) *as* $\tau \to \infty$.

Proof. Rescaling we can take $I = B_1$ without loss of generality and by (2.5) and (2.6) it is sufficient to show that

$$W_n(I + (x, z)) = o((x \log x)^{-3/2})$$

uniformly in z as $x \to \infty$ when $n = n_x = o(\sqrt{x \log x})$, where $W_n = \sum_{j=0}^{n-1} Q^{*j}$. Let $t = t_x = \inf\{n \geqq 1: \ S_n^{(1)} - 2nv \geqq x/3\}$ and $\tilde{t} = \inf\{n > t: \ S_n^{(1)} - 2nv \geqq 2x/3 + S_t^{(1)} - 2tv\} - t$. Conditioning on $\mathscr{F}_t$, for x sufficiently large

$$\begin{aligned} W_n(I + (x, z)) &= E[W_{n-t}(I + (x, z) - \mathbf{S}_t); t \leqq n] \\ &\leqq E[W_{n-t}(I + (x, z) - \mathbf{S}_t); t \leqq n, S_t^{(1)} < x/2] \\ &\quad + E[W_{n-t}(I + (x, z) - \mathbf{S}_t); t \leqq n, S_t^{(1)} \geqq x/2]. \end{aligned}$$

The first term is bounded for large x by a multiple of $P(\tilde{t} < \infty) = o(x^{-2})$ (as in the derivation leading to equation 2.3). For x sufficiently large, $\{t \leqq n\} \subset \{S_{t-1}^{(1)} < 2x/5\}$, so the second term is bounded by

$$\sum_{n=1}^{\infty} E[W(I + (x, z) - \mathbf{S}_n); S_n^{(1)} \geqq x/2, S_{n-1}^{(1)} < 2x/5].$$

Bounding $W(I + (x, z) - \mathbf{S}_n)$ by $K(2 + |S_n^{(1)} - x|)^{-1/2}$ the proof is completed by the same arguments leading to (2.4).

3. Proofs

Proof of Theorem 1.1. Let $r(x, y) = R(\{(x, y)\})$. Since P is lattice, $|\phi|$ is bounded away from one on any open subset of $(-\pi, \pi)^2 - \{(0, 0)\}$ and Theorem 2.1 with H a point mass at zero and $n \equiv 0$ gives

$$r(x, y) + r(-x, -y) = \hat{r}(x, y) + o(\tau^{-3})$$

as $\tau \to \infty$ uniformly for $x > 0$. By Theorem 2.5, $r(x, y) = o(\tau^{-3})$ as $\tau \to \infty$ uniformly for $x \leqq 0$ and these assertions combine to prove Theorem 1.

Proof of Theorem 1.2. By change of variables,

$$\begin{aligned}\iint f\,dR &= \iint f(x, z+\gamma x)\,dW(x, z)\\ &= \iint_{x \leqq 0} f(x, z+\gamma x)\,dW(x, z) + \iint_{x > 0} f(x, z+\gamma x)\,dV(x, z)\\ &\quad - \iint_{x<0} f(-x, -z-\gamma x)\,dW(x, z).\end{aligned}$$

Partitioning $\mathbf{R}^2$ into small rectangles and using equation (2.5), the first and third terms in magnitude are $o(1) \iint m_f(x, y, 1)\,\tau^{-3}(x, y)\,dx\,dy$ uniformly in f as $\tau_f \to \infty$.

Let $H^{(1)}$ be a fixed member of $\mathscr{H}$ with support B_1 with

$$\iint [|\hat{h}^{(1)}| + |\hat{h}_p^{(1)}| + |\hat{h}_{pp}^{(1)}| + |\hat{h}_q^{(1)}| + \ldots + |\hat{h}_{qqqq}^{(1)}|] = K < \infty.$$

Define $H^{(\varepsilon)}$ for $0 < \varepsilon < 1$ by $H^{(\varepsilon)}(A) = H^{(1)}(A/\varepsilon)$. Then $H^{(\varepsilon)}$ has support B_ε, $\hat{h}^{(\varepsilon)}(p, q) = \hat{h}^{(1)}(\varepsilon p, \varepsilon q)$ and

$$\iint [|\hat{h}^{(\varepsilon)}| + |\hat{h}_p^{(\varepsilon)}| + |\hat{h}_{pp}^{(\varepsilon)}| + |\hat{h}_q^{(\varepsilon)}| + \ldots + |\hat{h}_{qqqq}^{(\varepsilon)}|] \leqq K/\varepsilon^2.$$

Fix $0 < \eta < \eta' < 1/6$ and let $\varepsilon = \varepsilon_f = \exp(-\tau_f^\eta)$ and $n = n_f \sim \tau_f^{\eta'}$. Then by Theorem 2.1

$$\frac{d}{d\lambda} H^{(\varepsilon)} * V(x, z) = \hat{r}(x, z+\gamma x) + \frac{d}{d\lambda} H^{(\varepsilon)} * V_n(x, z) + o(\tau^{-3}(x, z))$$

as $\tau_f \to \infty$ uniformly for $\tau(x, z) \geqq \tau_f$. Define $f_+(x, y) = f(x, y)\,I\{x > 0\}$. Since $H^{(\varepsilon)}$ has support B_ε

$$\left|\iint f_+(x, z+\gamma x)\{dV - d(H^{(\varepsilon)} * V)\}\right| \leqq \iint \omega_{f_+}(x, z+\gamma x, (1+\gamma)\varepsilon)\,d(H^{(\varepsilon)} * V)$$

(note that $\{(x, z)\colon \|(x, z+\gamma x\| < (1+\gamma)\varepsilon\} \supset B_\varepsilon$). From Lemma 2.7, since $H^{(\varepsilon)}$ has support B_ε, for any bounded set I,

$$H^{(\varepsilon)} * V_n(I + (x, y)) = o(\tau^{-3}).$$

This implies for ε sufficiently small that

$$\iint f(x, z+\gamma x)\, d(H^{(\varepsilon)} * V_n) = o(1) \iint m_f(x, z+\gamma x, 1)\, \tau^{-3}(x, z)\, dx\, dz$$

and

$$\iint \omega_{f_+}(x, z+\gamma x, (1+\gamma)\varepsilon)\, d(H^{(\varepsilon)} * V_n) = o(1) \iint m_f(x, z+\gamma x, 1)\, \tau^{-3}(x, z)\, dx\, dz$$

as $\tau_f \to \infty$. The theorem now follows (a few minor details such as the legitimacy of changing f_+ to f and $(1+\gamma)\varepsilon$ to ε are easily checked. The constants d_{01}, d_{02} and d_{10} can be ignored since $\varepsilon_f \to 0$ very quickly).

Proof of Corollary 1.3. Let us begin by looking at the integral of $\hat{r}$ over $A = (x - \delta, x+\delta) \times (-\infty, y)$. Using the identity

$$\int_{-\infty}^{y} H_{k+1}(z)\, \phi(z)\, dz = -H_k(z)\, \phi(z),$$

integration over y' gives

$$\int_{x-\delta}^{x+\delta} dx' \int_{-\infty}^{y} dy'\, \hat{r}(x', y') = \frac{1}{v} \int_{x-\delta}^{x+\delta} dx' \Big\{ \Phi(y^*) - \phi(y^*) \sqrt{\frac{v}{x'}}\, [c_1 + c_2 H_2(y^*)]$$
$$- \phi(y^*) \frac{v}{x'} [c_3 H_1(y^*) + c_4 H_3(y^*) + c_5 H_5(y^*)] \Big\}, \tag{3.1}$$

where $y^* = (y - \gamma x')/\sqrt{x' \sigma^2/v}$. By Taylor expansion,

$$\frac{1}{\sqrt{x' \sigma^2/v}} = \frac{1}{\sqrt{x \sigma^2/v}} \left[1 - \frac{x'-x}{2x} + O\{(x'-x)^2/x^2\} \right],$$

as $(x'-x)/x \to 0$. Multiplying this by $y - \gamma x' = y - \gamma x - \gamma(x'-x)$ gives

$$y^* = \hat{y} - \frac{\hat{y}(x'-x)}{2x} - \frac{\gamma(x'-x)}{\sqrt{x\sigma^2/v}} + O\{x^{-3/2}(x'-x)^2\}$$

as $x \to \infty$ with $\hat{y} = O(\sqrt{x})$ and $x' - x = O(\sqrt{x})$. Taylor expansion of Φ gives

$$\Phi(y^*) = \Phi(\hat{y}) - \phi(\hat{y}) \left[\frac{\hat{y}(x'-x)}{2x} + \frac{\gamma(x'-x)}{\sqrt{x\sigma^2/v}} \right]$$
$$- \frac{1}{2} \hat{y}\, \phi(\hat{y}) \frac{\gamma^2 (x'-x)^2}{x\sigma^2/v} + O\{x^{-3/2} |x'-x|^3 + x^{-3/2} |\hat{y}| (x'-x)^2\} \tag{3.2}$$

as $x\to\infty$ with $\hat{y}=O(\sqrt{x})$ and $x'-x=O(\sqrt{x})$. In this expression, $O\{\cdot\}$ can be replaced by $O\{x^{-3/2}|x'-x|^3+x^{-5/4}(x'-x)^2\}$ for if $1/\hat{y}=O(x^{-1/4})$, both sides are exponentially close to 0 or 1. Since $\sqrt{v/x'}=\sqrt{v/x}+O(x^{-3/2}|x'-x|)$,

$$\begin{aligned}
&-\phi(y^*)\sqrt{\frac{v}{x'}}[c_1+c_2H_2(y^*)]\\
&\quad=-\phi(\hat{y})\sqrt{\frac{v}{x}}[c_1+c_2H_2(\hat{y})]-\frac{\gamma(x'-x)}{\sqrt{x\sigma^2/v}}\phi(\hat{y})\sqrt{\frac{v}{x}}[c_1H_1(\hat{y})+c_2H_3(\hat{y})]\\
&\qquad+O\{x^{-3/2}[(x'-x)^2+|x'-x|+|\hat{y}(x'-x)|]\}
\end{aligned}\tag{3.3}$$

as $x\to\infty$ with $\hat{y}=O(\sqrt{x})$ and $x'-x=O(\sqrt{x})$. As in (3.2), $O\{\cdot\}$ can be replaced by $O\{x^{-3/2}(x'-x)^2+x^{-5/4}|x'-x|\}$. Finally,

$$\begin{aligned}
&-\phi(y^*)\frac{v}{x'}[c_3H_1(y^*)+c_4H_3(y^*)+c_5H_5(y^*)]\\
&\quad=-\phi(\hat{y})\frac{v}{x}[c_3H_1(\hat{y})+c_4H_3(\hat{y})+c_5H_5(\hat{y})]+O\{x^{-3/2}|x'-x|\}
\end{aligned}\tag{3.4}$$

as $x\to\infty$ with $\hat{y}=O(\sqrt{x})$ and $x'-x=O(\sqrt{x})$. Integration, after approximating the integrands in (3.1) using (3.2)–(3.4), gives

$$\begin{aligned}
\int_{x-\delta}^{x+\delta}dx'\int_{-\infty}^{y}dy'\,\hat{r}(x',y')&=\frac{2\delta}{v}\Phi(\hat{y})-\frac{2\delta}{\sqrt{vx}}\phi(\hat{y})[c_1+c_2H_2(\hat{y})]\\
&\quad-\frac{2\delta}{x}\phi(\hat{y})\left[\left(c_3+\frac{\delta^2\gamma^2}{6\sigma^2}\right)H_1(\hat{y})+c_4H_3(\hat{y})+c_5H_5(\hat{y})\right]\\
&\quad+O\{x^{-3/2}(\delta^4+\delta^3+\delta^2)+x^{-5/4}(\delta^3+\delta^2)\},
\end{aligned}\tag{3.5}$$

as $x\to\infty$ with $\delta=O(\sqrt{x})$. If $x\to\infty$ with $\delta=o(\sqrt{x})$, the $O\{\cdot\}$ term is $o\{(1+\delta^3)/x\}$.

Turning to the other terms in Theorem 1.2, $\int\tau^{-3}(x,y)\,dy=1/x$, so with $f=1_A$,

$$\iint m_f(x,y,1)\tau^{-3}(x,y)\,dx\,dy\leqq\int_{x-\delta-1}^{x+\delta+1}dx'/x'=O\{(1+\delta)/x\},\tag{3.6}$$

as $x\to\infty$ with $\delta=O(\sqrt{x})$. Since $\int\hat{r}(x,y)\,dy=O(1)$ as $x\to\infty$ and $\sup_y\hat{r}(x,y)=O(1/\sqrt{x})$ as $x\to\infty$,

$$\iint\omega_f(x,y,\varepsilon)\,\hat{r}(x,y)=O(\varepsilon),\tag{3.7}$$

as $x\to\infty$ with $\delta=O(\sqrt{x})$ and $\varepsilon=o(1)$. With $\varepsilon=\varepsilon_f=\exp\sqrt{-\tau_f^{1/10}}$ this term is $o(1/x)$ as $x\to\infty$ with $\delta=o(\sqrt{x})$ since $\tau_f\geqq\sqrt{x-\delta}$. The corollary now follows from Theorem 1.2.

Theorem 1.4 follows immediately from Theorems 2.1 and 2.5, with $G=P^{*n}$. Integrability of $\hat{h}$ and its derivatives follows because the characteristic function of a bounded density is square integrable.

4. Boundary Crossing Problems

In this section, (X_i, Y_i) for $i\geqq 1$ will be i.i.d. P, $S_n=\sum_{i=1}^{n} X_i$, $W_n=\sum_{i=1}^{n} Y_i$, and $t=t_a=\inf\{n\colon S_n>a\}$. We will only treat the "positive" case where $P(X\geqq 0)=1$. The main result is an expansion for the distribution of W_t.

Theorem 4.1. *If $E|X|^3<\infty$, $EZ^4<\infty$, $P(X\geqq 0)=1$, and Cramer's condition holds, then*

$$P(W_t<c)=\Phi(\hat{w})+\phi(\hat{w})\sqrt{\frac{v}{a}}\,[D_1+D_2H_2(\hat{w})]$$

$$+\phi(\hat{w})\frac{v}{a}[D_3H_1(\hat{w})+D_4H_3(\hat{w})+D_5H_5(\hat{w})]+o(1/a)$$

as $a\to\infty$, uniformly in c, where $\hat{w}=(c-\gamma a)/\sqrt{a\sigma^2/v}$, $\kappa=v\gamma/\sigma$, $D_1=-\kappa M_{20}$, $D_2=-M_{03}+M_{11}/2$, $D_3=-M_{20}/2-\kappa^2M_{30}-\kappa M_{21}+\kappa M_{11}M_{20}$, $D_4=M_{12}/2-M_{20}/4-M_{04}-M_{11}^2/2+M_{11}M_{03}+\kappa M_{11}M_{20}/2-\kappa M_{03}M_{20}$, and $D_5=-M_{11}^2/8+M_{11}M_{03}/2-M_{03}^2/2$.

Proof. Since we are in the positive case, $\{t=n+1\}=(S_n\leqq a,\ S_n+X_{n+1}>a\}$ (with the convention $S_0=0$). So

$$\begin{aligned}P(W_t<c)&=\sum_{n=0}^{\infty}P(S_n\leqq a,\ S_n+X_{n+1}>a,\ W_n+Y_{n+1}<c)\\&=\sum_{n=0}^{\infty}\iint_{\substack{s\leqq a\\ s+x>a\\ w+y<c}}dP^{*n}(s,w)\,dP(x,y)\\&=\int R\{(a-x,a]\times(-\infty,c-y)\}\,dP(x,y).\end{aligned}$$

Using the same arguments as those used to prove Corollary 1.3,

$$\begin{aligned}R\{(a-x,a]\times(-\infty,c-y)\}=&\frac{x}{v}\Phi(\hat{y})+\frac{\phi(\hat{y})}{\sqrt{av}}\left[\frac{\gamma x^2}{2\sigma}-xc_1-xc_2H_2(\hat{y})\right]\\&+\frac{\phi(\hat{y})}{a}\left\{\frac{x^2\hat{y}}{4v}-\frac{\gamma^2x^3\hat{y}}{6\sigma^2}+\frac{\gamma x^2}{2\sigma}[c_1H_1(\hat{y})+c_2H_3(\hat{y})]\right.\\&\left.-x[c_3H_1(\hat{y})+c_4H_3(\hat{y})+c_5H_5(\hat{y})]\right\}+o\left(\frac{1+x^3}{a}\right)\end{aligned}\tag{4.1}$$

as $a\to\infty$ with $x=o(\sqrt{a})$, uniformly for $c\in\mathbf{R}$, where $\hat{y}=(c-y-\gamma a)/\sqrt{a\sigma^2/v}$. Call the approximation in this equation $\hat{R}(x,y)$. By the ordinary renewal theorem, for some K, $R\{(a-x,a]\times\mathbf{R}\}\leqq K(1+x)$ for all $a\in\mathbf{R}$ and $x\geqq 0$. Since $EX^3<\infty$ and

$$E(X;X>b)\leqq E(X^3;X>b)/b^2,$$

we can choose a sequence $b=b_a$ such that $b=o(\sqrt{a})$ and $E(X;X>b)=o(1/a)$. For this sequence,

$$E[R\{(a-X,a]\,x(-\infty,c-Y)\};X>b]=o(1/a)$$

as $a\to\infty$, uniformly for $c\in\mathbf{R}$. By dominated convergence and (4.1),

$$\begin{aligned}E[R\{(a-X,a]\,x(-\infty,c-Y)\};X\leqq b]&=E[\hat{R}(X,Y);X\leqq b]+o(1/a)\\&=E\hat{R}(X,Y)+o(1/a)\end{aligned}$$

as $a\to\infty$, uniformly in c. Now $\hat{y}=\hat{w}-y/\sqrt{a\sigma^2/v}$ and Taylor expansion gives

$$\Phi(\hat{y})=\Phi(\hat{w})-\frac{y\phi(\hat{w})}{\sqrt{a\sigma^2/v}}-\frac{\hat{w}\phi(\hat{w})y^2}{2a\sigma^2/v}+o(y^2/a)$$

and

$$H_k(\hat{y})\,\phi(\hat{y})=H_k(\hat{w})\,\phi(\hat{w})+\frac{yH_{k+1}(\hat{w})\,\phi(\hat{w})}{\sqrt{a\sigma^2/v}}+o(|y|/\sqrt{a})$$

as $a\to\infty$, uniformly in $\hat{w}$. So

$$\begin{aligned}E\hat{R}(X,Y)=\Phi(\hat{w})+\phi(\hat{w})\sqrt{\frac{v}{a}}\left[\frac{\gamma EX^2}{2v\sigma}-c_1-c_2H_2(\hat{w})-\frac{EXY}{v\sigma}\right]\\
+\frac{\phi(\hat{w})}{a}\left\{\frac{\hat{w}EX^2}{4v}-\frac{\gamma^2\hat{w}EX^3}{6\sigma^2}+\frac{\gamma EX^2}{2\sigma}[c_1H_1(\hat{w})+c_2H_3(\hat{w})]\right.\\
-vc_3H_1(\hat{w})-vc_4H_3(\hat{w})-vc_5H_5(\hat{w})\\
-\frac{\hat{w}EXY^2}{2\sigma^2}+\frac{\gamma H_1(\hat{w})EX^2Y}{2\sigma^2}\\
\left.-\frac{c_1H_1(\hat{w})EXY}{\sigma}-\frac{c_2H_3(\hat{w})EXY}{\sigma}\right\}+o(1/a)\end{aligned}$$

as $a\to\infty$ uniformly in $\hat{w}$. After some algebra this gives the result in the theorem.

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