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Continued Fractions and Unique Additive Partitions

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Abstract. A partition of the positive integers into sets A and B *avoids* a set $S \subset \mathbb{N}$ if no two distinct elements in the same part have a sum in S. If the partition is unique, S is *uniquely avoidable.* For any irrational $\alpha > 1$, Chow and Long constructed a partition which avoids the numerators of all convergents of the continued fraction for α, and conjectured that the set S_α which this partition avoids is uniquely avoidable. We prove that the set of numerators of convergents is uniquely avoidable if and only if the continued fraction for α has infinitely many partial quotients equal to 1. We also construct the set S_α and show that it is always uniquely avoidable.

Keywords: additive partition, best approximation, continued fraction, uniquely avoidable set

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1. Introduction

A partition of the positive integers into sets A and B *avoids* a set $S \subset \mathbb{N}$ if no two distinct elements in the same part have a sum in S. We say that S is *avoidable*; if the partition is unique, S is *uniquely avoidable.*

The Fibonacci numbers are uniquely avoidable [1]. Generalized Fibonacci sequences defined by $(s_1, s_2) = 1$, $s_n = s_{n-1} + s_{n-2}$, are also uniquely avoidable provided that $s_1 < s_2$ or $2|s_1 s_2$; Alladi, Erdős and Hoggatt [1] proved this for $s_1 = 1$, and Evans [3] proved the general case. This suggests a connection with continued fractions. Chow and Long [2] studied this connection, and proved that the set of the numerators of continued-fraction convergents to any irrational α with $1 < \alpha < 2$ (easily generalized to any irrational α) is avoidable, although not necessarily uniquely avoidable. Their partition uses the sets

$$A_\alpha = \{n \in \mathbb{N} : \text{the integer multiple of } \alpha \text{ nearest } n \text{ is greater than } n\},$$
$$B_\alpha = \{n \in \mathbb{N} : \text{the integer multiple of } \alpha \text{ nearest } n \text{ is less than } n\}.$$

Let S_α be the set avoided by the partition $\{A_\alpha, B_\alpha\}$. The main results of Chow and Long are that S_α contains the numerators of all convergents to α, and every other element of S_α is either the numerator of an intermediate fraction or twice the numerator of a convergent. They conjectured that S_α was uniquely avoidable.

We give a characterization of a large class of sets which are uniquely avoidable if they are avoidable at all. We use this to show that the set of numerators of convergents to α is uniquely avoidable for any α if and only if infinitely many partial quotients of α are 1. We can also use the best approximation property to determine S_α precisely, and show that S_α is uniquely avoidable for any irrational $\alpha > 1$.

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2. Results on continued fractions

We will use some elementary results on continued fractions, as given in [4] for example. We use the standard notation for continued fractions, in which $\alpha = [a_0, a_1, \ldots]$, $p_{-2} = 0$, $q_{-2} = 1$, $p_{-1} = 1$, $q_{-1} = 0$, and for $n \geq -1$, we have $p_{n+1} = a_{n+1}p_n + p_{n-1}$, $q_{n+1} = a_{n+1}q_n + q_{n-1}$. We will study continued fractions in terms of their approximation properties.

Definition 1. For any p, let $E(p) = p - q\alpha$ be the error in approximating p by the closest multiple of α. Thus A_α is the set of all p with $E(p) < 0$ and B_α the set of all p with $E(p) > 0$.

The function E is additive modulo α. Specifically, if $|E(x) + E(y)| < \alpha/2$, then $E(x+y) = E(x) + E(y)$; otherwise, $E(x+y) = E(x) + E(y) \pm \alpha$. Likewise, if $|E(x) - E(y)| < \alpha/2$, then $E(x-y) = E(x) - E(y)$; otherwise, $E(x-y) = E(x) - E(y) \pm \alpha$. In particular, if $E(x)$ and $E(y)$ have opposite signs, $E(x+y) = E(x) + E(y)$; if they have the same sign, $E(x-y) = E(x) - E(y)$. Repeated application of additivity shows that $E(kx)$ is congruent to $kE(x)$ modulo α, and if $k|E(x)| < \alpha/2$, then $E(kx) = kE(x)$.

We can restate several elementary results on continued fractions in terms of E. The best approximation property of convergents is that $|E(x)| < |E(y)|$ for all positive integers $y < x$ if and only if x is the numerator of a convergent. The property that alternate convergents approach α from opposite sides is that $E(p_n) < 0$ for n even and $E(p_n) > 0$ for n odd. The property that $|\alpha - p_n/q_n| < 1/(q_n q_{n+1})$ is that $|E(p_n)| < 1/q_{n+1}$. The coefficients a_{n+1} of the continued fraction are defined by $E((a_{n+1}+1)p_n + p_{n-1}) < 0 \leq E(a_{n+1}p_n + p_{n-1}) < E(p_{n-1})$ if n is even, and the reverse if n is odd.

We can use this property $|a_{n+1}E(p_n)| < |E(p_{n-1})| < |(a_{n+1}+1)E(p_n)|$ to show that $|kE(p_n)| < |E(p_{n-1})| < \alpha/2$ for all k with $0 \leq k \leq a_{n+1}$; it thus follows that $E(kp_n) = kE(p_n)$ if $0 \leq k \leq a_{n+1}$.

The properties of E give us the following characterization of those numbers which can be approximated well by multiples of α; this lemma appears in similar form in [2].

LEMMA 1 *If $p < p_{n+1}$ and $|E(p)| < |E(p_{n-1})|$, then p is either kp_n or the numerator $kp_n + p_{n-1}$ of an intermediate fraction for some $k \in \mathbb{N}$.*

Proof: For simplicity of notation, we will assume n is odd, so that $E(p_n) > 0$. If $E(p) > 0$, then let

$$k = \min\left(\left\lfloor \frac{E(p)}{E(p_n)} \right\rfloor, \left\lfloor \frac{p}{p_n} \right\rfloor\right).$$

Then $p' = p - kp_n$ still satisfies $0 \leq E(p') < |E(p_{n-1})|$ since $0 < kE(p_n) = E(kp_n) \leq E(p)$. If $k = \lfloor p/p_n \rfloor$, then $p' < p_n$, but no positive $p' < p_n$ can have $0 \leq E(p') < |E(p_{n-1})|$, by the best approximation property of p_{n-1}. Otherwise, we have $k = \lfloor E(p)/E(p_n) \rfloor$ and thus $0 \leq E(p') = E(p) - kE(p_n) < E(p_n)$, but no positive $p' < p_{n+1}$ can have $0 \leq E(p') < E(p_n)$. Thus, in either case, we must have $p' = 0$ and $p = kp_n$.

If $E(p) < 0$, then since $E(p_{n-1})$ is also negative, we have $E(p - p_{n-1}) = E(p) - E(p_{n-1})$, which is between 0 and $-E(p_{n-1})$, so we can apply the above result to $p - p_{n-1}$. ■

We will often need to use the following generalization of this lemma to cases with $p > p_{n+1}$.

LEMMA 2 *If $|E(p)| < |E(p_{n-1})|$, then $p = ip_{n-1} + jp_n$, where i and j are non-negative integers and $i \le \lceil p/p_{n+1} \rceil$.*

Proof: We again assume n is odd, so that $E(p_{n+1}) < 0 < E(p_n)$. The proof is by induction on p. The previous lemma proves the case $p < p_{n+1}$. If $E(p) > 0$, then we write $p = p' + p_n$, which gives $E(p') = E(p) - E(p_n)$. Since $E(p') < E(p) < |E(p_{n-1})|$ and $E(p') > -E(p_n) > -|E(p_{n-1})|$, we can apply the lemma inductively to p'. If $E(p) < 0$, then we write $p = p' + p_{n+1} = p' + p_{n-1} + a_{n+1}p_n$, which gives $E(p') = E(p) - E(p_{n+1})$. Since $E(p') > E(p) > -|E(p_{n-1})|$ and $E(p') < E(p_{n+1}) < |E(p_{n-1})|$, we can apply the lemma inductively to p', with $\lceil p'/p_{n+1} \rceil = \lceil p/p_{n+1} \rceil - 1$. ■

To simplify the arguments, we will assume $1 < \alpha < 2$; the following lemma shows that our results generalize to arbitrary α.

LEMMA 3 *If $1 < \alpha < 2$ and $\alpha' = \alpha/(\alpha - 1)$, then α' and α have the same set of numerators of convergents, the same set of numerators of intermediate fractions, and complementary partitions with $A_\alpha = B_{\alpha'}$ (so that $S_\alpha = S_{\alpha'}$).*

Proof: We have $\alpha = [1, a_1, a_2, \ldots]$, and $\alpha' = [a_1 + 1, a_2, \ldots]$. Thus α' and α have the same numerators of convergents; $p_{0,\alpha'} = p_{1,\alpha} = a_1 + 1$, and the lost numerator $p_{0,\alpha} = 1$ appears as $p_{-1,\alpha'}$. The numerators of intermediate fractions between $p_{0,\alpha} = p_{-1,\alpha'} = 1$ and $p_{1,\alpha} = p_{0,\alpha'} = a_1 + 1$ are all the integers in this interval. Thus the sets of p_n and of numerators of intermediate fractions for α and α' are the same.

If $m \in A_\alpha$, then $(k - 1/2)\alpha < m < k\alpha$ for some k; in other words, $k - 1/2 < m/\alpha < k$. Since $1/\alpha + 1/\alpha' = 1$, this gives us $k - 1/2 < m - m/\alpha' < k$. Multiplying through by $-\alpha'$ gives $(-k + 1/2)\alpha' > m\alpha' - m > -k\alpha'$. Since m is an integer, we can write this as $(m - k + 1/2)\alpha' > m > (m - k)\alpha'$, which shows that $m \in B_{\alpha'}$. Thus $A_\alpha = B_{\alpha'}$. ■

3. Characterization of S_α

The characterization of A_α and B_α in terms of E gives a natural characterization of the avoided set S_α in terms of E, which we can then use to characterize S_α in terms of the convergents and intermediate fractions.

THEOREM 1 *If $E(x) > 0$, then $x \in S_\alpha$ if and only if there is no even $z < 2x$ with $0 < E(z) < E(x)$; likewise, if $E(x) < 0$, then $x \in S_\alpha$ if and only if there is no even $z < 2x$ with $0 > E(z) > E(x)$.*

Proof: Take $x \notin S_\alpha$; that is, $x = y_1 + y_2$ for some $y_1 \ne y_2$ with y_1 and y_2 either both in A_α or both in B_α. Assume for simplicity of notation that $E(x) > 0$.

First, suppose $y_1, y_2 \in A_\alpha$, so that $E(y_1)$ and $E(y_2)$ are both negative. Since $E(x) = E(y_1 + y_2) > 0$, we must have $E(y_1 + y_2) = E(y_1) + E(y_2) + \alpha$. It thus follows that $E(y_1) + \alpha/2$ and $E(y_2) + \alpha/2$, which are both positive, must be less than $E(x)$. Let $E(y_1) + \alpha/2$ be the smaller of the two; they are not equal since $y_1 \neq y_2$. We have $E(y_1) < E(x)/2 - \alpha/2 < -\alpha/4$. We can thus take $z = 2y_1$, which gives $E(z) = E(2y_1) = 2E(y_1) + \alpha$, and thus $0 < E(z) < E(y_1) + E(y_2) + \alpha = E(x)$, proving the theorem in this case.

Similarly, suppose $y_1, y_2 \in B_\alpha$, so that $E(y_1)$ and $E(y_2)$ are both positive. Since $E(x) = E(y_1 + y_2) > 0$ is positive, we must have $E(y_1 + y_2) = E(y_1) + E(y_2)$, not $E(y_1 + y_2) = E(y_1) + E(y_2) - \alpha$. Let $E(y_1)$ be the smaller of $E(y_1)$ and $E(y_2)$; they are not equal since $y_1 \neq y_2$. We have $E(y_1) < E(x)/2 < \alpha/4$. We can thus take $z = 2y_1$, which gives $0 < E(z) = 2E(y_1) < E(y_1) + E(y_2) = E(x)$, proving the theorem in this case.

Each of the the two above arguments can be read bottom to top, together showing that if there is an even $z < 2x$ with $0 < E(z) < E(x)$, we can take $y_1 = z/2$ and $y_2 = x - y_1$, getting a sum which shows that $x \neq S_\alpha$. This proves the theorem for $E(x) > 0$; the case $E(x) < 0$ is analogous. ■

Setting $y = y_1$ in the proof of Theorem 1 gives us the following lemma. It is usually easier to use the theorem to show that a particular x does not occur as a sum, and the lemma to show that it does occur. Note that, if x occurs as a sum, then y is one of the two addends.

LEMMA 4 *If $E(x) > 0$, then $x \in S_\alpha$ if and only if there is no $y < x$ with $y \neq x/2$ and either $0 < E(y) < E(x)$ or $0 < E(y) + \alpha/2 < E(x)$; likewise, if $E(x) < 0$, then $x \in S_\alpha$ if and only if there is no $y < x$ with $y \neq x/2$ and either $0 > E(y) > E(x)$ or $0 > E(y) - \alpha/2 > E(x)$.*

The following two theorems, the main theorems of [2], follow immediately from Lemma 4 and Theorem 1.

THEOREM 2 *The numerator p_n of every convergent is in S_α.*

Proof: We apply Theorem 1 to $x = p_n$, looking for z with $0 < E(z) < E(p_n)$ or $0 > E(z) > E(p_n)$. We cannot have $z = p_{n+1}$ because this would give $E(z)$ and $E(p_n)$ opposite signs. Any other z with $|E(z)| < |E(p_n)|$ is at least $p_n + p_{n+1}$ by Lemma 1, which is larger than $2p_n$ and thus too large. ■

THEOREM 3 *If $p \in S_\alpha$, then p is either the numerator p_n of a convergent, twice the numerator of a convergent, or the numerator $p_{n-1} + kp_n$ of an intermediate fraction.*

Proof: If p is not the numerator of a convergent, then $p_n < p < p_{n+1}$ for some n and $|E(p)| > |E(p_n)|$. As in Lemma 1, we assume that n is odd for simplicity of notation (so $E(p_n) > 0$). If $0 < E(p_n) < E(p)$, then we can take $y = p_n$ in Lemma 4 unless $p = 2p_n$. If $E(p) < E(p_{n-1}) < 0$, then we can take $y = p_{n-1}$ in Lemma 4 unless $p = 2p_{n-1}$. Otherwise, we have $E(p_{n-1}) < E(p) < E(p_n)$, which implies $|E(p)| < |E(p_{n-1})|$. By Lemma 1, this implies that p is either the numerator of an intermediate fraction or kp_n for some k. In this case, p must be the numerator of an intermediate fraction; if we had instead $p = kp_n$, we would have $k \leq a_{n+1}$ to have $p < p_{n+1}$, and we know that for all such k, $E(kp_n) = kE(p_n) \geq E(p_n)$ rather than the $E(p) < E(p_n)$ we need. ∎

The cases which are not resolved by these two theorems can be checked similarly.

THEOREM 4 *For p_n the numerator of a convergent, $2p_n$ is in S_α if and only if p_n is odd, and either (i) p_{n+1} is odd and $a_{n+1} \geq 3$, or (ii) p_{n+1} is even and $a_{n+1} \geq 2$, or (iii) $p_n = 1$.*

Proof: We first assume $|E(p_n)| < \alpha/4$, so $E(2p_n) = 2E(p_n)$; we will deal with the special case $|E(p_n)| > \alpha/4$ at the end. In the general case, we will again assume that n is odd (and thus $E(p_n) > 0$) for simplicity of notation.

We will apply Theorem 1 to $x = 2p_n$. If p_n is even, then we can take $z = p_n$, which has $E(z) = E(p_n) < E(2p_n)$ and thus $2p_n \notin S_\alpha$.

If p_n is odd, Theorem 1 says that $x \neq S_\alpha$ if and only if there is an even $z < 4p_n$ with $0 < E(z) < 2E(p_n) = E(2p_n)$. If $E(z) > E(p_n)$ with $z < p_n$, then we apply Lemma 2 to $2p_n - z$, which has $0 < E(2p_n - z) = E(2p_n) - E(z) < E(p_n)$, and use $2p_n - z$ instead of z in this argument; we may thus assume either $E(z) < E(p_n)$ or $z > p_n$. If $E(z) < E(p_n)$, we can apply Lemma 2 to z to get $z = ip_n + jp_{n+1}$; let $z' = z$. If $E(z) > E(p_n)$ with $z > p_n$, then we can apply Lemma 2 to $z' = z - p_n$, which has $0 < E(z - p_n) = E(z) - E(p_n) < E(p_n)$, to show $z' = (i-1)p_n + jp_{n+1}$ for $i \geq 1$. Thus, in any of these cases, we can write $z = ip_n + jp_{n+1}$, for some $i, j \geq 0$, and $z' = i'p_n + jp_{n-1}$ for $i' = i$ or $i' = i - 1$. Since z and z' are both less than $4p_n$, $i + j \leq 3$; we have the additional conditions that z is even and not equal to $2p_n$, and $0 < E(z') < E(p_n)$ while we have only $0 < E(z) < 2E(p_n)$. The existence of z implies the existence of z', but not vice versa; thus we can show $x \in S_\alpha$ by showing either z or z' cannot satisfy these conditions, but must find z to show that $x \notin S_\alpha$.

We can exclude all cases except $i = 1$ or 2, $j = 1$. If $i = 0$, then we also have $i' = 0$, and $z = z' = jp_{n+1}$; we thus need $0 < E(z) < E(p_n)$. Note that $|E(p_{n+1})| < |E(p_n)|$ by the best approximation property. If $|E(p_{n+1})| < \alpha/6$, then $j = 1$ gives $E(z) = E(p_{n+1})$, $j = 2$ gives $E(z) = E(2p_{n+1}) = 2E(p_{n+1})$ and $j = 3$ gives $E(z) = E(3p_{n+1}) = 3E(p_{n+1})$, all of which are negative, but we need $E(z) > 0$. If $\alpha/6 < |E(p_{n+1})| < \alpha/4$, then $j = 1$ and $j = 2$ give $E(z) < 0$ as before, while $j = 3$ gives $E(z) = E(3p_{n+1}) = \alpha + 3E(p_{n+1}) > \alpha - 3\alpha/4 = \alpha/4$, but we need $E(z) < E(p_n) < \alpha/4$. Thus $i = 0$ is impossible.

If $j = 0$ or $j = 2$, then since p_n is odd, i must also be even to give even z; thus $j = 2$ requires $i = 0$ and thus $i' = 0$, and $j = 0$ requires $i = 2$ and $z = 2p_n$, which is forbidden. The case $j = 3$ also requires $i = 0$, which leaves only $j = 1$.

If $j = 1$, the condition $z < 4p_n$ is equivalent to $i < 3 - a_{n+1}$, since $z = p_{n+1} + ip_n = p_{n-1} + a_{n+1}p_n + ip_n$, and $p_{n-1} < p_n$ for $p_n \neq 1$. We also need to check that z is even; it is even if p_{n+1} is odd and $i = 1$, or p_{n+1} is even and $i = 2$. Thus, in case (i) or (ii), we cannot get any even $z < 4p_n$, and thus $x \in S_\alpha$.

If we are not in case (i) or (ii), we do get $0 < E(z) < 2E(p_n)$ and thus can run this argument in reverse to show that $2p_n \notin S_\alpha$. Both $E(p_n)$ and $E(2p_n) = 2E(p_n)$ are positive, and $E(p_{n+1})$ is negative but greater than $-E(p_n)$; we thus have $0 < E(p_n + p_{n+1}) < E(2p_n + p_{n+1}) < 2E(p_n) = E(x)$, as required. Thus $z = p_n + p_{n+1}$ or $z = 2p_n + p_{n+1}$

can be used in Theorem 1, provided that it is even and less than $4p_n$, which is exactly the condition that we are not in case (i) or (ii). This proves the theorem in the case $|E(p_n)| < \alpha/4$.

We now need to check the special cases with $|E(p_n)| > \alpha/4$. Recall that we may assume $\alpha < 2$. Since $|E(p_n)| < 1/q_{n+1}$, there can only be problems if $q_{n+1} \leq 3$, which implies $q_n \leq 2$; thus the only possible p_n/q_n are $1/1$, $2/1$, and $3/2$. Case (iii) covers $p_n = 1$; 2 cannot occur as a sum of any two distinct positive integers. If $p_n/q_n = 2/1$ is a convergent, then $3/2 < \alpha < 2$; the theorem says that the sum of 4 should occur since $p_n = 2$ is even. We have $1 \in A_\alpha$ because $\alpha/2 < 1 < \alpha$, and $3 \in A_\alpha$ because $3\alpha/2 < 3 < 2\alpha$; thus the sum of 4 occurs as $1 + 3$. If $p_n/q_n = 3/2$ is a convergent, then $4/3 < \alpha < 2$ and $q_{n+1} \geq 3$; we thus have $|E(p_n)| < 1/3 < \alpha/4$, so the special case does not occur. ■

THEOREM 5 *The numerator $x = p_n + kp_{n+1}$ of an intermediate fraction is in S_α if and only if either (i) p_{n+1} is even, or (ii) $k = 1$ and p_n is odd, or (iii) $k = a_{n+2} - 1$ (i.e., $x = p_{n+2} - p_{n+1}$) and p_{n+2} is odd.*

Proof: As in the previous proof, we will assume $|E(p_n)| < \alpha/3$, proving the special case $|E(p_n)| > \alpha/3$ at the end, and assume in the general case that n is odd (so $E(p_n) > 0$) for simplicity of notation.

Recall that $p_{n+2} = p_n + a_{n+2}p_{n+1}$, and $0 < E(p_{n+2}) < E(p_n)$. Since x is the numerator of an intermediate fraction with $k < a_{n+2}$, we have $0 < -E(p_{n+1}) < E(x) < E(p_n)$. We can apply Theorem 1 to see whether $x \in S_\alpha$; it is not in S_α if and only if there is an even $z < 2x$ with $0 < E(z) < E(x) = E(p_n) + kE(p_{n+1})$. Since $0 < E(z) < E(p_n)$ and $z < 2x < 2p_{n+2}$, we can apply Lemma 2 to write $z = ip_n + jp_{n+1}$ with $i \leq 2$, and $i = 2$ is possible only if $z > p_{n+2}$. We will exclude $i = 0$ and $i = 2$, and then show that we can find z with $i = 1$ if any only if none of the three conditions (i)-(iii) hold.

For all three values of i, we will use similar properties of $mE(p_{n+1})$ for $0 \leq m \leq 2a_{n+2}$. By the properties of the continued fraction, $E(p_n) + (a_{n+2} + 1)E(p_{n+1}) < 0 < E(p_n) + a_{n+2}E(p_{n+1})$, which gives $-a_{n+2}E(p_{n+1}) < E(p_n)$. Therefore, $mE(p_{n+1})$ is between $-2E(p_n)$ and 0. In addition, $E(p_n) + mE(p_{n+1})$ is between $-E(p_n)$ and $E(p_n)$ (and thus between $-\alpha/2$ and $\alpha/2$), so we have $E(p_n + mp_{n+1}) = E(p_n) + mE(p_{n+1})$ rather than the two sides differing by a multiple of α. In particular, $E(p_n + mp_{n+1})$ is a decreasing function of m for $0 \leq m \leq 2a_{n+2}$, since $E(p_{n+1})$ is negative.

We first exclude the case $i = 0$. Since we need $z < 2x = 2p_n + 2kp_{n+1} < (2k+2)p_{n+1}$, we must have $j \leq 2k + 1$. Since $k < a_{n+2}$, we have $j < 2a_{n+2}$, and thus $j|E(p_{n+1})| < 2E(p_n)$, which in turn is at most $2\alpha/3$ since $E(p_n) < \alpha/3$. If $j|E(p_{n+1})| < \alpha/2$, then we have $E(z) = E(jp_{n+1}) = jE(p_{n+1})$, which is negative, but we need $E(z) > 0$. If $\alpha/2 < j|E(p_{n+1})| < 2\alpha/3$, then $E(z) = E(jp_{n+1}) = \alpha + jE(p_{n+1}) > \alpha/3$, but we need $E(z) < E(x) < E(p_n) < \alpha/3$. Thus $i = 0$ is impossible.

We can also exclude $i = 2$. By Lemma 2, this case is only possible if $z > p_{n+2}$. We have $j \geq a_{n+2}$ since $z > p_{n+2}$, and $j < 2k < 2a_{n+2}$ since $z < 2x < 2p_n + 2kp_{n+1} < 2p_{n+2}$. Since we need $E(z) > 0$ and we have $E(p_{n+2}) > 0$, we can write $E(z) > E(z) - E(p_{n+2}) = E(z - p_{n+2}) = E(p_n + (j - a_{n+2})p_{n+1})$; thus, if we have $E(z) < E(x)$, we also have $E(p_n + (j - a_{n+2})p_{n+1}) < E(x)$. Since $E(p_n + mp_{n+1})$ is a decreasing function of m for $0 \leq m \leq 2a_{n+2}$, we can only have $E(p_n + (j - a_{n+2})p_{n+1}) < E(p_n + kp_{n+1}) =$

$E(x)$ if $j - a_{n+2} > k$. But this violates the condition $j < 2k$, which we needed in order to have $z < 2x$. Thus $i = 2$ is also impossible.

Thus the only possibility is $i = 1$, which gives $z = p_n + jp_{n+1}$. The bound on j given by $z < 2x$ is now $j \leq 2k$ since $p_n + 2kp_{n+1} < 2x = 2p_n + 2kp_{n+1} < p_n + (2k+1)p_{n+1}$. We again use the fact that $E(p_n + mp_{n+1})$ is a decreasing function of m for $0 \leq m \leq 2a_{n+2}$. This gives $E(z) < E(x)$ for $j > k$ (that is, for $z > x$), and $E(z) > 0$ for $j \leq a_{n+2}$. Any j which meets these three conditions gives a z we can use in Theorem 1 provided that z is even.

If p_{n+1} is even, then p_n is odd (since the numerators of consecutive convergents are relatively prime), and thus no choice of j gives even z; this is case (i). If p_{n+1} is odd and $k = 1$, the only choice allowed is $j = 2$; any other j gives $z > 2x$. This $z = p_n + 2p_{n+1}$ is odd if p_n is odd and even if p_n is even; this covers case (ii). If p_{n+1} is odd and $k = a_{n+2} - 1$, the only choice allowed is $j = a_{n+2} = k + 1$; any smaller j gives $E(z) \geq E(x)$ and any larger j gives $E(z) < 0$. This gives $z = p_{n+2}$; this covers case (iii). If p_{n+1} is odd and we have any other k, we can take either $j = k + 1$ or $j = k + 2$; one of these will give z even. Thus we have found a z that we can use in Theorem 1 if case (i), (ii), or (iii) does not occur, and shown that there is no such z if one of these three cases does occur.

We now deal with the special cases with $|E(p_n)| > \alpha/3$, and show that the theorem is still valid as stated in these cases. Recall that we may assume $\alpha < 2$, and that the approximation properties give $|E(p_n)| < 1/q_{n+1}$. Thus $|E(p_n)| > \alpha/3$ requires $q_{n+1} \leq 2$ and thus $q_n = 1$; the possible convergents are $1/1$ and $2/1$. If $p_n/q_n = 1/1$, then $n = 1$, and $|E(p_1)| > \alpha/3$ only for $\alpha > 3/2$. For such α, the next convergent is $2/1$, with $a_2 = 1$ and $p_2 = 2$, so the theorem says that all the numerators of intermediate fractions, which are all odd $x < 2a_3 + 1$, are in S_α by case (i). We have $2 - 1/(a_3 + 1) < \alpha < 2$ since $(2a_3 + 1)/(a_3 + 1)$ is the next convergent. Thus, if $j = 2k \leq 2a_3$ is even, then

$$k\alpha < 2k = j < (2k + 1) - (2k + 1)/(2a_3 + 2) < (k + 1/2)\alpha,$$

and thus $j \in B_\alpha$; if $j = 2k + 1 \leq 2a_3 + 1$ is odd, then

$$(k + 1/2)\alpha < 2k + 1 = j < (2k + 2) - (k + 1)/(a_3 + 1) < (k + 1)\alpha,$$

and thus $j \in A_\alpha$. Thus no odd number less than $2a_3 + 1$ can occur as a sum of two elements from the same set, as required by the theorem.

The other p_n/q_n which could give the special case is $p_n/q_n = 2/1$, but this is only a convergent to α if $\alpha > 3/2$, which implies $|E(2)| < 1/2 < \alpha/3$. Thus the special case does not apply here. ■

Theorems 2, 3, 4, and 5 give the complete characterization of the avoided set S_α.

4. Uniquely avoidable sets

The most natural characterization of unique avoidability is the graph-theoretic characterization of [1]. If $S \subset \mathbb{N}$, then the *graph* $G(S)$ of S is the graph with vertex set $\mathbb{N}$ and an edge between x and y if $x \neq y$ and $x + y \in S$. A partition which avoids S is a 2-coloring of

$G(S)$, and thus S is avoidable if and only if $G(S)$ is bipartite, and S is uniquely avoidable if and only if $G(S)$ is bipartite and connected.

Note that, if $a \in S$ and $x < a$, then x and $a - x$ are in the same connected component of $G(S)$; either $x = a - x$, or $x \neq a - x$ and their sum is a. Thus, for every $x < a$, if a and $a + b \in S$, then x and $x + b$ are in the same connected component as $x - a$, and thus in the same component. This allows us to prove the following result.

THEOREM 6 *If S contains a, b, and $a + b$ with $(a, b) = 1$ (or, more generally, if there are some $a' \geq a$ and $b' \geq b$ for which S contains a', $a' + b$, b', and $a + b'$), then all numbers less than $a + b$ are in the same connected component of $G(S)$. If S contains infinitely many such subsets, then $G(S)$ is connected, and therefore S is uniquely avoidable if it is avoidable at all.*

Proof: By the above argument, n and $n - a$ are in the same connected component for $a < n \leq a + b$ (and even for $n \leq a + b'$ in the more general case), and n and $n - b$ are in the same connected component if $b < n \leq a + b$ (or $n \leq a' + b$).

Assume $a < b$. By the Chinese Remainder Theorem, any integer m can be uniquely written $m = xa - yb$ for $0 \leq y \leq a - 1$; if $0 < m < a + b$, then $1 \leq x \leq b$. We prove connectivity for $0 < m < a + b$ by induction on $x + y$. The base case is $m = a$, in which $x + y = 1$. For any other m, if $m > a$, then m and $m - a$ are in the same component; if $m < a$, then m and $m + b$ are in the same component. Thus by induction, every m is in the same component as a.

For example, if $a = 5$, $b = 8$, and $m = 1 = 5a - 3b$, we have that 1 is in the same connected component as $1 + 8 = 9$, $9 - 5 = 4$, $4 + 8 = 12$, $12 - 5 = 7$, $7 - 5 = 2$, $2 + 8 = 10$, and finally $10 - 5 = 5$. ■

Our main results on unique avoidability follow.

THEOREM 7 *The set of numerators of convergents of α is uniquely avoidable if and only if infinitely many partial quotients a_n are 1.*

Proof: If infinitely many a_n are 1, then we can apply Theorem 6 to p_{n-2}, p_{n-1}, and $p_n = p_{n-2} + p_{n-1}$ to show that the graph is connected, and we have already shown that the set of numerators is avoidable.

If only finitely many a_n are 1, then there is some N such that for any $n > N$ we have $p_n > 2p_{n-1}$. Thus, regardless of the partition of integers less than p_{n-1}, we can extend the partition up to integers less than p_n by placing integers from p_{n-1} to $\lfloor p_n/2 \rfloor$ arbitrarily, and placing x in the set which does not contain $p_n - x$ for $\lfloor p_n/2 \rfloor < x < p_n$. We will never reach a contradiction because $x > p_{n-1}$ and we thus have only one constraint in placing x. We can continue inductively for all n. ■

THEOREM 8 *The set S_α avoided by the partition A_α, B_α is uniquely avoidable for any irrational $\alpha > 1$.*

Proof: If p_{n+1} is even, then $p_n + p_{n+1}$ is either p_{n+2} or the numerator of an intermediate fraction, and if it is the numerator of an intermediate fraction, it is in S_α by case (i) of Theorem 5. If p_{n+1} is odd, then $p_{n+1} - p_n$ is either p_{n-1} or the numerator $p_{n-1} +$

$(a_{n+1}-1)p_n$ of an intermediate fraction, and if it is the numerator of an intermediate fraction, it is in S_α by case (iii) of Theorem 5. In either case, we can apply Theorem 6 to $p_n, p_{n+1}, p_n+p_{n+1}$ or $p_n, p_{n+1}-p_n, p_{n+1}$ for all n to show that S_α is uniquely avoidable. ■

The unique avoidability of generalized Fibonacci sequences starting with arbitrary relatively prime s_1, s_2 [3] follows as a special case of these results. If $s_1 < s_2$, we can let s_1 and s_2 be the numerators p_{n-1} and p_n of two consecutive convergents; we can do this by backward induction, letting $a_k = \lfloor p_k/p_{k-1} \rfloor$, $p_{k-2} = p_k - a_k p_{k-1}$ until we reach $p_0 = 1$ [2]. Let $a_m = 1$ for all $m > n$, so that $s_i = p_{i+n-2}$, and apply Theorem 7. If $s_1 > s_2$, we can let $p_n = s_2$, $p_{n+1} = s_1 + s_2$; the set of all the s_i other than s_1 is thus uniquely avoidable, and s_1 is the intermediate fraction $p_{n+1} - p_n$, which is an avoided sum by case (i) of Theorem 5 if $s_2 = p_n$ is even and by case (iii) of Theorem 5 if $s_1 + s_2 = p_{n+1}$ is odd. Thus the sequence of s_i is uniquely avoidable if $s_1 < s_2$ or either s_1 or s_2 is even, and not avoidable at all otherwise.

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