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THE GRAPH OF AN ABSTRACT POLYTOPE*

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Recently a generalization of simple convex polytopes to combinatorial entities known as abstract polytopes has been proposed. The graph of an abstract polytope of dimension d is a regular connected graph of degree d. Given a connected regular graph Γ of degree d, it is interesting to find out whether it is the graph of some abstract polytope P. We obtain necessary and sufficient conditions for this, in terms of the existence of a class of simple cycles in Γ satisfying certain properties. The main result in this paper is that if a pair of simple convex polytopes or abstract polytopes have the same two-dimensional skeleton, then they are isomorphic. Every two-dimensional face of a simple convex polytope or an abstract polytope is a simple cycle in its graph. Given the graph of a simple convex polytope or an abstract polytope and the simple cycles in this graph corresponding to all its two-dimensional faces, then we show how to construct all its remaining faces. Given a regular connected graph Γ and a class of simple cyles $\mathcal{D}$ in it, we provide necessary and sufficient conditions under which $\mathcal{D}$ is the class of two-dimensional faces of some abstract polytope which has Γ as its graph.

1. Introduction

1.1. Let K be the set of all feasible solutions of

$$Ax = b, \qquad x \geqq 0, \tag{1}$$

where A is a matrix of order $m \times n$ and rank m. Suppose K is nonempty and bounded. Also assume that the column vector b is *nondegenerate in* (1), i.e., that it does not lie in any subspace generated by $m-1$ or less column vectors of A. Under these assumptions, K is a *simple convex polytope* of dimension $d=n-m$ and $\tilde{x} \in K$ is an extreme point iff exactly m of the $\tilde{x}_j$ are positive and the remaining $n-m$ are zero.

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Let $S = \{1, \ldots, n\}$. Corresponding to each extreme point $\tilde{x}$ of K, define

$$V(\tilde{x}) = \{j\colon\ j \text{ such that } \tilde{x}_j = 0\}.$$

Each $V(\tilde{x})$ is a subset of S of cardinality $d=n-m$ and will be called a *vertex*. Let C be the class of all the vertices. Let $P(K)$ denote the pair (S,C). Clearly $S^* = \cup_{V \in C} V$ may not be all of S.

In analogy with the definition of adjacency of extreme points on K, we define that two vertices V_1 and V_2 in $P(K)$ are *adjacent* iff $V_1 \cap V_2$ has cardinality equal to the cardinality of any vertex minus one. From the theory of linear programming, we know that $P(K)$ satisfies the following:

1.1.1. The class C of vertices is nonempty.

1.1.2. Every vertex is a subset of S of the same cardinality.

1.1.3. If E is a subset of S of cardinality equal to the cardinality of a vertex minus one, then the number of vertices which contain E as a subset is either 0 or 2.

1.1.4. If V_0 and V_* are a pair of vertices, then there exists a sequence of vertices $V_0, V_1, V_2, \ldots, V_t, V_*$, known as an *adjacency path* or *edge path* between V_0 and V_*, such that every pair of consecutive vertices in the sequence are adjacent and every vertex in the sequence contains $V_0 \cap V_*$ as a subset.

1.2. The pair $P(K)=(S,C)$ is known as an *abstract polytope* corresponding to the simple convex polytope K. $P(K)$ captures the essential adjacency structure between the extreme points of K.

In general, any pair (S,C) satisfying 1.1.1–1.1.4 defines an *abstract polytope*. In analogy to simple convex polytopes, we define the *dimension* of the abstract polytope (S,C) to be the cardinality of any vertex in C.

1.3. *Faces*. A nonempty face of the convex polytope K is a nonempty set of feasible solutions of (1) after setting some of the variables x_j to zero. Analogously, let $P=(S,C)$ be an abstract polytope and $F \subset S$. Let $C_F = \{V \setminus F\colon V \in C,\ V \supseteq F\}$, where $\setminus$ indicates set-theoretic difference. If C_F is nonempty, it is clear that (S,C_F) is itself an abstract polytope, and we call it the nonempty *face* of P generated by the subset F of S. The dimension of this face is equal to the dimension of P minus the cardinality of F. Two faces (S,C_{F_1}) and (S,C_{F_2}) are distinct if $F_1 \neq F_2$.

Clearly a nonempty face of dimension 0 of an abstract polytope P=(S,C) is generated by a subset of S which is itself a vertex, and hence it contains only that vertex.

A nonempty face of dimension 1 is generated by a subset of S of cardinality equal to the dimension of P minus one, and it contains exactly two adjacent vertices of P. It will be called an *edge* of P.

1.4. *Graph*. The graph of an abstract polytope P=(S,C) is obtained by representing each vertex in C by a vertex of the graph and joining a pair of vertices by an edge of the graph iff these vertices are adjacent on P. The graph is just the *one-dimensional skeleton* of P, and is a connected regular graph of degree equal to the dimension of P.

1.5. The axioms defining abstract polytopes were first outlined by Dantzig and discussed in [3]. References [1–3, 6] contain several results on abstract polytopes analogous to known results on convex polytopes.

1.6. Suppose that Γ is a given connected regular graph of degree d. If we find a finite set of symbols $S=\{1,2,\ldots,n\}$ and a *labelling* of the vertices of Γ by subsets of S of cardinality d such that when C is the class of labels on the vertices of Γ, (S,C) is an abstract polytope and two vertices of Γ are connected by an edge of Γ iff the labels on these vertices represent a pair of adjacent vertices of (S,C), then we say that the vertices of Γ have been labelled to be the vertices of an abstract polytope (S,C). We obtain necessary and sufficient conditions for the existence of such a labelling.

We also obtain the result that two abstract polytopes which have the same two-dimensional skeleton are isomorphic. Since the class of abstract polytopes includes all simple convex polytopes, this result also holds for simple convex polytopes. For other related results on the k-skeletons of convex polytopes, see [5, ch. 12].

Finally, the connections between abstract polytopes and simplicial complexes in combinatorial topology are easy to see. In the $P(K)$=(S,C) derived from the simple convex polytope K, the vertices in C are the d–1 simplices of a simplicial complex which is isomorphic to the boundary complex of the simplicial polytope dual to K. See [4, chs. 4,5].

2. The results

2.1. *Lemma. The graph of an abstract polytope P of dimension 2 is a simple cycle.*

Proof. By definition, the graph of P is a connected graph of degree 2 and is therefore a simple cycle.

2.2. *Lemma. Let Γ be the graph of an abstract polytope $P=(S,C)$ of dimension $d \geqq 3$. Let $\mathcal{D} = \{c_1, \ldots, c_T\}$ be the class of all the simple cycles in Γ determined by the two-dimensional faces of P. Then*:

(i) *If a pair of simple cycles in $\mathcal{D}$ contain two common vertices, then they must be adjacent and these cycles have no other vertex in common. Thus, a pair of simple cycles in $\mathcal{D}$ contain either exactly one common edge or no common edges at all.*

(ii) *Each edge of Γ occurs in exactly* $d-1$ *simple cycles in $\mathcal{D}$; or, equivalently, for every pair of edges of Γ with a common vertex, there is a unique simple cycle in $\mathcal{D}$ containing both of them.*

Proof. (i) Let c_1, c_2 be the simple cycles in $\mathcal{D}$ which are the two-dimensional faces generated by $F_1 \subset S$, and $F_2 \subset S$, respectively, $F_1 \neq F_2$. If V_0, V_1 are two distinct vertices which are on both c_1 and c_2, then both V_0, V_1 must contain both F_1, F_2 as subsets. Since both F_1 and F_2 have cardinality $d-2$, this implies that $F_1 \cup F_2 = V_0 \cap V_1$ has cardinality $d-1$ and hence V_0, V_1 must be adjacent. Since every common vertex of c_1 and c_2 must contain $F_1 \cup F_2$ as a subset, 1.1.3, V_0 and V_1 are the only two common vertices of c_1 and c_2. This also implies that any pair of simple cycles in $\mathcal{D}$ can have at most one common edge.

(ii) Let e_1 be an edge of Γ which is the one-dimensional face of P generated by F_3. Both the vertices on e_1 contain F_3 as a subset. Hence every subset of F_3 of cardinality $d-2$ generates a two-dimensional face of P. Obviously there are $d-1$ of these two-dimensional faces, and by (i) these are the only two-dimensional faces containing e_1.

Γ is a regular graph of degree d. Let V_1 be a vertex on e_1, and let $e_2, \ldots, e_d$ be the other edges incident at V_1. By the above arguments, e_1 is contained among $d-1$ simple cycles in the class $\mathcal{D}$ and the edge e_1 is the only common edge between any pair of these simple cycles. This implies that there is a unique simple cycle in $\mathcal{D}$ passing through each pair of the edges e_1, e_j for $2 \leqq j \leqq d$.

2.3. *Lemma. Let* Γ *be the graph of an abstract polytope* $P=(S,C)$ *of dimension d. Let* $e_1, \ldots, e_r$ *be the edges of* Γ *incident at a vertex of* Γ. *Then there is a unique r-dimensional face of* Γ *containing all the edges* $e_1, \ldots, e_r$.

Proof. Let $V_0 = \{1, \ldots, d\}$ be the common vertex on $e_1, \ldots, e_r$. Each subset of V_0 of cardinality $d-1$ generates an edge containing V_0. Let e_i be the edge generated by the subset $\{1, \ldots, i-1, i+1, \ldots, d\}$, for $i=1, \ldots, r$. Then clearly the r-dimensional face of P generated by the subset $\{r+1, \ldots, d\}$ is the unique r-dimensional face of P which contains all the edges $e_1, \ldots, e_r$.

2.4. We will now explore the properties of the class of simple cycles which are the two-dimensional faces of an abstract polytope. In particular, we will develop a procedure for generating the faces of all dimensions of an abstract polytope using only the class of its two-dimensional faces.

2.5. *Theorem. Let* Γ *be the graph of an abstract polytope P of dimension r. Let* $\mathcal{D}$ *be the class of simple cycles in* Γ, *which are the two-dimensional faces of P. Let* V_0 *be a vertex with edges* $e_1, \ldots, e_r$ *incident at it. If a subclass* γ *of* $\mathcal{D}$ *is obtained by the rules*

(a) *for every pair of edges* e_i, e_j, $1 \leqq i < j \leqq r$ *the unique simple cycle in* $\mathcal{D}$ *which contains both* e_i, e_j *is in* γ,

(b) *any simple cycle in* $\mathcal{D}$ *which contains two edges from the edges on the simple cycles which are already in* γ *is itself in* γ,

then $\gamma = \mathcal{D}$.

Proof. Let V_1 be the other vertex at the end of edge e_1 and suppose $e_{r+1}, \ldots, e_{2r-1}$ are the other edges incident at V_1 (see fig. 1).

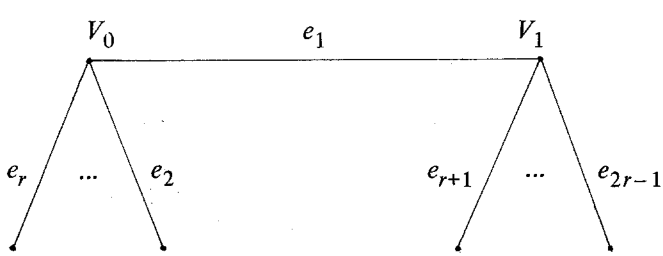

Fig. 1.

By Lemma 2.2, the simple cycles in $\mathcal{D}$ through the edges e_1, e_i for i=2, . . . ,r must each contain one distinct edge among $e_{r+1}, \ldots, e_{2r-1}$. Each one of these simple cycles is in γ. Hence the fact that γ contains all the simple cycles in $\mathcal{D}$ which pass through the vertex V_0 implies by (b) that γ contains all the simple cycles in $\mathcal{D}$ passing through each adjacent vertex of V_0. Since Γ is connected, this argument applied repeatedly yields that $\gamma = \mathcal{D}$.

2.6. *Theorem. Let Γ be the graph of an abstract polytope $P=(S,C)$ of dimension d. Let $\mathcal{D} = \{c_1, \ldots, c_T\}$ be the class of simple cycles in Γ which are all the two-dimensional faces of P. Let $e_1, \ldots, e_r$ be edges of Γ with a common vertex V_0. Let $\mathcal{D}\ \{e_1, \ldots, e_r\}$ be the subclass of $\mathcal{D}$ obtained by the rules*

(a) *for every pair of edges e_i, e_j, $1 \leqq i < j \leqq r$, the unique simple cycle in $\mathcal{D}$ which contains both e_i, e_j is in $\mathcal{D}\{e_1, \ldots, e_r\}$,*

(b) *any simple cycle in $\mathcal{D}$ which contains two edges from the edges on the simple cycles which are already in $\mathcal{D}\ \{e_1, \ldots, e_r\}$ is itself in $\mathcal{D}\{e_1, \ldots, e_r\}$,*

then $\mathcal{D}\ \{e_1, \ldots, e_r\}$ is the class of all the two-dimensional faces of the unique r-dimensional face of P containing $e_1, \ldots, e_r$.

Proof. The fact that $\mathcal{D}\{e_1, \ldots, e_r\}$ constructed in the above manner contains all the simple cycles corresponding to the two-dimensional faces of the unique r-dimensional face of P containing $e_1, \ldots, e_r$ follows from Lemma 2.3 and Theorem 2.5.

Let $V_0 = \{1,2, \ldots, d\}$. For $1 \leqq i \leqq r$, let e_i be the edge of P generated by the subset $\{1, \ldots, i-1, i+1, \ldots, d\}$ of S. Then the r-dimensional face of P generated by the subset $\{r+1, \ldots, d\}$ is the unique r-dimensional face of P which contains all the edges $e_1, \ldots, e_r$.

If F_1 and F_2 are two subsets of S which generate edges of P, then from 1.3 it is clear that both these edges lie on a two-dimensional face of P iff $F_1 \cap F_2$ has cardinality $d-2$, and in this case all the vertices on this two-dimensional face contain $F_1 \cap F_2$ as a subset and all the edges on this two-dimensional face are generated by subsets of S wich contain $F_1 \cap F_2$ as their subset.

Hence all the edges on the two-dimensional faces which are included in $\mathcal{D}\ \{e_1, \ldots, e_r\}$ in step (a) are generated by subsets of S which contain $\{r+1, \ldots, d\}$ as a subset. Applying this argument repeatedly, from (b) it is clear that every edge contained on the simple cycles included in $\mathcal{D}\ \{e_1, \ldots, e_r\}$ is generated by some subset of S which contains

$\{r+1, \ldots, d\}$ as its subset. Hence none of the simple cycles in $\mathcal{D}\{e_1, \ldots, e_r\}$ contain an edge which does not lie in the r-dimensional face of p generated by $\{r+1, \ldots, d\}$.

These things imply that $\mathcal{D}$ $\{e_1, \ldots, e_r\}$ is the set of all simple cycles corresponding to the two-dimensional faces of the unique r-dimensional face of P containing $e_1, \ldots, e_r$.

The partial subgraph of Γ consisting of the vertices and edges appearing among the simple cycles in $\mathcal{D}$ $\{e_1, \ldots, e_r\}$ is the graph of the r-dimensional face of P containing $e_1, \ldots, e_r$.

2.7. *Procedure.* The two-dimensional skeleton of an abstract polytope consists of its graph Γ and the class $\mathcal{D}$ of simple cycles in Γ which are all the two-dimensional faces of P. Theorem 2.6 provides an easy procedure for generating the faces of all dimensions of P given only its two-dimensional skeleton.

Pick a set of r edges of Γ, say $e_1, \ldots, e_r$, containing a common vertex. Then $\mathcal{D}\{e_1, \ldots, e_r\}$ obtained by using (a) and (b) of Theorem 2.6 determines an r-dimensional face of P. By picking all possible subsets of r edges containing a common vertex, all r-dimensional faces of P can be generated in this manner. However, in this list, each r-dimensional face of P is likely to appear several times. Duplication can be avoided by picking each time a set of r edges with a common vertex, so that all these r edges together did not appear in any r-dimensional face generated so far.

By varying r from 3 to the dimension of P, all the faces of P can be generated.

2.8. *Theorem. A pair of abstract polytopes which have the same 2-dimensional skeleton are isomorphic.*

Proof. By Theorem 2.6, and 2.7, it is clear that if two abstract polytopes have the same two-dimensional skeleton, then they have the same facial structure. Hence they are isomorphic.

2.9. *Corollary. For the class of simple convex polytopes, the two-dimensional skeleton is unambiguous.*

Proof. Since each simple convex polytope is an abstract polytope, this result follows from Theorem 2.8. See [5, ch 12] for other related results on k-dimensional skeletons of convex polytopes.

2.10. We will now use these results to obtain necessary and sufficient conditions for a regular connected graph to be the graph of an abstract polytope. These turn out to be very simple when the degree of the graph is 3 and we discuss this case now.

2.11. *Theorem. A regular connected graph* Γ *of degree* 3 *is the graph of an abstract polytope iff there exists in* Γ *a class* $\mathcal{D} = \{c_1, \ldots, c_T\}$ *of simple cycles satisfying properties* (i), (ii) *of Lemma* 2.2 *with d equal to* 3.

Proof. The necessity part of this theorem follows from Lemma 2.2. To prove the sufficiency, suppose the class $\mathcal{D} = \{c_1, \ldots, c_T\}$ of simple cycles in Γ satisfying (i), (ii) of Theorem 2.2 with d equal to 3 is given. Associate the symbol j with the simple cycle c_j in $\mathcal{D}$, and let $S = \{1, \ldots, T\}$. By (ii), each vertex in Γ lies in exactly 3 simple cycles in $\mathcal{D}$. Let the label on each vertex be the subset of 3 symbols associated with the three simple cycles in $\mathcal{D}$ containing that vertex. Let C be the class of all the labels on the vertices of Γ. We will now prove that (S,C) is an abstract polytope of dimension 3 and that Γ is its graph. Obviously the pair (S,C) satisfies the axioms 1.1.1 and 1.1.2.

Let E be any subset of S of cardinality 2, say $E = \{1,2\}$, and suppose there is a vertex whose label contains E as a subset, say $V_0 = \{1,2,3\}$. Let e_1, e_2, e_3 be the edges incident at V_0. Suppose the symbols 1,2,3 are associated with the simple cycles c_1, c_2, c_3, containing the pairs of edges (e_1, e_2), (e_2, e_3), (e_3, e_1), respectively. Then e_2 is common to both the simple cycles c_1 and c_2 and by (i) the two vertices on e_2 are the only two vertices whose labels contain E as a subset. Hence axiom 1.1.3 is satisfied and adjacent vertices in Γ have labels which have two symbols in common.

Let V_0 and V_* be the labels on two vertices. If $V_0 \cap V_*$ has cardinality 2, then by the above argument these vertices are adjacent on Γ. If $V_0 \cap V_*$ has cardinality 1, then both these vertices lie on a simple cycle in $\mathcal{D}$, and this simple cycle contains a path between these two vertices along the edges of Γ such that the label on every vertex along the path contains $V_0 \cap V_*$ as a subset.

If $V_0 \cap V_*$ is empty, since Γ is connected, there is a path between these two vertices along the edges of Γ. Thus axiom 1.1.4 is also satisfied.

Hence the pair (S,C) is an abstract polytope of dimension 3 and Γ is its graph.

2.12. When Γ is a regular connected graph of degree $d > 3$, Adler [2] has constructed an example to show that the mere fact of the existence of a class of simple cycles in Γ satisfying (i), (ii) of Lemma 2.2 is not sufficient to conclude that Γ is the graph of an abstract polytope. So when $d > 3$ we have to impose some more conditions. For this we introduce the following notation:

Let Γ be a regular connected graph of degree $d > 3$. Let $\mathcal{D}$ be a class of simple cycles in Γ satisfying (i) and (ii) of Lemma 2.2. For any r such that $2 \leqq r \leqq d-1$, let $e_1, \ldots, e_r$ be a set of r edges of Γ with a common vertex. Define the subclass $\mathcal{D}\{e_1, \ldots, e_r\}$ of $\mathcal{D}$ by (a), (b) of Theorem 2.6. There is a subset $\mathcal{D}\{e_1, \ldots, e_r\}$ of $\mathcal{D}$ for every set of distinct edges $\{e_1, \ldots, e_r\}$ of Γ with a common vertex. Make a list of all these subsets. In this list each subset $\mathcal{D}\{e_1, \ldots, e_r\}$ is likely to appear several times . Let F_r denote the family of all the distinct subsets of the form $\mathcal{D}\{e_1, \ldots, e_r\}$.

By this procedure we therefore get the families $F_2, F_3, \ldots, F_{d-1}$ of subsets of $\mathcal{D}$. Obviously each subset in the family F_2 consists of just one simple cycle in $\mathcal{D}$.

2.13. *Theorem. If Γ is a connected regular graph of degree $d > 3$, it is the graph of an abstract polytope iff there exists in Γ a class $\mathcal{D} = \{c_1, \ldots, c_T\}$ of simple cycles satisfying the following properties*:

$\mathcal{D}$ satisfies properties (i) *and* (ii) *of Lemma* 2.2.

The families $F_2, \ldots, F_{d-1}$ of subsets of $\mathcal{D}$ generated as in 2.12 *satisfy the following properties:*

(iii) *The partial subgraph of Γ consisting of all the vertices and edges appearing among the simple cycles in any subset of $\mathcal{D}$ belonging to the family F_{d-1} is a connected regular graph of degree $d-1$.*

In addition to this, the conditions listed below are satisfied with respect to each vertex in Γ. Let V be any vertex in Γ with edges $e_1, \ldots, e_d$ incident at it. Then:

(iv) *V is the only vertex which appears among the simple cycle in each of the subsets $\mathcal{D}\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_d\}$ for $i=1, \ldots, d$.*

(v) *e_j is the only edge, and the two vertices on e_j are the only two vertices which appear among the simple cycles in each of the subsets $\mathcal{D}\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_d\}$ for all $i=1, \ldots, d$, $i \neq j$.*

(vi) *Let $\{e_{j_1}, \ldots, e_{j_r}\}$ be any subset of the edges $e_1, \ldots, e_d$ with cardinality r between 2 and $d-2$. Then*

$$\bigcap_{\substack{1 \leq i \leq d \\ i \notin \{j_1, \ldots, j_r\}}} \mathcal{D}\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_d\} = \mathcal{D}\{e_{j_1}, \ldots, e_{j_r}\}.$$

Proof. To prove the necessity: If Γ is the graph of an abstract polytope, let $\mathcal{D}$ be the set of all its two-dimensional faces, each of which is a simple cycle in Γ. If $e_1, \ldots, e_r$ are edges of Γ with a common vertex, then $\mathcal{D}\{e_1, \ldots, e_r\}$ is the set of two-dimensional faces of the r-dimensional face of Γ containing $e_1, \ldots, e_r$ by Theorem 2.6. Hence by Lemmas 2.2 and 2.3, and Theorem 2.6, all the conditions (i) to (vi) are satisfied.

To prove the sufficiency: Let $\mathcal{D}$ be a class of simple cycles in Γ, and $F_2, \ldots, F_{d-1}$ the families of subsets of $\mathcal{D}$ generated as in 2.12, which satisfy all the conditions (i) to (vi). For convenience in referring to them, let $f^1, f^2, \ldots, f^n$ be all the distinct subsets in the family F_{d-1}. Associate the symbol j with the subset f^j in F_{d-1}, for $j=1, \ldots, n$. Let $S=\{1, \ldots, n\}$. Label each vertex of Γ by a subset of S by the rule: j is contained in the label on a vertex iff that vertex appears among the simple cycles in f^j. Let C be the class of all the labels on the vertices of Γ. We will now prove that (S,C) is an abstract polytope and that Γ is its graph.

Axiom 1.1.1 is obviously satisfied by the pair (S,C). Let V be any vertex in Γ with edges $e_1, \ldots, e_d$ incident at it. Then by (iii) each of these subsets, $\mathcal{D}\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_d\}$ for $i=1, \ldots, d$, is distinct and these are the only subsets in the family F_{d-1} which contain a simple cycle through V. Hence the label on V consists of the symbols associated with these subsets, and hence has cardinality d. Also by (iv), V is the only vertex with this label. Hence axiom 1.1.2 is satisfied by the pair (S,C).

Suppose i is the index associated with the subset $\mathcal{D}\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_d\}$ for $i=1, \ldots, d$. Then the label on V is $\{1, \ldots, d\}$. Let E be any subset of V of cardinality $d-1$, say $E=\{1, \ldots, j-1, j+1, \ldots, d\}$. By (v), the two vertices on edge e_j are the only vertices of Γ whose labels contain E as a subset. Hence axiom 1.1.3 is satisfied by the pair (S,C) and a pair of vertices are adjacent on Γ if the labels on them contain $d-1$ symbols in common.

Let V^* be another vertex in Γ such that the intersection of the labels on V and V^* is $\{1, \ldots, r\}$ of cardinality r between 1 and $d-2$. Then V^* is also contained among the simple cycles in the subset $\mathcal{D}\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_d\}$ for $1 \leqq i \leqq r$. From (vi) this implies that both V and V^* are contained among the simple cycles in $\mathcal{D}\{e_{r+1}, \ldots, e_d\}$, and the label on every vertex appearing among the simple cycles in $\mathcal{D}$ $\{e_{r+1}, \ldots, e_d\}$ contains $\{1, \ldots, r\}$ as a subset. By the definition of the subset $\mathcal{D}$ $\{e_{r+1}, \ldots, e_d\}$, the partial subgraph consisting of the vertices and edges appearing among the simple cycles in it is connected. This implies that there exists an edge path between V and V^* such that the label on every vertex along the path contains $\{1, \ldots, r\}$ as a subset. This, and the fact that Γ itself is connected, imply that the pair (S,C) satisfies axiom 1.1.4 also.

Hence (S,C) is an abstract polytope of dimension d with Γ as its graph. Condition (vi) implies that $\mathcal{D}$ is the class of the two-dimensional faces of this abstract polytope and that each subset in the family F_r is the class of two-dimensional faces of some r-dimensional face of this abstract polytope.

2.14. *Corollary. Given a regular connected graph* Γ, *and a class of simple cycles* $\mathcal{D}$ *in it, the necessary and sufficient conditions under which* $\mathcal{D}$ *is the class of two-dimensional faces of an abstract polytope are* (i) *to* (vi) *of Theorem* 2.13.

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