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Gilberto Bini · Claudio Fontanari

Calculating cohomology groups of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ ⋆

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Abstract. Here we investigate the rational cohomology of the moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ of degree d stable maps from n-pointed rational curves to $\mathbb{P}^r$. We obtain partial results for small values of d with an inductive method inspired by a paper of Enrico Arbarello and Maurizio Cornalba.

1. Introduction

Let $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ be the moduli space of degree d pointed stable maps to the projective space $\mathbb{P}^r$. As proved in [6], this moduli space is a projective variety of complex dimension $(d+1)(r+1)+n-4$, with finite quotient singularities. From a topological point of view, $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ can be alternatively viewed as a smooth orbifold. This space has been intensively studied in the past few years for its various applications to enumerative geometry and quantum cohomology (see, for instance, [1]). However, a systematic study of the geometric properties of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ – specifically of its rational cohomology – has been only partially accomplished (see [3], [4], and the references cited therein). In particular, in [4] we prove that when $r = 1$ all odd cohomology groups vanish; additionally, we give generators and relations for the second cohomology group. Hereafter, instead, we deal with the case $r \geq 2$. We are able to prove some partial results similar to the $r = 1$ case for small values of r and d.

Our methods rely on an inductive strategy inspired by [2], where the vanishing of some cohomology groups of the moduli space of n-pointed genus g stable curves is carried through in a very simple way. In fact, if one can prove that the cohomology with compact support of the moduli space of smooth curves $\mathcal{M}_{g,n}$ vanishes in low degree, then the long exact sequence of cohomology with compact support and a bit of Hodge theory imply that the cohomology of the compactification of $\mathcal{M}_{g,n}$ injects

G. Bini: University of Michigan, Department of Mathematics, East Hall, 525 East University, Ann Arbor, MI 48109-1109, USA, e-mail: gbini@math.lsa.umich.edu

C. Fontanari: Dipartimento di Matematica, Università degli Studi di Trento, Via Sommarive 14, 38050 Povo, Trento, Italy, e-mail: fontanar@science.unitn.it (current address)

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into the direct sum of the corresponding cohomology of the boundary components. The explicit description of such components allows one to apply induction.

These very same arguments can be applied to moduli spaces of stable maps too. However, a different approach is required for a couple of problems, namely:

1) the vanishing of the cohomology with compact support of $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$ in sufficiently low degree, which for $\mathcal{M}_{g,n}$ follows from a cellular decomposition by means of Strebel differentials (cf. [8]);
2) the study of the cohomology of the boundary components, which for moduli spaces of stable curves follows directly from the Künneth formula.

As for problem 1), in [4] we prove that $\mathcal{M}_{0,n}(\mathbb{P}^1, d)$ is almost always affine. Indeed, a map from $\mathbb{P}^1$ to $\mathbb{P}^1$ is given by a pair of homogeneous polynomials without common roots. Consequently, such maps are parametrized by the complement of a resultant hypersurface in a projective space. However, for maps to $\mathbb{P}^r$, $r \geq 2$, it is no longer true that $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$ is affine. More precisely, maps of degree d from $\mathbb{P}^1$ to $\mathbb{P}^r$ are parametrized by the complement in a suitable projective space of a resultant variety R which corresponds to $(r+1)$-tuples of degree d polynomials having a common zero. By the Elimination theory (see [10], Proposition 3, and [13], I, § 6), R can be explicitly described in terms of algebraic equations in the coefficients of the $r+1$ polynomials, so exhibiting its complement as a union of a bounded number of affine open subsets. Unfortunately, this approach yields a number of equations which is too high in order to provide any information on the cohomology of $\mathcal{M}_{0,n}(\mathbb{P}^r, d)$. To circumvent this problem, we introduce a natural open dense subset $\mathcal{M}^*_{0,n}(\mathbb{P}^r, d) \subset \mathcal{M}_{0,n}(\mathbb{P}^r, d)$, which turns out to be a union of a lower number of affine open subsets (see Proposition 3). A little variant of the inductive argument outlined above (see Lemma 4) yields the needed vanishing result.

As for problem 2), it is well known that the boundary components of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ are fibered products (and not simply products as in the case of stable curves) of moduli spaces of stable maps with either a lower number of marked points or lower degree. Although the Künneth formula can not be applied here, an elementary spectral sequence argument makes induction still work.

As a consequence, we obtain the vanishing of $H^1(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 2))$ and of $H^3(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 2))$, and a complete description in terms of generators and relations of $H^2(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 2))$ for small values of r (see Proposition 5 and Corollary 6). Moreover, we are able to prove that all odd cohomology of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ vanishes whenever $d \leq 1$ (see Theorem 1), $d = 2$ and either $r = 2$ or $r = 3$ and n is odd (see Theorem 7 and Proposition 8). We conjecture that the vanishing of the odd cohomology of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ may hold for all r's and d's, but such a result is probably out of reach with the elementary methods implemented in the present paper. In the future, we hope to carry out further investigation on this topic by applying different techniques.

Throughout, we work over the field $\mathbb{C}$ of complex numbers; all cohomology groups are intended to have rational coefficients.

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2. Stable maps of degree 0 and 1

In this section we briefly discuss all the cohomology groups of stable maps of degree $d \leq 1$.

For every $n \geq 3$ and $r \geq 1$, $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 0)$ is isomorphic to $\overline{\mathcal{M}}_{0,n} \times \mathbb{P}^r$, where $\overline{\mathcal{M}}_{0,n}$ is the moduli space of n-pointed rational stable curves. Hence, by the Künneth formula,

$$H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 0)) = \bigoplus_{p+q=k} H^p(\overline{\mathcal{M}}_{0,n}) \otimes H^q(\mathbb{P}^r).$$

Since $H^*(\overline{\mathcal{M}}_{0,n})$ was computed in [11], we have a complete description of the rational cohomology of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 0)$. In particular, from the vanishing of the odd cohomology of both $\overline{\mathcal{M}}_{0,n}$ and $\mathbb{P}^r$, we deduce

$$H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 0)) = 0 \tag{1}$$

for every odd k.

Analogous results hold for $d = 1$. Indeed, recall from [6, § 0.4], that $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 1)$ is the Grassmannian $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^r)$, and if $n \geq 1$, then $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 1)$ is a locally trivial fibration over $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^r)$ with fiber the configuration space $\mathbb{P}^1[n]$ defined in [5]. Hence, if $n = 0$ we simply refer to [7, Proposition on p. 196]; if, instead, $n \geq 1$ we introduce the Leray spectral sequence associated with the fibration

$$\pi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 1) \longrightarrow \mathbb{G}(\mathbb{P}^1, \mathbb{P}^r),$$

with E_2-term

$$E_2^{p,q} = H^p(\mathbb{G}(\mathbb{P}^1, \mathbb{P}^r)) \otimes H^q(\mathbb{P}^1[n]).$$

Since $H^{\mathrm{odd}}(\mathbb{P}^1[n]) = 0$ (see [4, proof of Proposition 9], for a detailed explanation of this fact), it follows that the differential

$$d_2 : E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$$

is identically zero, so the spectral sequence abuts at E_2. Thus, for every k we have

$$H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 1)) = \bigoplus_{p+q=k} H^p(\mathbb{G}(\mathbb{P}^1, \mathbb{P}^r)) \otimes H^q(\mathbb{P}^1[n]). \tag{2}$$

This gives an explicit description of the rational cohomology of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 1)$, since $H^*(\mathbb{P}^1[n])$ is determined in [5, Theorem 6] (with the intersection ring taken to be the cohomology ring with rational coefficients). In particular, we point out the following fact:

Theorem 1. *If k is odd, then $H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, 1)) = 0$ for every $n \geq 0$, $r \geq 1$.*

Proof. Recall that, for each odd i, we have both $H^i(\mathbb{G}(\mathbb{P}^1, \mathbb{P}^r)) = 0$ and $H^i(\mathbb{P}^1[n]) = 0$. Hence, if $n = 0$ the thesis follows from the isomorphism $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 1) \cong \mathbb{G}(\mathbb{P}^1, \mathbb{P}^r)$, and if $n \geq 1$ it is a direct consequence of (2). □

3. The general set-up

Let $Q_{r,d}$ be the set of degree d maps from $\mathbb{P}^1$ to $\mathbb{P}^r$. As an algebraic variety, $Q_{r,d}$ is a Zariski-open subset of the projective space $\mathbb{P}(\oplus_{i=0,\dots,r} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)))$. Its complement has codimension r, and corresponds to $(r+1)$-tuples of degree d homogeneous polynomials in two variables having at least a common root.

Assume $d \leq r$. Set

$$Q^*_{r,d} := \{f \in Q_{r,d} : \mathrm{Im}(f) \text{ spans a } \mathbb{P}^d \text{ in } \mathbb{P}^r\},$$

and

$$\mathcal{M}^*_{0,n}(\mathbb{P}^r, d) := \{[f] \in \mathcal{M}_{0,n}(\mathbb{P}^r, d) : \mathrm{Im}(f) \text{ spans a } \mathbb{P}^d \text{ in } \mathbb{P}^r\}.$$

Define $N := \binom{r+1}{d+1}$.

Lemma 2. *The algebraic variety $Q^*_{r,d}$ is covered by N affine open subsets.*

Proof. First, recall that all points $P \in \mathbb{P}^1$ are linearly equivalent and that by the Riemann–Roch theorem, $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(dP)) = d + 1$. Therefore,

$$Q^*_{r,d} \cong \mathbb{G}(\mathbb{P}^d, \mathbb{P}^r) \times \mathrm{PGL}_{d+1}(\mathbb{C}).$$

Now, $\mathbb{G}(\mathbb{P}^d, \mathbb{P}^r)$ is covered by N affine open subsets, since $\mathbb{G}(\mathbb{P}^d, \mathbb{P}^r) \subset \mathbb{P}^{N-1}$ via the Plücker embedding. Moreover, $\mathrm{PGL}_{d+1}(\mathbb{C})$ is affine, since it is the complement in $\mathbb{P}(\mathrm{Mat}_{d+1,d+1}(\mathbb{C})) \cong \mathbb{P}^{(d+1)^2-1}$ of the hypersurface defined by the vanishing of the determinant. Hence the claim follows. □

Lemma 2 also provides a bound on the number of affine open sets which may cover $\mathcal{M}^*_{0,n}(\mathbb{P}^r, d)$ for any n. In fact, the following holds:

Proposition 3. *For every $d \geq 2$, $\mathcal{M}^*_{0,n}(\mathbb{P}^r, d)$ is covered by N affine open subsets.*

Proof. We are going to mimic the proof of Proposition 1 in [4]. If $n \geq 3$, then

$$\mathcal{M}^*_{0,n}(\mathbb{P}^r, d) \cong \mathcal{M}_{0,n} \times Q^*_{r,d}.$$

Therefore, the claim follows from Lemma 2 since $\mathcal{M}_{0,n}$ is affine. If, instead, $n \leq 2$, fix $3 - n$ points in $\mathbb{P}^r$, $P_1, \dots, P_{3-n}$, and define X to be the locus

$$X := \nu^{-1}_{n+1}(P_1) \cap \dots \cap \nu^{-1}_3(P_{3-n}),$$

where

$$\nu_i : \overline{\mathcal{M}}_{0,3}(\mathbb{P}^r, d) \longrightarrow \mathbb{P}^r, \quad 1 \leq i \leq 3 - n,$$

are the natural evaluation maps. Consider the map

$$\rho : X \longrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d),$$

which forgets the last $3-n$ marked points. Since $d \geq 2$, the map ρ is surjective and generically finite. Note also that

$$\rho^{-1}\big(\mathcal{M}^*_{0,n}(\mathbb{P}^r, d)\big) = \mu_{n+1}^{-1}(P_1) \cap \ldots \cap \mu_3^{-1}(P_{3-n}),$$

where

$$\mu_i : \mathcal{M}^*_{0,3}(\mathbb{P}^r, d) \longrightarrow \mathbb{P}^r, \quad 1 \leq i \leq 3-n,$$

are the corresponding evaluation maps. If $U_1, \ldots, U_N$ are open affine subsets which cover $\mathcal{M}^*_{0,3}(\mathbb{P}^r, d) \cong Q^*_{r,d}$, then we have

$$\rho^{-1}\big(\mathcal{M}^*_{0,n}(\mathbb{P}^r, d)\big) = \bigcup_{i=1}^{N} \big(\rho^{-1}\big(\mathcal{M}^*_{0,n}(\mathbb{P}^r, d)\big) \cap U_i\big),$$

where every $\rho^{-1}(\mathcal{M}^*_{0,n}(\mathbb{P}^r, d)) \cap U_i$ is affine since it is a closed subset of the affine open set U_i. In order to conclude, just notice that for every i, $\rho(\rho^{-1}(\mathcal{M}^*_{0,n}(\mathbb{P}^r, d)) \cap U_i)$ is open, since the forgetful map ρ is flat, and affine by Chevalley's theorem (see [9, Corollary 1.5 on p. 63]), since the restricted map

$$\rho^{-1}\big(\mathcal{M}^*_{0,n}(\mathbb{P}^r, d)\big) \longrightarrow \mathcal{M}^*_{0,n}(\mathbb{P}^r, d)$$

is surjective and finite. □

Theorem 4. *Let $d \leq r$ and fix an odd integer k such that $k \leq (d+1)(r+1) + n - 4 - N$. Assume that for every $n \geq 0$ and for every odd $h \leq k$ we have*

$$H^h(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^s, d)) = 0, \quad 1 \leq s < d, \tag{3}$$

and

$$H^h(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, t)) = 0, \quad 0 \leq t < d. \tag{4}$$

Then $H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) = 0$ for every $n \geq 0$.

Proof. Since every map of degree d to $\mathbb{P}^r$ whose image does not span a $\mathbb{P}^d$ is in fact a map to a $\mathbb{P}^s$, $1 \leq s < d$, embedded in $\mathbb{P}^r$, there is a natural identification:

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \setminus \mathcal{M}^*_{0,n}(\mathbb{P}^r, d) = \partial\mathcal{M}_{0,n}(\mathbb{P}^r, d) \cup \bigcup_{1 \leq s < d} \mathbb{G}(\mathbb{P}^s, \mathbb{P}^r) \times \overline{\mathcal{M}}_{0,n}(\mathbb{P}^s, d).$$

Consider now the long exact sequence:

$$\begin{aligned} \ldots \to H^k_c\big(\mathcal{M}^*_{0,n}(\mathbb{P}^r, d)\big) &\to H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)) \to \\ \to H^k\left(\partial\mathcal{M}_{0,n}(\mathbb{P}^r, d) \cup \bigcup_{1 \leq s < d} \mathbb{G}(\mathbb{P}^s, \mathbb{P}^r) \times \overline{\mathcal{M}}_{0,n}(\mathbb{P}^s, d)\right) &\to \ldots . \end{aligned}$$

Since a variety Y, which is covered by $(q+1)$ affine open subsets, has the homotopy type of a finite complex of dimension $\leq q + \dim(Y)$, from Proposition 3 we deduce that

$$H^k_c\big(\mathcal{M}^*_{0,n}(\mathbb{P}^r, d)\big) = 0, \;\; k \leq (d+1)(r+1) + n - 4 - N.$$

Thus, we have

$$H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)) \hookrightarrow H^k\left(\partial\mathcal{M}_{0,n}(\mathbb{P}^r,d) \cup \bigcup_{1\le s<d} \mathbb{G}(\mathbb{P}^s,\mathbb{P}^r)\times\overline{\mathcal{M}}_{0,n}(\mathbb{P}^s,d)\right).$$

As usual (see [2, Lemma 2.6], and [4, Lemma 4]), it follows that the map

$$H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)) \longrightarrow \bigoplus_{d_1+d_2=d,n_1+n_2=n} H^k(\overline{\mathcal{M}}_{0,n_1+1}(\mathbb{P}^r,d_1)\times_{\mathbb{P}^r}\overline{\mathcal{M}}_{0,n_2+1}(\mathbb{P}^r,d_2))$$
$$\oplus \bigoplus_{1\le s<d} H^k(\mathbb{G}(\mathbb{P}^s,\mathbb{P}^r)\times\overline{\mathcal{M}}_{0,n}(\mathbb{P}^s,d))$$

is injective too. By the vanishing of the odd cohomology of $\mathbb{G}(\mathbb{P}^s,\mathbb{P}^r)$ (see, for instance, Proposition on p. 196 in [7]) and by assumption (3), the Künneth formula yields

$$H^k(\mathbb{G}(\mathbb{P}^s,\mathbb{P}^r)\times\overline{\mathcal{M}}_{0,n}(\mathbb{P}^s,d)) = 0$$

for every $1\le s<d$. Therefore, we get the injective map:

$$H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)) \longrightarrow \bigoplus_{d_1+d_2=d,n_1+n_2=n} H^k(\overline{\mathcal{M}}_{0,n_1+1}(\mathbb{P}^r,d_1)\times_{\mathbb{P}^r}\overline{\mathcal{M}}_{0,n_2+1}(\mathbb{P}^r,d_2)).$$

Now, we can apply Lemma 7 and Remark 5 in [4] so that the claim follows easily by induction on n from assumption (4), since if $d_i=0$ then $n_i\ge 2$ (see [6, § 6.1]). □

4. Stable maps of degree 2

In this section we apply the results proved in Section 2 to calculate cohomology groups of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,2)$ for small values of r.

Proposition 5. *For every $n\ge 0$:*

i) $H^1(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,2))=0$, $2\le r\le 4$;
ii) $H^3(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,2))=0$, $2\le r\le 3$.

Proof. By the assumptions on r we have $k\le 3(r+1)+n-4-N$; so we can apply Theorem 4, because (3) holds by [4, Proposition 8], and (4) holds by (1) and Theorem 1. This proves i) and ii). □

Corollary 6. *There is a complete description with generators and relations of* $H^2(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,2))$, $2\le r\le 4$, $n\ge 0$.

Proof. By Proposition 5, if $d\le 4$, then $H^1(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,2))=0$. Hence the arguments used to prove Proposition 14 in [4] apply verbatim so to yield

$$H^2(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,2)) \cong A_{3(r+1)+n-5}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d))\otimes\mathbb{Q}.$$

The thesis follows from this isomorphism, since generators and relations of $A_{(r+1)(d+1)+n-5}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d))\otimes\mathbb{Q}$ are given in [12] for any $n\ge 0$, $r\ge 2$, and $d\ge 0$. □

Theorem 7. *For every $n \geq 0$, if k is odd then $H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, 2)) = 0$.*

Proof. If $k \leq n+4$, then the thesis directly follows from Theorem 4, since all its hypotheses are satisfied by (1), Theorem 1, and Theorem 10 in [4].

If $k \geq n+6$, we reduce to the previous case, since $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, 2) = n+5$ and we can apply the Poincaré duality.

Now we are left with the case $k = n+5 = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, 2)$. If $n = 0$, just recall (see [6, § 0.4]), the natural identification,

$$\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2) \cong \mathbb{P}^5.$$

If, instead, $n \geq 1$ we can invoke Proposition 3 and Remark 5 in [4] to obtain the fibration

$$\pi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, 2) \longrightarrow \mathbb{P}^2,$$

with fiber F and the Leray spectral sequence abutting at E_2. Hence

$$H^{n+3}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, 2)) = H^{n+3}(F) \oplus H^{n+1}(F) \oplus H^{n-1}(F) \tag{5}$$

$$H^{n+5}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, 2)) = H^{n+5}(F) \oplus H^{n+3}(F) \oplus H^{n+1}(F) \tag{6}$$

$$H^{n+7}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, 2)) = H^{n+7}(F) \oplus H^{n+5}(F) \oplus H^{n+3}(F). \tag{7}$$

On the other hand, by the previous cases, we have:

$$H^{n+7}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, 2)) = H^{n+3}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, 2)) = 0.$$

Hence, from (5) and (7) we deduce that $H^{n+5}(F) = H^{n+3}(F) = H^{n+1}(F) = 0$. Now the thesis follows from (6). □

Proposition 8. *If both k and n are odd, then $H^k(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^3, 2)) = 0$.*

Proof. If $k \leq n+4$, then the thesis directly follows from Theorem 4, since all its hypotheses are satisfied by (1), Theorem 1, and Theorem 10 in [4].

If $k \geq n+12$, we reduce to the previous case, since $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,n}(\mathbb{P}^3, 2) = n+8$, so we can apply the Poincaré duality.

Suppose now $n+4 < k < n+12$. By our assumption on n, we have $n \geq 1$. Thus, we can invoke Proposition 3, and Remark 5 in [4] to obtain the fibration

$$\pi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^3, 2) \longrightarrow \mathbb{P}^3,$$

with fiber Z and the Leray spectral sequence abutting at E_2. Hence

$$H^{n+4}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^3, 2)) = H^{n+4}(Z) \oplus H^{n+2}(Z) \oplus H^{n}(Z) \oplus H^{n-2}(Z) \tag{8}$$

$$H^{n+6}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^3, 2)) = H^{n+6}(Z) \oplus H^{n+4}(Z) \oplus H^{n+2}(Z) \oplus H^{n}(Z) \tag{9}$$

$$H^{n+8}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^3, 2)) = H^{n+8}(Z) \oplus H^{n+6}(Z) \oplus H^{n+4}(Z) \oplus H^{n+2}(Z) \tag{10}$$

$$H^{n+10}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^3, 2)) = H^{n+10}(Z) \oplus H^{n+8}(Z) \oplus H^{n+6}(Z) \oplus H^{n+4}(Z) \tag{11}$$

$$H^{n+12}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^3, 2)) = H^{n+12}(Z) \oplus H^{n+10}(Z) \oplus H^{n+8}(Z) \oplus H^{n+6}(Z). \tag{12}$$

On the other hand, by the previous cases, we have:

$$H^{n+12}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^3, 2)) = H^{n+4}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^3, 2)) = 0.$$

Hence, from (8) and (12) we deduce that $H^{n+10}(Z) = H^{n+8}(Z) = H^{n+6}(Z) = H^{n+4}(Z) = H^{n+2}(Z) = H^{n}(Z) = 0$. Now the thesis follows from (9), (10), and (11). □

References

1. Aluffi, P.: Quantum cohomology notes at the Mittag-Leffler Institute. Appunti dei Corsi tenuti dai Docenti della Scuola, Scuola Normale Superiore, Pisa 1997
2. Arbarello, E., Cornalba, M.D.T.: Calculating cohomology groups of moduli spaces of curves via algebraic geometry. Publ. Math., Inst. Hautes Étud. Sci. **88**, 97–127 (1998)
3. Behrend, K., O'Halloran, A.: On the cohomology of stable map spaces. Preprint http://xxx.lanl.gov/abs/math.AG/0202288
4. Bini, G., Fontanari, C.: On the cohomology of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, d)$. Comm. Cont. Math. **4**, 751–761 (2002)
5. Fulton, W., MacPherson, R.: A compactification of configuration spaces. Ann. Math. **139**, 183–225 (1994)
6. Fulton, W., Pandharipande, R.: Notes On Stable Maps And Quantum Cohomology. Proc. Sympos. Pure Math. **62**, Part 2. Providence, RI: Am. Math. Soc. 1997
7. Griffiths, P., Harris, J.: Principles of Algebraic Geometry. John Wiley and Sons, Inc. 1978
8. Harer, J.L.: The virtual cohomological dimension of the mapping class group of an orientable surface. Invent. Math. **84**, 157–176 (1986)
9. Hartshorne, R.: Ample Subvarieties of Algebraic Varieties. Lect. Notes Math. **156**. Berlin, New York: Springer 1970
10. Kakié, K.: The resultant of several homogeneous polynomials in two indeterminates. Proc. Am. Math. Soc. **54**, 1–7 (1976)
11. Keel, S.: Intersection theory of moduli space of stable n-pointed curves of genus zero. Trans. Am. Math. Soc. **330**, 545–574 (1992)
12. Pandharipande, R.: Intersection of $\mathbb{Q}$-divisors on Kontsevich's moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ and enumerative geometry. Trans. Am. Math. Soc. **351**, 1481–1505 (1999)
13. Segre, B.: Prodromi di geometria algebrica. Roma: Edizioni Cremonese 1972