J. Phys. A: Math. Gen. **18** (1985) L833–L838. Printed in Great Britain

LETTER TO THE EDITOR

Renormalisation theory of the self-avoiding Lévy flight

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Received 9 July 1985

Abstract. The self-avoiding Lévy flight (SALF) in d dimensions with Lévy exponent μ is formulated as a geometrical equilibrium statistical mechanical problem. A direct renormalisation theory, based on modern field theoretic techniques, is used to derive the critical exponents and the end-to-end distance probability function through first order in $\varepsilon = 2\mu - d$. The non-perturbative structure of the probability function is characterised by a universal scaling function. The SALF represents a simple many-body system that can assume a continuum of values of ε near zero.

Random walks in which the step length is a random variable with an infinite mean-squared displacement per step are called Lévy flights. The probability distribution $P_0(R, N)$ to be at a distance R away from the origin after N steps for this general class of random walks was investigated by Lévy (1937). This Lévy flight distribution function in d dimensions is most easily represented in Fourier space where it assumes the form (Khinchine and Lévy 1936, Montroll and West 1979, Shlesinger 1983):

$$P_0(q, N) = \int d^dR \exp(iq \cdot R) P_0(R, N) = \exp(-q^\mu N) \qquad 0 < \mu < 2. \tag{1}$$

This probability function characterises the Lévy flight process of order μ and is uniquely defined by the Lévy exponent μ. The quantity q^μ is a shorthand notation for $|q|^\mu l^\mu$ where l is a (microscopic) length scale. Although $P_0(q, N)$ assumes a simple form for general d and μ, only the asymptotic ($R \to \infty$) behaviour of $P_0(R, N)$ can be evaluated in closed form (Montroll and West 1979, Seshadri and West 1982):

$$P_0(R, N) \sim R^{-d-\mu}. \tag{2}$$

Thus the moments $\langle R^m \rangle$ of order m of this distribution are infinite if $m \geq \mu$. The long-ranged power law tail in this distribution is the trademark of the Lévy process that leads to 'superdiffusive' behaviour. The special Gaussian case $\mu = 2$, via the central limit theorem, represents the limiting distribution for all random walks with finite mean-squared displacement per step and leads to the ordinary and the well understood diffusive behaviour.

Recently there has been considerable interest in the Lévy flight process. The self-similar structure of the Lévy flight has been characterised by a fractal dimension $D = \mu$ (Mandelbrot 1979, Hughes *et al* 1981, Seshadri and West 1982). The connection between the Lévy flight and the spherical model of critical phenomena with long-range interactions has been noted (Joyce 1972, Hioe 1984). The effect of a self-avoiding constraint on the Lévy process has been studied via numerical simulations (Halley and Nakanishi 1984, Grassberger 1985). In analogy to the isomorphism (De Gennes 1972) between the self-avoiding random walk and a zero-component spin system, it has been noted (Halley and Nakanishi 1984) that the self-avoiding Lévy flight is equivalent to a zero-component spin model with long-range (power law) interactions. The renormalisation group theory of this magnetic model (Fisher *et al* 1972) provides theoretical critical exponents which have been compared (Halley and Nakanishi 1984, Grassberger 1985) to the numerical simulations of the self-avoiding Lévy flight.

It is more natural to directly represent the self-avoiding Lévy flight (SALF) as a geometrical equilibrium statistical mechanical model without reference to any spin system. Consider a SALF consisting of N (steps) bonds on a d-dimensional cubic lattice with lattice spacing l. The state of the SALF is uniquely defined by the set of points $\{r_0 r_1 r_2 \ldots r_N\}$ where r_i is the d-dimensional vector labelling the ith point along the flight. This state describes a random walk configuration and will be denoted by $\{r_N\}$. The set of these microscopic states forms the state space for an equilibrium statistical mechanical treatment.

The Hamiltonian defined on this state space is

$$H\{r_N\} = H_0\{r_N\} + H_I\{r_N\}$$

where

$$\begin{aligned} H_0\{r_N\} &= \sum_{i=1}^{N} \ln \tau(r_i - r_{i-1}) \\ H_I\{r_N\} &= \tfrac{1}{2}\sum_{ij} v(r_i - r_j). \end{aligned} \tag{3}$$

The connectivity of particles is described by the normalised bond probability function $\tau(r_i - r_{i-1})$ and the self-avoiding condition is described by the two-body repulsive interaction $v(r_i - r_j)$. The Fourier transform (characteristic function) of $\tau(x)$ is the Lévy step distribution $\tau(q) = \exp(-q^\mu)$. The Fourier transform of the short-ranged interaction $v(x)$ is denoted by v. This definition of the Hamiltonian in equation (3) ensures that an equilibrium ensemble of SALFs will be generated when each random walk state $\{r_N\}$ is weighted by $\exp(H\{r_N\})$.

The equilibrium statistical mechanics of this SALF model is characterised by the partition function (endpoint correlation function):

$$Z(R, N) = \int_0^R \mathrm{d}\{r_N\} \exp(H\{r_N\}). \tag{4}$$

The integral represents a sum-over-states with endpoints fixed at $r_0 = 0$ and $r_N = R$. This partition function has the same form as those used to understand the statistics of polymer molecules (Yamakawa 1971). It is more convenient to represent the partition function in Fourier space:

$$Z(q, N) = \int \mathrm{d}^d R \exp(\mathrm{i}q \cdot R) Z(R, N). \tag{5}$$

The probability function is then given by

$$P(q, N) = Z(q, N)/Z(0, N). \tag{6}$$

For $H_I = 0$, the Lévy distribution (equation (1)) is recovered:

$$P_0(q, N) = \tau^N(q) = e^{-q^\mu N}. \tag{7}$$

The SALF ($H_I \neq 0$) represented a non-trivial non-Markovian process. The effect of the self-avoiding condition on the Lévy process may be understood using the same theoretical methods that have provided insight into the self-avoiding random walk problem in recent years (De Gennes 1972, Witten and Schafer 1981, Oono *et al* 1981, Des Cloizeaux 1981). These methods are based on perturbation theory and the renormalisation group.

The diagrammatic structure of the perturbation theory for the SALF problem is identical to that of the self-avoiding random walk, being uniquely determined by H_I. The differing connectivities H_0 in the two problems manifest themselves in a different free (Lévy flight) propagator $Z_0(q, N)$. The perturbation theory is conveniently performed in the Laplace-transformed representation where the free Lévy flight propagator (equation (7)) assumes the form

$$Z_0(q, s) = \int_0^\infty \mathrm{d}N\, e^{-sN} Z_0(q, N) = \frac{1}{q^\mu + s}. \tag{8}$$

Expanding the SALF partition function (equation (4)) in H_I, one finds

$$Z(q, t) = \frac{1}{q^\mu + t}\left(1 - g\frac{t}{q^\mu + t}\int^{l^{-1}} \mathrm{d}^d p \frac{1}{p^\mu(p^\mu + t)} + O(g^2)\right). \tag{9}$$

A change of variables has been performed from s to $t = (s - s_c)/l^\mu$ where t measures the distance away from the critical point s_c at which $\langle N \rangle = \infty$. The natural expansion variable for the perturbation theory is

$$g = (2\pi)^{-d} l^{d-2\mu} v. \tag{10}$$

The dimension of this interaction constant indicates that the upper critical dimension for the SALF problem is $d_c = 2\mu$. The domain of integration is the first Brillouin zone of the hypercubic lattice and has been denoted by the cutoff l^{-1}. If this microscopic length scale vanishes, then the resulting divergent integral renders the perturbation theory meaningless if $d \geqslant 2\mu$.

The renormalisation group theory is designed to extract non-perturbative scaling information from the ill behaved perturbation theory. The renormalised theory is obtained from the bare theory via the transformation:

$$Z_R = z_1 Z, \qquad t_R = z_2 t, \qquad g_R = z_3 \Omega L^{2\mu - d} g/\mu. \tag{11}$$

z_1, z_2 and z_3 are the renormalisation constants (independent of q and t), L is the renormalised length scale and Ω is the surface area of the unit sphere in d dimensions. The renormalisation constants ensure that the renormalised theory is well behaved in the limit $(l/L) \to 0$ by absorbing the non-universal short-distance (ultraviolet) divergences. The SALF perturbation theory is renormalisable to all orders because of its identical structure to the renormalisable self-avoiding random walk theory. A consequence of the renormalisability of the SALF theory is that the partition function satisfies the renormalisation group equation:

$$\left(L\frac{\partial}{\partial L} + \beta(g_R)\frac{\partial}{\partial g_R} + \gamma_2(g_R) t_R \frac{\partial}{\partial t_R} - \gamma_1(g_R)\right) Z_R(q, t_R, g_R, L) = 0$$

where

$$\beta(g_R) = \partial g_R/\partial \ln L$$
$$\gamma_1(g_R) = \partial \ln z_1/\partial \ln L \qquad (12)$$
$$\gamma_2(g_R) = \partial \ln z_2/\partial \ln L.$$

This equation expresses the invariant structure of the theory at different length scales. The solution to this equation at the fixed point g_R^*, defined by $\beta(g_R^*) = 0$, yields the non-perturbative scaling behaviour of the SALF partition function for large N:

$$Z(q, N) = e^{s_c N} N^{\gamma - 1} F(q^\mu N^{\nu\mu}). \qquad (13)$$

$F(x)$ is a universal scaling function such that $F(0) = 1$ and the critical exponents γ and ν are obtained from the renormalisation constants as follows:

$$\nu^{-1} = \mu + \gamma_2(g_R^*)$$
$$\gamma = [\mu - \gamma_1(g_R^*)]\nu. \qquad (14)$$

This proof of the scaling behaviour of the partition function provides information on the non-perturbative structure of the SALF probability function:

$$P(q, N) = F(q^\mu N^{\nu\mu}). \qquad (15)$$

In this formalism, the critical exponent η is calculated from the relation $\eta = \gamma_1(g_R^*)$. Thus we find the scaling relation

$$\gamma = (\mu - \eta)\nu. \qquad (16)$$

The form of this relation differs from the conventional one $\gamma = (2 - \eta)\nu$ and is a simple manifestation of our definition of η. The above relation is consistent with the modern field theory interpretation of η (Amit 1978) as the anomalous dimension associated with the two-point correlation function $Z(q, t)$. Since the canonical dimension is μ (not 2), the natural definition of η is to characterise the anomalous scaling behaviour $Z(q, t) \sim q^{\eta - \mu}$ $(qt^{-\nu} \gg 1)$. This definition ensures that $\eta = 0$ to lowest order as it is for other critical phenomena. For the SALF, it is conjectured (Sak 1977) that $\eta = 0$ to all orders and thus $\gamma = \mu\nu$.

The critical exponents are calculated from a renormalisation group analysis of the primitively divergent SALF diagrams. This exponent information is contained in the partition function of equation (9) together with the partition function of a system consisting of two SALFs. The perturbation expansion of the two-SALF partition function (primitive diagrams only) is

$$Z^{(2)}(q_1 q_2 q_3 q_4, t) = l^{2\mu - d} g[1 + g(I(q_1 + q_3, t) + 2I(q_1 + q_2, t) + I(q_1 + q_4, t)) + O(g^2)] \qquad (17)$$

where

$$I(q, t) = \int d^d p \frac{1}{p^\mu + t} \frac{1}{(q + p)^\mu + t}.$$

To implement the renormalisation program, we utilise the modern field theoretic techniques of dimensional regularisation and minimal subtraction renormalisation (Amit 1978). Details of this program, as applied to the self-avoiding random walk, are described elsewhere (Prentis 1982). Upon introducing the parameter

$$\varepsilon = 2\mu - d \tag{18}$$

and expanding the partition functions in ε, the singularities (poles in ε) plaguing the perturbation theory can be removed via renormalisation and absorbed into the renormalisation constants which we find to be

$$z_1 = 1 + O(g_R^2), \qquad z_2 = 1 + (\mu/\varepsilon)g_R + O(g_R^2), \qquad z_3 = 1 + (4\mu/\varepsilon)g_R + O(g_R^2). \tag{19}$$

From these results and equations (12) and (14), the fixed point and the critical exponents of the SALF theory can be calculated

$$\begin{aligned} g_R^* &= -(\varepsilon/4\mu) + O(\varepsilon^2) \\ \nu &= (1/\mu)[1 + (\varepsilon/4\mu) + O(\varepsilon^2)] \\ \gamma &= 1 + (\varepsilon/4\mu) + O(\varepsilon^2). \end{aligned} \tag{20}$$

The universal critical exponents are identical with the exponents characterising the zero component, long-range interacting spin magnetic model (Fisher *et al* 1972), thus providing support for the SALF-magnet analogy.

The exponent ν provides a measure of the average size of the SALF. This interpretation requires caution because of the existence of infinite moments which characterise the Lévy flight process. More precisely, from the scaling form of the probability function, the generalised moments of order m are finite $\langle R^m \rangle \sim N^{m\nu}$ if $m < \mu$ and infinite otherwise.

The SALF probability function $P(q, N)$ can also be calculated as an expansion in ε where $\varepsilon \geq 0$. Upon renormalising (equation (11)) the partition function (equation (9)) at the fixed point (equation (20)) and converting back to N space, we obtain the expansion

$$P(q, N) = e^{-q^\mu N}\left(1 + \frac{\varepsilon}{4\mu}[e^{q^\mu N} - 1 + (1 - q^\mu N)D(q^\mu N) - q^\mu N(\gamma + \ln N)] + O(\varepsilon^2)\right). \tag{21}$$

In this equation, $\gamma = 0.577$ is Euler's constant and $D(x)$ is the analytic function

$$D(x) = \int_0^x dy \frac{e^y - 1}{y}. \tag{22}$$

In terms of the SALF exponent ν (equation (20)), this expansion can be exponentiated into a form consistent with (through first order in ε) the non-perturbative scaling behaviour (equation (15)). The final result for the SALF probability function is

$$P(q, N) = F(q^\mu N^{\nu\mu}) \tag{23}$$

where the universal scaling function is

$$F(x) = e^{-x}\{1 + (\varepsilon/4\mu)[e^x - 1 + (1 - x)D(x) + \gamma x)]\} \tag{24}$$

In summary, the renormalisation group symmetry of the SALF has been exploited to derive the critical exponents and the probability function through first order in ε. This formulation, based on the direct renormalisation of a geometric model, avoids analogy to the zero-component spin system and thus is self-contained and more transparent. The critical exponents agree with those of the zero-component spin system, thus providing support for the magnet analogy. The SALF probability function is a new result, characterised by a universal scaling function, that provides more detailed information on the non-Markovian (self-avoiding) effects. The ε expansion of a scaling function, whose scaling variable contains the unperturbed critical exponent, may be more reliable than the ε expansion of a critical exponent. In addition, the SALF system can assume infinitesimal values of ε near zero. It is hoped that numerical simulations and experimental realisations of the SALF system, in conjunction with the theoretical predictions, will provide further insight into this random process and the ε expansion in critical phenomena.

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