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LETTER TO THE EDITOR

Performance bounds on depth estimation of bioelectrical sources

Thomas G Xydis and Andrew E Yagle

Department of Electrical Engineering and Computer Science, The University of Michigan, Ann Arbor, MI 48109-2122, USA

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Abstract. The unknown depth of bioelectrical sources confined to a horizontal plane in a horizontally layered volume conductor is estimated from noisy measurements of electrical potential on another, parallel plane (e.g. the surface). The Cramer–Rao bound is computed and discussed. Numerical simulations suggest the maximum likelihood depth estimate is asymptotically efficient.

We consider the following volume conductor source location estimation problem. Given knowledge of the functional form of a two-dimensional planar bioelectrical source at an unknown depth in a horizontally layered volume conductor with known structure, and measurements of the resulting extracellular field on a plane parallel to the source, determine the depth of the planar source.

This location estimation problem can be applied to problems frequently encountered in cardiology research. Healthy cardiac muscle consists of multiple, electrically-active layers. Following myocardial infarction, many of these layers are dead. The problem of determining the depth of any layer still active (i.e. alive) from measurements at or near the surface of the pericardium can be formulated as the above location estimation problem.

Although the surface of the heart is obviously not a plane, the brush electrode used to perform electric field measurements and the scale of the problem itself are both so small (of the order of millimetres) that regarding the heart as locally planar is reasonable. The planar heart approximation has been used in the past for the volume conductor forward problem at small distances, as well as in the interpretation and simulation of tissue bath experiments in which the planar assumption is rigorously met [1–5]. Di Persio and Barr [6] solved the one-dimensional version of the above problem by using a template-matching algorithm to estimate the location of an action potential along a strand.

We make the following assumptions, common in many volume conductor studies:

1. The extracellular medium is linear and consists of homogeneous layers parallel to the plane $z = 0$ of measurements. The assumption of a layered medium is appropriate and useful for the proposed application of this algorithm: the determination of the location of potential sources in hearts damaged by patchy infarction, i.e. epicardial tissue with several discrete active layers separated by regions of fibrous growth.

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2. The source is modelled as a two-dimensional distributed potential $s(x, y, z')$ lying in a plane $z = z'$ parallel to the plane of measurements. Our goal is to estimate z'.
3. The problem is quasistatic, i.e. the medium has negligible reactance and propagation delay. Thus all potentials are computed at an instant in time. The quasistatic assumption ensures that each sample will be independent of the other samples [7].

Our data consist of noisy measurements $r(x, y, z')$ of the surface potential $\phi(x, y, z')$

$$r(x, y, z') = \phi(x, y, z') + n(x, y) \tag{1}$$

where $n(x, y)$ is a zero-mean white Gaussian noise field. Fourier transforms of these quantities are denoted using capital letters, e.g. $S(k_x, k_y, z') = \mathcal{F}_{x \to k_x} \mathcal{F}_{y \to k_y} \{s(x, y, z')\}$.

The surface potential $\phi(x, y, z')$ is related to the source potential $s(x, y, z')$ by [8]

$$\begin{aligned} \phi(x, y, z') &= \mathcal{F}^{-1}_{k_x \to x} \mathcal{F}^{-1}_{k_y \to y} \{S(k_x, k_y, z') H(k_x, k_y, z')\} \\ &= \frac{1}{(2\pi)^2} \int\int S(k_x, k_y, z') H(k_x, k_y, z') \mathrm{e}^{-\mathrm{i}k_x x} \mathrm{e}^{-\mathrm{i}k_y y} \, \mathrm{d}k_x \mathrm{d}k_y \end{aligned} \tag{2}$$

where the *medium filter* $H(k_x, k_y, z')$ represents the effect of the volume conductor between the planes $z = z'$ and $z = 0$. For a homogeneous medium, $H(k_x, k_y, z')$ is [8]

$$H(k_x, k_y, z') = \mathrm{e}^{-\sqrt{k_x^2 + k_y^2} z'} \tag{3}$$

for a layered medium, $H(k_x, k_y, z')$ may be computed in a layer-recursive manner [8]. Note that the depth z' of the source affects the surface potential $\phi(x, y, z')$ through $H(k_x, k_y, z')$.

Using Parseval's theorem, the likelihood function may be written as

$$\begin{aligned} \ln \Lambda[R(k_x, k_y), (x, y, z')] = \frac{2}{N_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(k_x, k_y) S(k_x, k_y) H(k_x, k_y, z') \, \mathrm{d}k_x \mathrm{d}k_y \\ - \frac{1}{N_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S(k_x, k_y) H(k_x, k_y, z')|^2 \, \mathrm{d}k_x \mathrm{d}k_y . \end{aligned} \tag{4}$$

The maximum likelihood estimate $\hat{z}$ of depth z' is found by maximizing (4) over the depth z'. Note that the strength $N_0/2$ of the observation noise field is irrelevant.

The Cramer–Rao bound is a lower bound on the variance of any unbiased estimator $\hat{a}$ of parameter A in the nonlinear estimation problem

$$r(t) = s(t, A) + n(t) \qquad 0 \leqslant t \leqslant T \tag{5}$$

where $n(t)$ is zero-mean white noise of strength $N/2$ and $s(t, A)$ is a nonlinear function of parameter A. The variance of $\hat{a}$ is bounded by [9]

$$E[(\hat{a} - A)^2] \geqslant \frac{N}{2 \int_0^T [\partial s(t, A)/\partial A]^2 \, \mathrm{d}t} . \tag{6}$$

Applying the two-dimensional version of (5) and (6) to the present problem yields

$$E[(z'-\hat{z})^2] \geqslant \frac{N}{(2\pi)^2}\left(\int_0^\infty \int_0^{2\pi} \left|\frac{\partial H(k,\phi,z')}{\partial z'} S(k,\phi)\right|^2 k\,\mathrm{d}\phi\mathrm{d}k\right)^{-1} \tag{7}$$

where $\hat{z}$ is the estimate of depth z', $k = (k_x^2 + k_y^2)^{1/2}$ is radial wavenumber, and $S(k,\phi)$ is the Fourier transform of the source potential in polar coordinates.

For an infinite homogeneous volume conductor, inserting (3) in (7) yields

$$E[(z'-\hat{z})^2] \geqslant \frac{N}{(2\pi)^2}\left(\int_0^\infty \int_0^{2\pi} k^3 \mathrm{e}^{-2kz'} |S(k,\phi)|^2 \mathrm{d}\phi\mathrm{d}k\right)^{-1}. \tag{8}$$

The Cramer–Rao bound (8) illustrates several interesting points:

1. As depth z' increases, the weighting factor $\mathrm{e}^{-2kz'}$ decreases exponentially. Although this factor is inside the integral, this accounts for the presence of a performance threshold, and the rapid degradation of performance beyond it.
2. Low-wavenumber components of the source potential are important, due to the factor of $\mathrm{e}^{-2kz'}$ which attenuates high-wavenumber components of the source potential. This is reasonable, since these components are strongly attenuated by the medium.
3. Very low-wavenumber components in the source potential are not helpful, since they are attenuated by the factor k^3. This is reasonable: the medium has little effect on these components, so they are less useful in estimating z'.

Several numerical simulations were performed to compare the performance of the maximum-likelihood estimator (4) to the Cramer–Rao lower bound (8). The medium filter was used to solve the forward problem and generate the surface potential $\phi(x,y,z')$. The source potential was a rectangular function in the frequency domain. Since only a small portion of the wavenumber spectrum of the source significantly affects either the bound or the estimator, both are robust to uncertainties in the form of the source. Numerical simulations not included here show that approximating the source by a constant function having the same spatial support as the source, and with amplitude equal to the RMS value of the source, also gives good performance in the depth estimation algorithm.

To test the performance of the maximum likelihood estimator with respect to the Cramer–Rao bound, depth estimates were made with 50 different two-dimensional noise fields and several signal-to-noise ratios. Each noise field was Gaussian with unit variance; at each signal-to-noise ratio the noise fields were scaled to the required variance. The source potential was uniform with amplitude 50 for $-15 \leqslant k_x, k_y \leqslant +15$, and zero otherwise; this allowed simple evaluation of the Cramer–Rao bound given above.

The mean square error in estimating depth z' was approximated by the average of the squared errors for each run. A graph of the Cramer–Rao bound and experimental variance is shown in figure 1, which shows that until the signal-to-noise ratio becomes very negatively large, the performance of the estimator attains the Cramer–Rao bound. Thus for these signal-to-noise ratios and this source, the nonlinear depth estimator is asymptotically efficient. This is no surprise, for reasons discussed in [9].

In conclusion we summarize our results. The Cramer–Rao bound on the mean square error in estimating the depth of a plane of bioelectrical sources has been

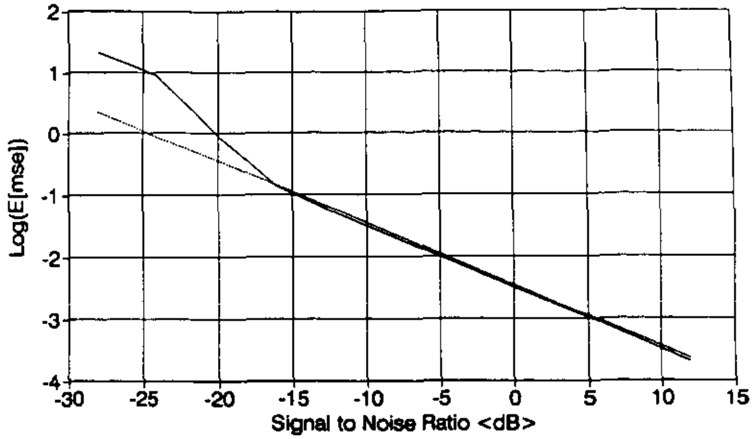

Figure 1. A graph of the Cramer–Rao bound (dotted curve) and experimaental variance (full curve). To approximate the variance in the estimation of z, each $\hat{z}$ was assumed to be generated by an ergodic process, thus allowing the variance to be approximated as the average of the squared errors of each run. Note that until the signal-to-noise ratio becomes quite large, the performance of the algorithm compares almost exactly with the Cramer–Rao bound. Thus for these signal-to-noise ratios, the nonlinear depth estimator can be considered efficient.

derived, and explicitly evaluated for an infinite homogeneous volume conductor. The maximum likelihood estimator for the unknown depth was also determined. Both results used the two-dimensional medium filter to provide simple expressions. Numerical simulations showed that the estimator achieves the bound for surprisingly high noise levels. The form of the Cramer–Rao bound makes these results reasonable.

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