INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 15, 1557-1579 (1980)

SHORT COMMUNICATIONS

REMARKS ON RELATIONS BETWEEN PENALTY AND MIXED FINITE ELEMENT METHODS FOR A CLASS OF VARIATIONAL INEQUALITIES

NOBORU KIKUCHI

The University of Michigan, Ann Arbor, Michigan, U.S.A.

AND

YOUNG JOON SONG

Korea Military Academy, Seoul, Korea

INTRODUCTION

This short communication is intended to describe a characterization of a penalty–finite element approximation for a variational inequality. Particularly, we will show that the solution u_ε^h of the penalty–finite element approximation and the contact pressure p_ε^h converge to the solution $\{u^h, p^h\}$ of a mixed formulation of the variational inequality.

Theories of equivalence between penalized and mixed finite element methods have been studied by Malkus,[1] Hughes,[2] Malkus and Hughes[3] and Bercovier[4] for the Stokes problem, and by Lee and Pian[5] for plate and shell problems. The present study is an extension of these works, which are concerned with constraints represented by equations, to penalty methods resolving constraints given by inequalities.

The first result of the present paper is that the penalized solution u_ε^h converges to the one of the variational inequality in the discrete form. The second is the convergence of $\{u_\varepsilon^h, p_\varepsilon^h\}$ to the solution $\{u^h, p^h\}$ of a mixed formulation of the variational inequality. The last result is an error estimate of approximations with respect to the penalty parameter ε, for a fixed mesh size h of the finite element model.

A UNILATERAL PROBLEM

Suppose that the variational inequality

$$u \in K: a(u, v-u) \geqslant f(v-u), \qquad \forall v \in K \tag{1}$$

is given together with the constrained set

$$K = \{v \in V:\ v - s \leqslant 0 \text{ in } \Omega\} \tag{2}$$

where V is a closed subspace of the Sobolev space $H^m(\Omega)$, $m \geqslant 0$, and Ω is a bounded open domain in R^3. Suppose that the boundary Γ of Ω is smooth enough. The variational inequality (1) represents the obstacle problem of an elastic membrane, Signorini's problem of a plate or a linearly elastic body, and so on, see Duvaunt and Lions.[6]

Suppose that the bilinear form $a(.,.)$ is coercive and continuous on V and the linear form $f(.)$ is continuous on V, i.e.

$$a(u, u) \geqslant m\|u\|_m^2 \qquad a(u, v) \leqslant M\|u\|_m\|v\|_m \qquad f(v) \leqslant \|f\|_m^*\|v\|_m \tag{3}$$

0029-5981/80/1015-1557$01.00

Received 27 November 1979
Revised 14 February 1980

where $\|.\|_m$ is the norm of $H^m(\Omega)$, and $\|.\|_m^*$ is its dual norm. Let the distance function s in the constrained set be an element of $H^m(\Omega)$, i.e.

$$s \in H^m(\Omega) \tag{4}$$

A PENALTY-FINITE ELEMENT APPROXIMATION

One of practical methods of approximation of the variational inequality (1) is a penalty–finite element approximation.

Let V_h be the finite element approximation of the space V. Let b_h be an approximation of the penalty function to the constraint given by

$$b_h(v^h) = (v^h - s)(x)^+ \tag{5}$$

where $g^+ = \mathrm{Max}\,(0., g)$ for $g \in R$. Let $(.,.)$ be the $L^2(\Omega)$-inner product defined by

$$(f, g) = \int_\Omega fg\,\mathrm{d}x \tag{6}$$

and let I_G be the operation of numerical integration. Then, the penalty-finite element approximation of (1) is given by

$$u_\varepsilon^h \in V_h: a(u_\varepsilon^h, v^h) + \frac{1}{\varepsilon} I_G(b_h(u_\varepsilon^h), v^h) = f(v^h), \qquad \forall v^h \in V_h \tag{7}$$

where ε is a penalty parameter and

$$I_G(b_h(u_\varepsilon^h), v^h) = \sum_{e=1}^{E} \sum_{i=1}^{G} w_i b_h(u_\varepsilon^h)(x_i^e) v^h(x_i^e) \tag{8}$$

Here E is the total number of elements, G is the total number of integration points within an element, w_i and x_i^e are weights and points of numerical integration, and $i = 1, 2, \ldots, G$, $e = 1, 2, \ldots, E$. The reason why the rule of numerical integration I_G is applied to the term of penalty is that the exact integration of it is impossible because of the special nonlinearity of the penalty function $b_n(u_\varepsilon^h)$.

We first show that the sequence u_ε^h of the solution of the penalized formulation (7) converges to a solution of an approximation of the variational inequality (1), as ε goes to zero.

Theorem 1. Suppose that (3) and (4) hold. Then the solution u_ε^h of (3) is uniformly bounded in ε, and the sequence u_ε^h converges to a solution u^h of the variational inequality

$$\begin{cases} u^h \in K_h: a(u^h, v^h - u^h) \geq f(v^h - u^h), \qquad \forall v^h \in K_h & (9) \\ K_h = \{v^h \in V_h: (v^h - s)(x_i^e) \leq 0, i = 1, .., G, e = 1, .., E\} & (10) \end{cases}$$

as the penalty parameter ε goes to zero.

(*Proof*) (Uniform Boundedness) Let $v^h \in K_h$. From (7),

$$a(u_\varepsilon^h, v^h - u_\varepsilon^h) + \frac{1}{\varepsilon} I_G(b_h(u_\varepsilon^h), v^h - u_\varepsilon^h) = f(v^h - u_\varepsilon^h)$$

Since $b_h(v^h) = 0$.

$$\begin{aligned} I_G(b_h(u_\varepsilon^h), v^h - u_\varepsilon^h) &= -I_G(b_h(v^h) - b_h(u_\varepsilon^h), v^h - u_\varepsilon^h) \\ &\leq -I_G((v^h - s)^+ - (u_\varepsilon^h - s)^+, (v^h - s)^+ - (u_\varepsilon^h - s)^+) \leq 0 \end{aligned} \tag{11}$$

Thus we have

$$a(u_\varepsilon^h, v^h - u_\varepsilon^h) \geq f(v^h - u_\varepsilon^h), \qquad \forall v^h \in K_h \tag{12}$$

This inequality also implies the estimate

$$\|u_\varepsilon^h\|_m \leq C, \qquad \varepsilon > 0 \tag{13}$$

(Convergence) Since V_h is a (closed) finite dimensional subspace of a reflexive Banach space $H^m(\Omega)$, there exists a subsequence of u_ε^h, still denoted by u_ε^h, which converges to u^h in V_h. Passing to the limit $\varepsilon \to 0$ in (12), we have

$$a(u^h, v^h - u^h) \geq f(v^h - u^h), \qquad \forall v^h \in K_h$$

Thus we need show only that $u^h \in K_h$. Putting $v^h - u_\varepsilon^h$ in (7),

$$a(u_\varepsilon^h, v^h - u_\varepsilon^h) + \frac{1}{\varepsilon} I_G((u_\varepsilon^h - s)^+, v^h - u_\varepsilon^h) = f(v^h - u_\varepsilon^h)$$

If $v^h \in K_h$, i.e. if $v^h - s \leq 0$ at x_i^e,

$$I_G((u_\varepsilon^h - s)^+, v^h - u_\varepsilon^h) = I_G((u_\varepsilon^h - s)^+, v^h - s - (u_\varepsilon^h - s)) \leq -I_G((u_\varepsilon^h - s)^+, (u_\varepsilon^h - s)^+)$$

Then we have the inequality

$$a(u_\varepsilon^h, v^h - u_\varepsilon^h) - \frac{1}{\varepsilon} I_G((u_\varepsilon^h - s)^+, (u_\varepsilon^h - s)^+) \geq f(v^h - u_\varepsilon^h)$$

i.e.

$$I_G((u_\varepsilon^h - s)^+, (u_\varepsilon^h - s)^+) \leq \varepsilon(a(u_\varepsilon^h, v^h) + f(u_\varepsilon^h - v^h))$$

Since u_ε^h is uniformly bounded in ε and since v^h is an arbitrary but fixed element in K_h, we can conclude that

$$|(u_\varepsilon^h - s)(x_i^e)^+|^2 \leq C\varepsilon, \qquad \text{then} \qquad (u^h - s)(x_i^e)^+ = 0$$

Therefore $u^h \in K_h$. □

We have shown that the penalty solution u_ε^h converges to a solution u^h of the variational inequality (9) which may be considered as an approximation of (1). We now explore substance of (9). To this end, we first find a meaning of the set K_h in which the contact condition is satisfied pointwise.

Let q^h be an arbitrary element of the piecewise polynomial space Q_h such that

$$(q^h, v^h) = I_G(q^h, v^h), \qquad \forall v^h \in V_h \tag{14}$$

If $G = 3$ and V_h consists of piecewise cubic polynomials in the one-dimensional domain $\Omega = (0, 1)$, q^h is a piecewise quadratic polynomial. Then, if $s \in V_h$,

$$v^h \in K_h \quad \text{if and only if} \quad (q^h, v^h - s) \geq 0, \qquad \forall q^h \in N_h \tag{15}$$

where

$$N_h = \{q^h \in Q_h : q^h(x_i^e) \leq 0, i = 1, .., G, e = 1, .., E\} \tag{16}$$

Using the dual N_h of the constrained set K_h, we can characterize the set K_h by

$$K_h = \{v^h \in V_h : (q^h, v^h - s) \geq 0, \qquad \forall q^h \in N_h\} \tag{17}$$

It is now clear that q^h is the Lagrangian multiplier to the constraint $v^h - s \leq 0$, and represents the approximation of the contact pressure due to contact, physically. We will find relations between the penalty and Lagrangian multiplier (or mixed) methods, more precisely. Define $p^h \in Q_h$ by

$$p_\varepsilon^h(x_i^e) = -\frac{1}{\varepsilon}(u_\varepsilon^h - s)(x_i^e)^+ \tag{18}$$

Then, by (14) and (18), the penalty form (7) can be written as

$$a(u_\varepsilon^h, v^h) - (p_\varepsilon^h, v^h) = f(v^h), \qquad \forall v^h \in V_h \tag{19}$$

The form (19) suggests that $p_\varepsilon^h \in Q_h$ is an approximation of the contact pressure.

Theorem 2. Suppose that (14) and $s \in V_h$ hold. Further suppose that

$$C_h > 0: C_h \|q^h\|_0 \leq \sup_{v^h \in V_h} \frac{(q^h, v^k)}{\|v^h\|_m}, \qquad \forall q^h \in Q_h \tag{20}$$

Then $p_\varepsilon^h \in Q_h$, defined by (18), is uniformly bounded in ε, and converges to $p^h \in Q_h \cap N_h$ which satisfies

$$a(u^h, v^h) - (p^h, v^h) = f(v^h), \qquad \forall v^h \in V_h \qquad (q^h - p^h, u^h - s) \geq 0, \qquad \forall q^h \in N_h \tag{21}$$

(*Proof*) Using (19) and (20),

$$C_h \|p_\varepsilon^h\|_0 \leq \sup_{v^h \in V_h} \frac{a(u_\varepsilon^h, v^h) - f(v^h)}{\|v^h\|_m} \leq M\|u_\varepsilon^h\|_m + \|f\|_m^*$$

Since u_ε^h is uniformly bounded in ε, p_ε^h is also uniformly bounded in $L^2(\Omega)$-sense. Then there is a subsequence of p_ε^h, still denoted by p_ε^h, which converges to p^h in Q_h. Passing to the limit $\varepsilon \to 0$ in (19) yields $(21)_1$. We will show $(21)_2$. By (18),

$$(q^h - p_\varepsilon^h, u_\varepsilon^h - s) = (q^h, u_\varepsilon^h - s) + \frac{1}{\varepsilon} I_G((u_\varepsilon^h - s)^+, u_\varepsilon^h - s) \geq (q^h, u_\varepsilon^h - s)$$

Passing to the limit $\varepsilon \to 0$ in the above for $q^h \in N_h$,

$$(q^h - p^h, u^h - s) \geq (q^h, u^h - s) \geq 0, \qquad \forall q^h \in N_h$$

since $u^h \in K_h$. □

Similar results of convergence for the penalty–finite element approximation can be also obtained for the case that a proper space Q_h of the contact pressure, satisfying the relation (14), does not exist. The corresponding convergence theorem for such cases is stated by the mixed formulation (21) with the operation of numerical integration I_G, see Kikuchi.[7]

So far we could characterize the penalty-finite element approximation (7) as an approximation of a mixed (Lagrangian multiplier) formulation to the variational inequality (1) under the additional condition (20), which is called the discrete L.B.B. condition in the literature of mixed finite element methods.

We finally give rates of convergence of u_ε^h and p_ε^h for the penalty–finite element approximation.

Theorem 3. Same conditions in Theorem 2 hold. Then

$$\|u^h - u_\varepsilon^h\|_1 \leq (M/m)\|p^h\|_0\varepsilon/C_h, \qquad \|p^h - p_\varepsilon^h\|_0 \leq M\|u^h - u_\varepsilon^h\|_1/C_h \tag{22}$$

where C_h is the constant in (20).

(*Proof*) From (19) and (21),

$$a(u^h - u_\varepsilon^h, v^h) = (p^h - p_\varepsilon^h, v^h) \tag{23}$$

Taking $v^h = u^h - u_\varepsilon^h$ in (23) yields

$$\begin{aligned} a(u^h - u_\varepsilon^h, u^h - u_\varepsilon^h) &= (p^h - p_\varepsilon^h, u^h - u_\varepsilon^h) \\ &= (p^h - p_\varepsilon^h, u^h - s) + (p^h - p_\varepsilon^h, (u_\varepsilon^h - s)^-) - (p^h - p_\varepsilon^h, (u_\varepsilon^h - s)^+) \\ &\leq -(p^h - p_\varepsilon^h, p^h - p_\varepsilon^h)\varepsilon + (p^h - p_\varepsilon^h, p^h)\varepsilon \end{aligned}$$

since (14), (18) and (21). Here $v^- = v^+ - v$.

Applying (3), we have

$$m\|u^h - u_\varepsilon^h\|_1^2 \leq \|p^h - p_\varepsilon^h\|_0\|p^h\|_0\varepsilon \tag{24}$$

On the other hand, from (23)

$$C_h\|p^h - p_\varepsilon^h\|_0 \leq M\|u^h - u_\varepsilon^h\|_1 \tag{25}$$

Here (3) and (20) have been applied. Therefore, (22) follows from (24) and (25). □

ACKNOWLEDGEMENTS

The authors are grateful to Prof. J. Tinsley Oden for his discussions. During this work, the authors were supported by AFOSR F-49620-78-C-0083.

REFERENCES

1. D. S. Malkus, 'A finite element displacement model valid for any value of the compressibility', *Int. J. Solids Struct.* **12**, 731–738 (1976).
2. T. J. R. Hughes, 'Equivalence of finite elements for nearly incompressible elasticity', *LBL5237*, University of California (Aug. 1976).
3. D. S. Malkus and T. J. R. Hughes, 'Mixed finite element methods—reduced and selective integration techniques: a unified concepts', *Comp. Meth. Appl. Mech. Engng.* **12**, 63–81 (1978).
4. M. Bercovier, 'Pertubation of a mixed variational problem; application to mixed finite element methods', *RAIRO, Numer. Anal.* **12**, 211–236 (1978).
5. S. W. Lee and T. H. H. Pian, 'Improvements of plate and shell finite elements by mixed formulations', *AIAA/ASME 18th Structures, Structural Dynamics and Material Conference*, San Diego, California (21–23 March, 1977).
6. G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.
7. N. Kikuchi, 'Convergence of a penalty-finite element approximation for an obstacle problem', in review.