DETERMINATION OF THE OPTIMAL PROCESS MEAN AND THE UPPER LIMIT FOR A CANNING PROBLEM

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Abstract

We deal with a canning problem where the amount of an expensive ingredient put into a can is a random variable whose mean can be set by the canner. Underfilled (below a specified lower limit) and overfilled (above a controllable upper limit) cans are emptied and refilled. We assume that the fill is normally distributed with known variance, and assuming a reasonable cost structure obtain optimal values for the process mean and the upper limit. Simple approximate analytical expressions relating these optimal values to fundamental parameters are also established. An explicit measure is given for the value of being able to impose an upper limit on the fill.

INTRODUCTION

There is a vast literature on quality control that focuses on defining economic upper and lower control limits for a process. A common assumption is that the process is set optimally and the problem is usually to detect deviation from the "normal" performance of the process at an early stage [see Montgomery (1980)]. In this paper we, instead, address the issue of finding the optimal process settings in conjunction with a pre-determined control limit.

In particular, we consider a canning problem where cans are filled with an expensive ingredient called "fill". The amount put in a can is a random variable, with a mean (the "process mean") set by the canner. Filled cans are weighed: underfilled cans (those weighing below a specified limit) are emptied and refilled at the expense of a reprocessing cost (this might include, among other costs, the cost of production time lost); cans weighing above the specified limit are sold in a regular market for a fixed price.

If the process mean is set very high then the probability of underfilled cans becomes small, and the canner saves reprocessing cost at the expense of sending out too much ingredient at no benefit to him. (The cost of the excess fill is sometimes referred to as "give-away" cost). On the other hand, if the canner sets the process mean too low then he will save on the give-away cost but the reprocessing cost will increase because more cans will be underfilled. An immediate problem then is to set the process mean at the most economic level.

In addition, overfilled cans bring a fixed price in a regular market. Hence, it is not profitable to sell cans that are filled too excessively. The canner may have an option of reprocessing cans that are filled over a controllable upper limit, in which case it is desirable to know the most economic upper limit. Thus, the canner faces the combined problem of finding optimal values for both the process mean <u>and</u> an upper limit.

Bettes (1962) studied the problem of finding optimal values for both the process mean and the upper limit. However, his procedure is based on trial and error, is computationally tedious, and does not give accurate values.

Hunter and Kartha (1977) found the optimal process mean only, with the assumption that underfilled cans can be sold in a secondary market for a fixed price. Bisgaard, Hunter and Pallesen (1984) later suggested that this assumption is unrealistic because it implies that empty cans can be sold for as much as almost acceptably full ones, and instead looked at the situation where underfilled cans are sold in the secondary market at a price proportional to the amount of ingredient in a can.

However, in some cases (such as pharmaceuticals) the product may be sold only in the regular market. The secondary market may also be so far away that transportation and other related costs may make is prohibitive to sell a substandard product. In such cases if a can is underfilled then the only available alternative is to empty and refill it with an associated reprocessing cost. Golhar (1986) formulated such a problem and found the optimal process mean setting. But his assumption that the overfilled cans, no matter the amount of fill, be sold for a fixed price is unrealistic. At times, it may not be profitable to sell overfilled cans at a fixed price and there can be a controllable upper limit such that overfilled cans (i.e., weighing above this limit) can be reprocessed.

Here, we extend Golhar's (1986) model to a process where both the process mean <u>and</u> the upper limit can be controlled and show the superiority of such a policy over controlling the process mean only.

THE PROBLEM

Let $g(X;\mu,\sigma^2)$ represent a normal density function for the random fill X, and $f(X)$ and $F(X)$ be the standard normal density and distribution functions respectively. L is a pre-specified minimum weight limit and U a controllable upper limit. Thus, a can weighing between L and U is sold in a regular market, at a price A. If a can weighs below L or above U, it is emptied and refilled at the (reprocessing) cost R. C denotes the cost of the contents/unit. The objective of a canner is to find the optimal process setting μ^* and the upper limit U^* that will maximize the expected profit $\overline{P}$ per can sold.

THE SOLUTION

Let $P(X;\mu,U)$ denote the profit for a can sold with contents X and $\overline{P}(\mu,U)$ its expected value. If the amount of fill X is such that $L \le x \le U$ then the can is sold for A and the net profit is A - Cx. On the other hand, if the can weighs below L or above U then the can is emptied and is refilled at cost R. The refilled can will then realize the expected profit $\overline{P}(\mu,U)$. Hence, for a refilled can the net expected profit is $\overline{P}(\mu,U)$ - R. The profit per can sold is therefore:

$$P(X;\mu,U) = \begin{cases} A - CX & \text{for } L \le X \le U \\ \\ \overline{P}(\mu,U) - R & \text{otherwise} \end{cases}$$

Hence, the expected profit is:

$$\overline{P}(\mu,U) = \int_L^U (A-Cx)\ g(x;\mu,\sigma^2)\ dx + \int_0^L \{\overline{P}(\mu,U) - R\}\ g(x;\mu,\sigma^2)dx$$
$$+ \int_U^\infty \{\overline{P}(\mu,U) - R\}\ g(x;\mu,\sigma^2)dx \qquad (1)$$

Using the well known result that:

$$\int_L^U x\ g(x;\mu,\sigma^2)\ dx = \mu F\left[\frac{U-\mu}{\sigma}\right] - \sigma f\left[\frac{U-\mu}{\sigma}\right] - \mu F\left[\frac{L-\mu}{\sigma}\right] + \sigma f\left[\frac{L-M}{\sigma}\right] \qquad (2)$$

and letting $t_1 = \frac{U-\mu}{\sigma}$ and $t_2 = \frac{L-\mu}{\sigma}$ we get:

$$\overline{P}(\mu,U) = A - C\mu + R - \frac{R+C\sigma[f(t_2)-f(t_1)]}{F(t_2)-F(t_2)} \qquad (3)$$

Note that without an upper control limit, i.e., $U = \infty$, equation (3) becomes:

$$\bar{P}(\mu,\infty) = A - C\mu + R - \frac{R+C\sigma f(t_2)}{1-F(t_2)} \tag{4}$$

which is essentially the same relationship obtained by Golhar (1986). We will show later the degree to which the process with upper control (given by equation (3)) is more profitable than without upper control (equation (4)).

In order to find μ^* and U^* (the most economic levels of μ and U), we first show that $\bar{P}(\mu,U)$ in equation (3) is a concave function of μ and U. Since the last term of equation (3) has $f(t_1)$ and $f(t_2)$ in the numerator, and $F(t_1)$ and $F(t_2)$ in the denominator, it is difficult to show the desired concavity analytically. However, the appendix numerically establishes the concavity of $\bar{P}(\mu,U)$ over a wide range of values of μ and U. Taking the first derivative of $\bar{P}(\mu,U)$ with respect to U, and equating to zero, gives:

$$\frac{\delta \bar{P}(\mu,U)}{\delta U} = 0 = t_1\left[F(t_1)-F(t_2)\right] + f(t_1)-f(t_2) - M \tag{5}$$

where $M \equiv R/(C\sigma)$, a constant for any given process.

Similarly, equating the first derivative of $\bar{P}(\mu,U)$ with respect to μ to zero, and combining the result with equation (5), we get:

$$\frac{\delta \bar{P}(\mu,U)}{\delta \mu} = 0 = F(t_1)-F(t_2)-f(t_2)\left[t_1-t_2\right] \tag{6}$$

Optimal values of t_1 and t_2 can be obtained numerically by solving equations (5) and (6) simultaneously.

RESULTS

Table 1 gives t_1* and t_2*, the optimal values for t_1 and t_2 solving equations (5) and (6) as a function of M. The optimal process setting μ^* is obtained through the relation $\mu^* = L - \sigma t_2^*$ and the optimal upper limit U^* is obtained via $U^* = \mu^* + \sigma t^*_1$. For M between 0.1 and 2, these optimal values are plotted (in units of σ) in figure 1 .

A convenient way of examining the resulting expected profit is to look at the excess over what would be obtained if each can could be filled to exactly L (achievable only when $\sigma \to 0$), in which case the profit would be A - CL. We can define the minimum expected excess cost per can as

$$\bar{E} = A - CL - P(\mu^*, U^*)$$

For different M, values of E are computed (in units of $c\sigma$) and are given in table 2. For M between 0.1 and 3, Figure 2 shows the behavior of $\bar{E}$.

To see how the parameters affect μ^*, U^* and the resulting costs, consider the following example: suppose initially C = \$0.5 per ounce, R = \$0.2, σ =.4 ounces, and L = 3 ounces. The constant M = 1, and from table 1, $t_1* = 1.657$ and $t_2* = -.75$. Hence, $\mu^* = L + .75\,\sigma = 3.30$ ounces and $U^* = L + 2.425\,\sigma = 3.97$ ounces. From table 2, this results in a cost per can of $\bar{E}$ = (.5)(.4)(1.409) = \$0.28 per can. Now, suppose due to process

Table 1.

Optimal values of t1 and t2 for a given M.

M	t_1*	t_2*
0.1	0.478	-0.236
0.2	0.682	-0.334
0.3	0.843	-0.410
0.4	0.983	-0.474
0.5	1.111	-0.530
0.6	1.230	-0.581
0.7	1.342	-0.628
0.8	1.450	-0.671
0.9	1.555	-0.711
1.0	1.657	-0.750
1.1	1.757	-0.786
1.2	1.855	-0.820
1.3	1.952	-0.853
1.4	2.049	-0.884
1.5	2.145	-0.913
1.6	2.240	-0.942
1.7	2.335	-0.969
1.8	2.430	-0.995
1.9	2.524	-1.020
2.0	2.619	-1.044
2.2	2.809	-1.088
2.4	2.998	-1.130
2.6	3.189	-1.168
2.8	3.380	-1.204
3.0	3.572	-1.237
3.2	3.764	-1.268
3.4	3.957	-1.298
3.6	4.151	-1.325
3.8	4.344	-1.351
4.0	4.539	-1.375
4.5	5.026	-1.432
5.0	5.515	-1.482
5.5	6.006	-1.526
6.0	6.498	-1.567
7.0	7.483	-1.639
8.0	8.472	-1.700
9.0	9.462	-1.754
10.0	10.454	-1.801

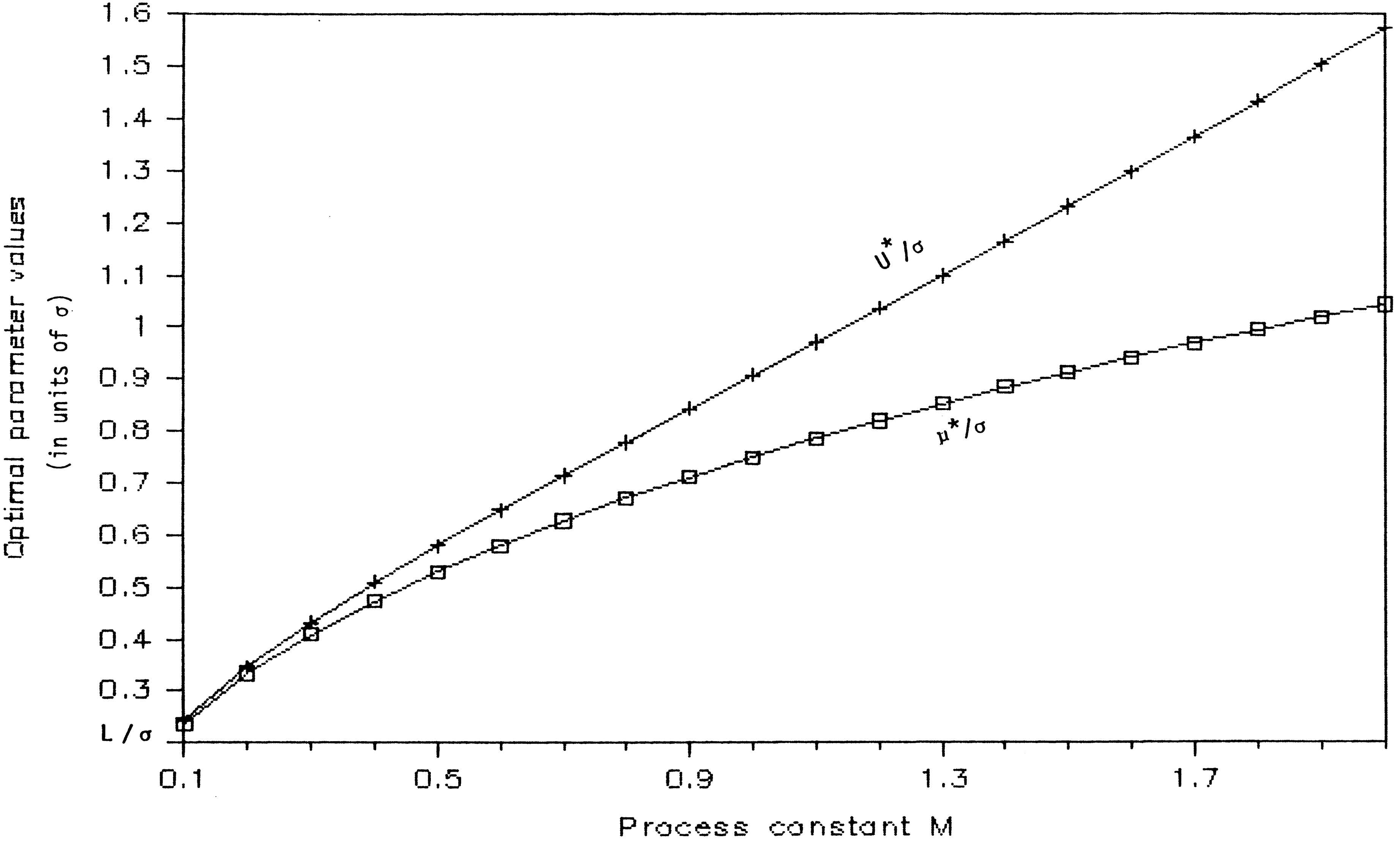

Figure 1: Optimal U and μ values as functions of M

innovations the process standard deviation is halved. Therefore, $M = 2$ which gives $t_1{}^* = 2.619$ and $t_2{}^* = -1.044$, with corresponding $\mu^* = 3.10$ and $U^* = 3.36$, and $\bar{E} = (.5)(.2)(1.662) =$ \$0.16 per can, a 43% saving.

PROFITABILITY OF A PROCESS WITH UPPER CONTROL LIMIT

It is of interest to compare the expected profit obtained above to the case where there is no upper control limit, in which case the objective is to find the optimal value of $t_2 = (L-\mu^*)/\sigma$ that will maximize the expected profit given by equation (4). Golhar (1986) has solved this problem, with results given in table 2 and plotted in figure 3. Notice that having an upper limit gives a higher expected profit, with the advantage decreasing as M increases. An upper limit allows a tighter control on the fill, resulting in higher profit.

However, controlling both parameters (μ and U) might be expensive and/or time consuming. These results allow the canner to determine the value of seeking to control both parameters, as a function of M.

APPROXIMATIONS TO OBTAIN OPTIMAL VALUES FOR THE PARAMETERS

In this section we develop simple approximate relationship between $t_1{}^*$ and $t_2{}^*$ as a function of M, that can be used as an alternative to table 1 for $M < 1$.

It is reasonable to assume that the process constant M for real processes would be small. (For values of M greater than

Table 2

Comparative evaluation of the advantage of being able to have an upper central limit. $\bar{E}$ = excess cost per can.

M	No Upper Limit		Upper Limit Available		
	t_2^*	$\bar{E}$ (units of $c\sigma$)	t_1^*	t_2^*	$\bar{E}$ (units of $c\sigma$)
0.1	0.364	0.858	0.478	-0.236	0.613
0.2	0.059	0.998	0.682	-0.334	0.816
0.3	-0.126	1.091	0.843	-0.410	0.954
0.4	-0.261	1.165	0.983	-0.474	1.058
0.5	-0.366	1.224	1.111	-0.530	1.141
1.0	-0.701	1.433	1.657	-0.750	1.409
1.5	-0.899	1.565	2.145	-0.913	1.559
2.0	-1.040	1.664	2.619	-1.044	1.663
2.5	-1.149	1.742	3.094	-1.149	1.742
3.0	-1.237	1.808	3.572	-1.237	1.808
3.5	-1.311	1.865	4.054	-1.311	1.865
4.0	-1.375	1.914	4.539	-1.375	1.913
5.0	-1.482	1.996	5.515	-1.482	1.998
6.0	-1.567	2.065	6.498	-1.567	2.065
7.0	-1.639	2.121	7.483	-1.639	2.121
8.0	-1.700	2.172	8.472	-1.700	2.172
9.0	-1.754	2.215	9.462	-1.754	2.215

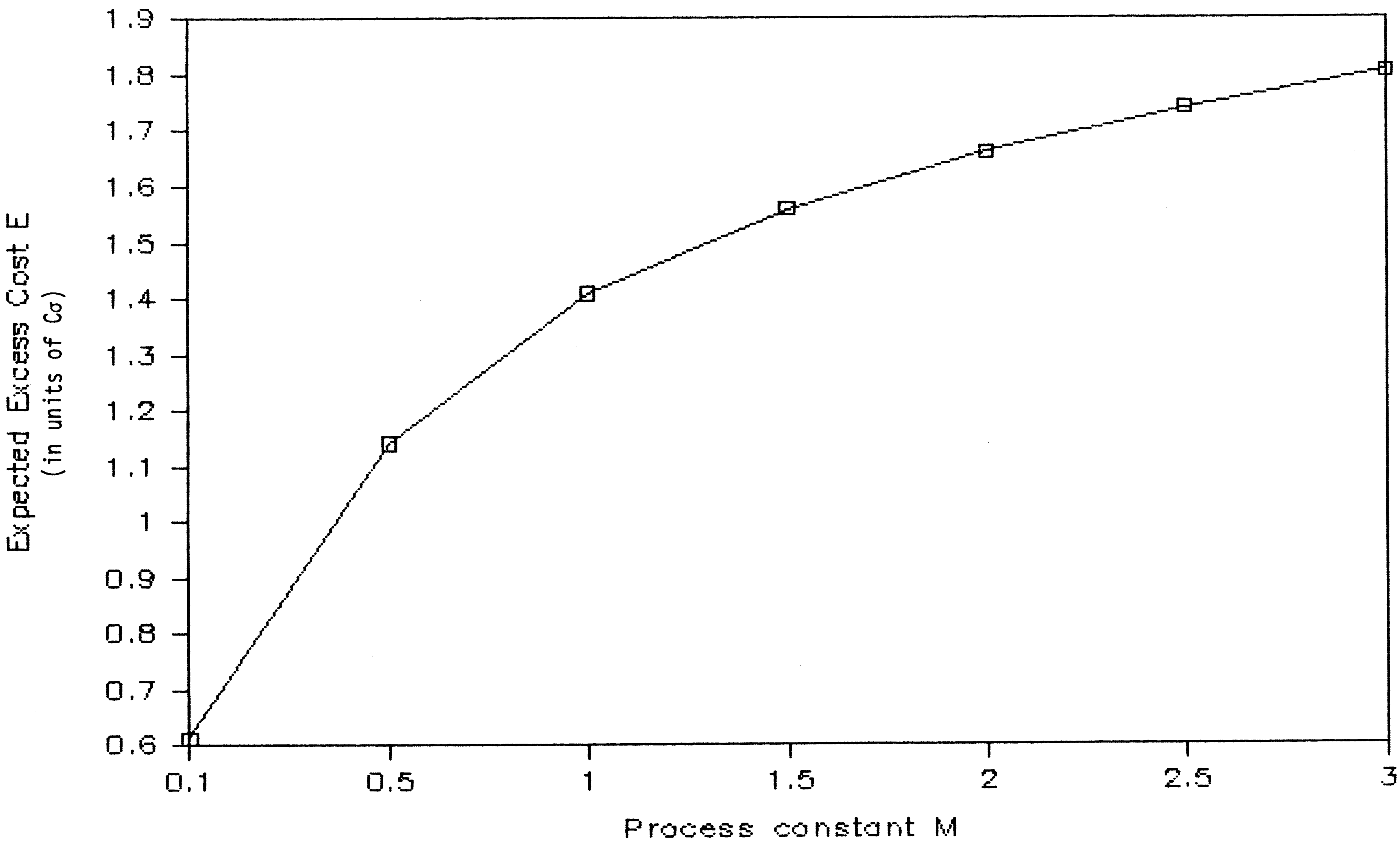

Figure 2: Expected excess cost as a function of M
when upper limit is available

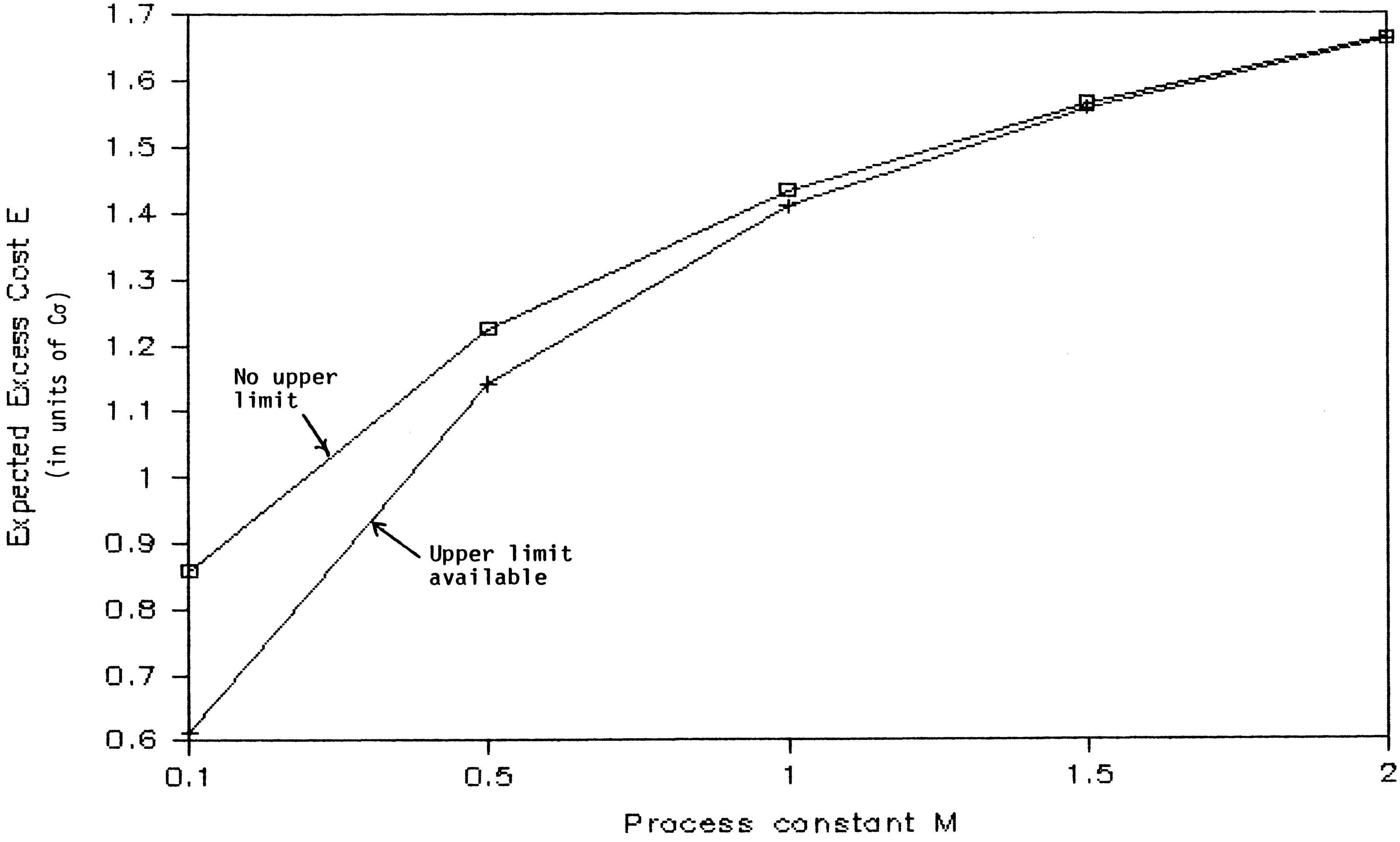

Figure 3: Expected excess costs for a process with and without upper limit

about 2, the optimal process setting is more than one standard deviation above the minimum weight limit L, a situation that would not be readily tolerated in most practical situations). It can be seen from table 1 that as M approaches 0 the optimal values of t_1 and t_2 also approach 0. Using Taylor series expansions, the standard normal density and distribution functions can be approximated for $y \rightarrow 0$, as

$$f(y) \approx \frac{1}{\sqrt{(2\pi)}} \left[1 - \frac{Y^2}{2} \right]$$

and $$F(y) \approx \frac{1}{2} + \frac{(Y - Y^3/6)}{\sqrt{(2\pi)}}$$

Let y_1 and y_2 represent the values of t_1^* and t_2^* respectively, obtained using this approximation.

Equation (5) can then be rewritten as :

$$\frac{1}{\sqrt{(2\pi)}} Y_1(Y_1-Y_2) + \frac{1}{\sqrt{(2\pi)}} \left[\frac{Y_2^2-Y_1^2}{2} \right] \approx M \tag{7}$$

Solving equation (7) we get:

$$Y_1 - Y_2 = \sqrt{(2M')} \tag{8}$$

where $M' = \sqrt{(2\pi)}\, M$

Similarly, equation (6) becomes :

$$\frac{1}{\sqrt{(2\pi)}} \left[Y_1 - Y_1^3/6 - Y_2 + Y_2^3/6 \right] - \frac{1}{\sqrt{(2\pi)}} \left[1-Y_2^2/2\right](Y_1 - Y_2)=0, \tag{9}$$

which reduces to :

$$3\, Y_1\, Y_2^2 = 2Y_2^3 + Y_1^3 \tag{10}$$

Simultaneously solving equations (8) and (10) gives:

$$Y_2 = -0.746 \sqrt{M} \tag{11}$$

and

$$Y_1 = -2\ Y_2 \tag{12}$$

For values of $M \leq 2$, the values of y_1 and y_2 are plotted against actual t_1^* and t_2^* (obtained from table 1) in figure 4. It is seen that the approximations fit very well. In particular, approximation (11) gives values of t_2^* that are within 1% of the actual, for $M \leq 2$.

DISCUSSION

The canning problem is in general subject to ever-changing values of the process constants R, C and σ. With changes in technology the reprocessing cost R and fill precision σ should change. The cost of ingredients, C, is a volatile function of market forces and should change often. Finally, with aging of the process itself the precision σ will change. Industry should, in turn, respond by adjusting the optimal process setting μ^* and the upper limit U*. Table 1 and relations (11) and (12) should prove useful for finding these settings without carrying out the detailed calculations.

However, adjusting these two parameters simultaneously could prove to be expensive and/or time consuming; in which case industry might want to cost-out the value of having the capability of controlling the upper limit. Table 2 can then be used to determine the cost effectiveness of the upper control limit.

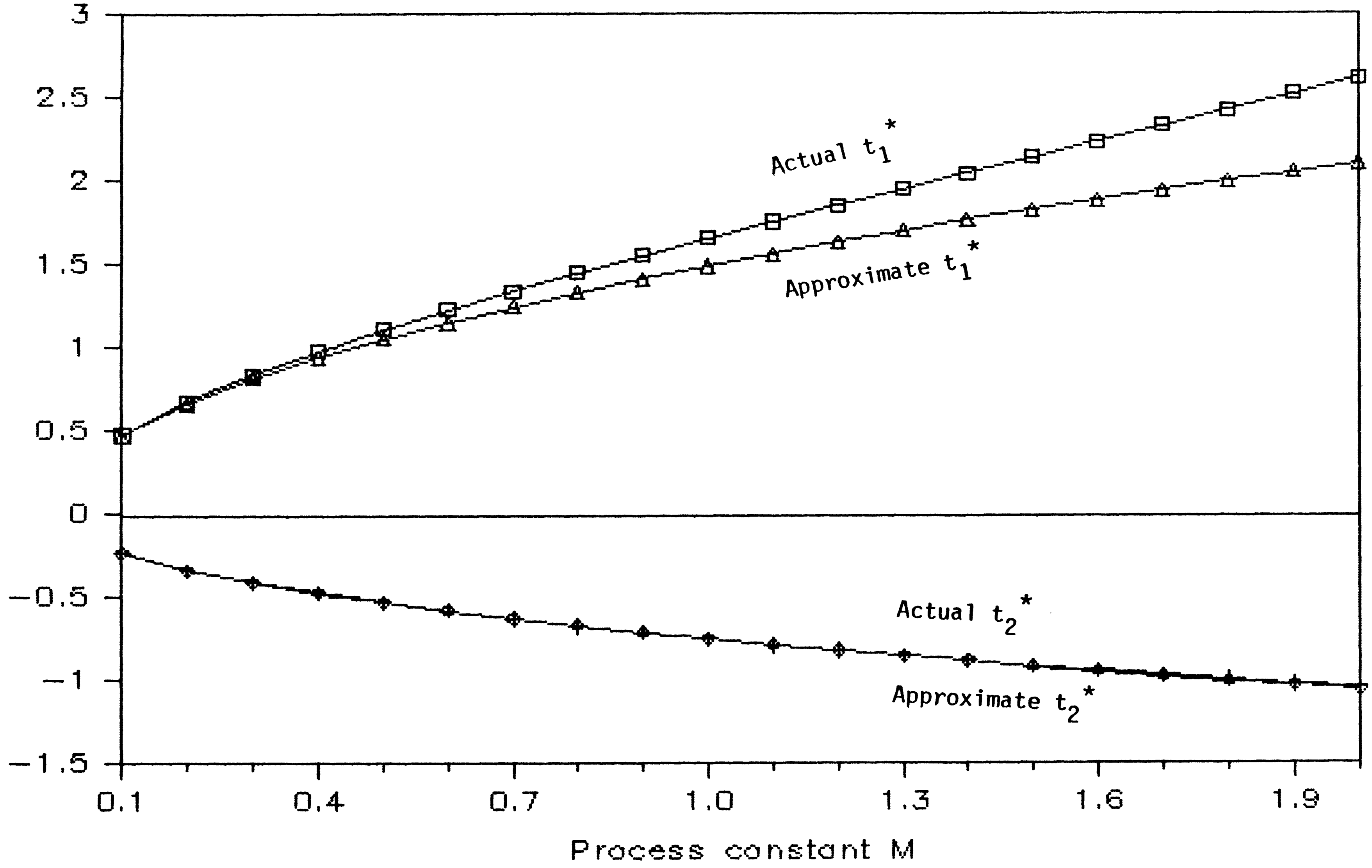

Figure 4: Actual and approximate t_1^*, t_2^* as functions of M values

Appendix

We show numerically that the function :

$$\bar{P}(\mu,U) = A - C\mu + R - \frac{R + C\sigma[f(t_2)-f(t_1)]}{F(t_1) - F(t_2)} \tag{A1}$$

is concave over parameters U and μ, where $t_1 = (U-\mu)/\sigma$ and $t_2 = (L-\mu)/\sigma$.

Hence, we show that the function :

$$S_1(\mu,U) = \frac{-\bar{P}(X;\mu,U) + A + R}{C\sigma} = \frac{\mu}{\sigma} + \frac{\frac{R}{C\sigma} + f(t_2)-f(t_1)}{F(t_1)-F(t_2)} \tag{A2}$$

is convex over U and μ. This is equivalent of showing :

$$S_2 = S_1 - \frac{L}{\sigma} = -t_2 + \frac{M + f(t_2)-f(t_1)}{F(t_1)-F(t_2)} \tag{A3}$$

is convex with respect to t_1 and t_2.

To show that equation (A3) is convex with respect to t_1, we fix t_2 and show that :

$$S_3(t_1) \equiv \frac{M + f(t_2)-f(t_1)}{F(t_1)-F(t_2)} \tag{A4}$$

is convex over t_1.

For different t_1, $S_3(t_1)$ was computed for $0 \leq M \leq 10$ and fixed $-0.1 \leq t_2 \leq -2.1$ (for a fixed $t_2 = -0.6$, sample computations are given in table A1) . It was seen that $S_3(t_1)$ was convex over t_1 for a wide range of M and t_2 values.

To show that equation (A3) is convex with respect to t_2, we first express t_1 in terms of t_2 :

$$t_1 = \frac{U-L}{\sigma} + t_2 = K + t_2 \tag{A5}$$

Hence equation (A3) becomes :

$$S_2(t_2) = -t_2 + \frac{M + f(t_2) - f(t_2 + K)}{F(t_2 + K)-F(t_2)} \qquad (A6)$$

For different t_2, $S_2(t_2)$ values were computed for $0.1 \le M \le 10$ and $0.1 \le K \le 3$. (Table A2 gives a sample of these computations for a fixed $K = 0.5$). It was seen that equation (A6) was convex over t_2 for a wide range of M and K values.

Table A1

Sample computations of $S_3(t_1)$ for a fixed t_2 = -0.6

T_1 \ M	0.5	1.0	2.0	3.0	4.0	5.0	7.5	10.0
0	1.92416	4.13949	8.57015	13.0008	17.4315	21.8621	32.9388	44.0154
0.4	1.22003	2.53202	5.15601	7.77999	10.404	13.028	19.5879	26.1479
0.8	1.05787	2.03101	3.97729	5.92358	7.86986	9.81614	14.6818	19.5476
1.2	1.04657	1.86544	3.50317	5.14091	6.77864	8.41637	12.5107	16.605
1.6	1.07662	1.82189	3.3124	4.80296	6.29349	7.78403	11.5104	15.2367
2.0	1.1086	1.81994	3.24261	4.66529	6.08797	7.51065	11.0673	14.624
2.4	1.13008	1.82694	3.22067	4.6144	6.00813	7.40185	10.8862	14.3705
2.8	1.14135	1.83282	3.21575	4.59869	5.98162	7.36455	10.8219	14.2792
3.2	1.14599	1.83564	3.21495	4.59426	5.97357	7.35288	10.8012	14.2494
3.6	1.14764	1.83682	3.21518	4.59354	5.9719	7.35026	10.7962	14.2421
4.0	1.14798	1.83697	3.21495	4.59293	5.97091	7.34889	10.7938	14.2388

Table A2

Sample computations of $S_2(t_2)$ for a fixed K = 0.5

T_1 \ M	0.5	1.0	2.0	3.0	4.0	5.0	7.5	10.0
-3.5	454.81	909.355	1818.45	2727.54	3636.63	4545.72	6818.45	9091.17
-3.0	102.368	204.409	408.491	612.572	816.654	1020.74	1530.94	2041.14
-2.5	30.4239	60.5444	120.785	181.026	241.267	301.508	452.111	602.713
-2.0	11.6471	23.0108	45.738	68.4653	91.1926	113.92	170.738	227.55
-1.5	5.71705	11.1577	22.0391	32.9205	43.8019	54.6833	81.8868	109.09
-1.0	3.60284	6.94062	13.6162	20.2918	26.9673	33.6429	50.3318	67.0207
-0.5	2.86618	5.47714	10.6991	15.921	21.1429	26.3649	39.4197	52.4745
0	2.85575	5.46672	10.6887	15.9106	21.1325	26.3544	39.4093	52.4641
0.4	3.35193	6.4672	12.6977	18.9283	25.1588	31.3893	46.9656	62.542
0.8	4.57205	8.9161	17.6042	26.2923	34.9804	43.6685	65.3887	87.109
1.2	7.31258	14.4048	28.5892	42.7736	56.958	71.1424	106.603	142.064
1.6	13.7642	27.3143	54.4146	81.5148	108.615	135.715	203.466	271.217
2.0	30.317	60.4375	120.678	180.919	241.16	301.401	452.004	602.606
2.4	79.575	158.94	317.67	476.401	635.131	793.861	1190.69	1587.51
2.8	238.244	476.34	952.531	1428.72	1904.91	2381.1	3571.58	4762.06
3.2	1000.73	2000.74	4000.77	6000.8	8000.82	10000.8	15000.9	20001

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