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Multigrid third-order least-squares solution of Cauchy–Riemann equations on unstructured triangular grids

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SUMMARY

In this paper, a multigrid algorithm is developed for the third-order accurate solution of Cauchy–Riemann equations discretized in the cell-vertex finite-volume fashion: the solution values stored at vertices and the residuals defined on triangular elements. On triangular grids, this results in a highly overdetermined problem, and therefore we consider its solution that minimizes the residuals in the least-squares norm. The standard second-order least-squares scheme is extended to third-order by adding a high-order correction term in the residual. The resulting high-order method is shown to give sufficiently accurate solutions on relatively coarse grids. Combined with a multigrid technique, the method then becomes a highly accurate and efficient solver. We present some results to demonstrate its accuracy and efficiency, including both structured and unstructured triangular grids. Copyright © 2006 John Wiley & Sons, Ltd.

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KEY WORDS: multigrid; high-order; unstructured grids; Cauchy–Riemann

1. INTRODUCTION

In this paper, we present a multigrid third-order accurate Cauchy–Riemann solver on triangular grids. This work is motivated by the wish to develop a highly accurate and efficient algorithm for solving the elliptic (Cauchy–Riemann type) subsystem of the two-dimensional Euler equations. The decomposition approach, where a partial differential equation is split into elliptic and hyperbolic equations, is an attractive strategy as we can apply a technique specially tailored to the physics of each subproblem. Such a strategy has been shown to yield highly accurate numerical solutions over the conventional finite-volume scheme [1–4]. It has also been shown that such decomposition can lead to optimal multigrid convergence of the Euler equations on structured

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grids [5], employing suitable grid hierarchy for each part: full-coarsening for elliptic part and semi-coarsening for hyperbolic part. This work contributes toward the same optimal convergence of the Euler computation on unstructured grids.

The least-squares method has been successfully used to solve the elliptic part of the Euler equations [6], demonstrating its superior properties over the Lax–Wendroff-type solver, which is known in particular to break the symmetry of the solution and not to attain a typical multigrid convergence rate [1, 7]. The advantage of the least-squares method is not only that it provides a symmetric stencil, but also that the spurious entropy generation can be suppressed, in a natural way, in the form of a constrained minimization [1]. Also, an adaptive grid technique can be devised, again in the framework of minimization [8, 9]. Moreover, it can be upgraded to higher-order by a simple modification; the third-order scheme has already been developed in Reference [6] following the work by Caraeni and Fuchs [10].

This work focuses on the efficiency of the least-squares method, and demonstrate that the method can achieve an optimal convergence when combined with a multigrid method. In particular, we show that the required CPU time to obtain a converged solution is $O(N)$ where N is the number of unknowns. This means tremendous saving in CPU time over direct solvers which require $O(N^3)$ especially for large-scale problems. Similar work has been reported by Borzi *et al.* [11]. Our method has three distinctive features. First, our method is third-order accurate while their method is second-order. Second, our least-squares formulation is directly applicable to unstructured grids while such extension is not clear for their method. Third, we restrict the nodal residual to define the coarse grid problems while they restrict the cell-residual. We propose also a simple fix to resolve the convergence difficulty with bad coarsening ratio that may arise for unstructured grids.

2. LEAST-SQUARES SCHEME

Consider the Cauchy–Riemann equations

$$\partial_x u + \partial_y v = 0, \quad \partial_x v - \partial_y u = 0 \tag{1}$$

where (u, v) is taken to be two velocity components. On the boundary, zero normal component $(u_n = 0)$ is enforced. To discretize the system, we triangulate the domain of interest into a set of triangles $\{T\}$. Storing the solutions at the vertices, we define the residual on a triangular element T as integrals of (1). Assuming piecewise linear variation of u and v within the element, we obtain the second-order version of the residuals.

$$\Theta_T = \frac{1}{2}\sum_{i\in\{j_T\}} (u_i \Delta y_i + v_i \Delta x_i), \quad \Omega_T = \frac{1}{2}\sum_{i\in\{j_T\}} (-v_i \Delta y_i + u_i \Delta x_i) \tag{2}$$

where $\{j_T\}$ is the set of vertices forming the triangle T and $\Delta\{\}_i$ denotes a difference taken counterclockwise along the side opposite to j. On triangular grids, the number of triangles is typically twice the number of vertices. This results a highly overdetermined problem, and therefore residuals cannot be driven to zero everywhere. We then seek to find a solution that minimizes the residuals in a least-squares norm in the form

$$\mathscr{F} = \sum_{T\in\{T\}} F_T = \frac{1}{2}\sum_{T\in\{T\}} (\Theta_T^2 + \Omega_T^2) \tag{3}$$

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We remark that by weighting F_T with $1/S_T$, where S_T is the area of the triangle T, we obtain the standard Galerkin discretization of the associated Laplace equations for u and v. This discretization however loses its accuracy, yielding even less than first-order accuracy. Although equivalent on uniform grids, the norm without such a weight as in (3) cures the problem, and retain the formal accuracy of the method. See Reference [9] for details.

To minimize the norm, we employ the steepest descent method,

$$u_j^{n+1} = u_j^n - c_j \frac{\partial \mathscr{F}}{\partial u_j} = u_j^n - \frac{c_j}{2} \sum_{T \in \{T_j\}} (\Delta y_T \Theta_T - \Delta x_T \Omega_T) \tag{4}$$

similarly for v, where $\{T_j\}$ is the set of triangles that share the vertex j, c_j is a constant and $\Delta\{\}_T$ denotes a difference taken counterclockwise along the side opposite to j on the triangle T. This method is however very slow to converge. To improve on it, we use Newton's method, but with only diagonal elements of the Hessian matrix, which defines c_j as

$$c_j = \omega \Big/ \frac{\partial^2 \mathscr{F}}{\partial u_j^2} = \omega \Big/ \frac{\partial^2 \mathscr{F}}{\partial v_j^2} = \omega / L_j^2, \quad L_j^2 = \frac{1}{4} \sum_{T \in \{T_j\}} \{(\Delta x_T)^2 + (\Delta y_T)^2\} \tag{5}$$

where ω is a relaxation parameter. On the boundary where the tangency condition must be imposed, we simply ignore the updates in the normal component of the solution.

To extend the method to third-order, we first recover the solution gradients at each vertex by the Green–Gauss formula, interpolate the solutions at the midpoint of each edge (Hermite interpolation), and then evaluate the residuals to the fourth-order accuracy. The results are

$$\Theta_T^{\mathrm{H}} = \Theta_T - \frac{1}{12} \sum_{\text{edges}} (\Delta p \Delta y - \Delta q \Delta x), \quad \Omega_T^{\mathrm{H}} = \Omega_T - \frac{1}{12} \sum_{\text{edges}} (\Delta q \Delta y + \Delta p \Delta x) \tag{6}$$

where Δ on the right-hand side denotes the difference taken clockwise along the edge, and p_i and q_i are the gradients evaluated at vertex i projected onto the edge. These residuals are however third-order accurate in reality because the gradient estimate is at best second-order. As clearly seen, the second terms on the right can be considered as a third-order correction to the second-order residual. With this correction taken as a numerical correction, the solution procedure remains identical to the second-order method except that we replace the second-order residuals (Θ_T, Ω_T) by the third-order counterparts $(\Theta_T^{\mathrm{H}}, \Omega_T^{\mathrm{H}})$ in the update formulas (4). See Reference [9] for more details.

3. MULTIGRID ALGORITHM

We now describe the multigrid algorithm that accelerates the convergence of the method to reach the ultimate $O(N)$ convergence. Coarse grids are generated by removing every other point on a fine grid, resulting vertex-nested structure. Coarse grid problems are then independently discretized on their grids. It is crucial at this point that the source term in the coarse grid problem (i.e. the restricted nodal residual) must be properly scaled [12]. Defining the nodal residual on a fine grid

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$\{T^h\}$ to be restricted onto a coarse grid $\{T^{2h}\}$ by

$$R_j^h = \frac{\sum_{T\in\{T_j^h\}} (\Delta y_T \Theta_T^{\mathrm{H}} - \Delta x_T \Omega_T^{\mathrm{H}})}{2(L_j^h)^4} + R^{h/2}|_j^h \tag{7}$$

where $R^{h/2}|_j^h$ is the nodal residual restricted onto the node j on $\{T^h\}$ from the next finer grid (which is zero if $\{T^h\}$ is the finest grid), we have the update formula for the solution on the coarse grid,

$$(u_j^{2h})^{n+1} = (u_j^{2h})^n - \omega(L_j^{2h})^2[R_j^{2h} + \alpha R^h|_j^{2h}] \tag{8}$$

similarly for the other variable. Note that the coarse grid problems can be badly scaled if the coarsening ratio is locally too large. We may modify the coarse grids to fix this problem, but instead we modify the algorithm. The parameter α has been introduced in (8) precisely for this purpose and defined by

$$\alpha = \begin{cases} 1, & (L_j^{2h}/L_j^h)^2 \leqslant 4.0 \\ 4(L_j^h/L_j^{2h})^2, & (L_j^{2h}/L_j^h)^2 > 4.0 \end{cases} \tag{9}$$

which effectively limit the maximum ratio to 4.

As for intergrid transfer operators, following Reference [13], we construct node-based intergrid transfer operators as follows. Prolongation is based on injection and linear interpolation,

$$u^{2h}|_j^h = \begin{cases} u_j^{2h} & \text{if } j \text{ is a nested vertex} \\ \sum\limits_{i\in\{j_T\}} u_i^{2h} w_i^{\mathrm{T}} & \text{otherwise} \end{cases} \tag{10}$$

where $u^{2h}|_j^h$ denotes the value interpolated from a coarse grid $\{T^{2h}\}$ to a fine grid $\{T^h\}$, T is a coarse-grid triangle enclosing a fine-grid node j and $\{w^{\mathrm{T}}\}$ is a set of weights for the linear interpolation over that triangle. Restriction is taken as the transpose of the prolongation

$$R^h|_j^{2h} = \frac{R_j^h + \sum_{T\in\{T_j^{2h}\}} \sum_{\{i_T\}} R_i^h w_j^{\mathrm{T}}}{1 + \sum_{T\in\{T_j^{2h}\}} \sum_{\{i_T\}} w_j^{\mathrm{T}}} \tag{11}$$

where $\{i_T\}$ is a set of fine-grid vertices in a coarse-grid triangle T. These operators are accurate enough to satisfy the well-known necessary condition for the multigrid efficiency [12] for the Cauchy–Riemann equations.

4. RESULTS

For all the results here, we use W-cycle with two relaxation sweeps on each grid level after both restriction and prolongation. The method is taken to be converged when the actual L_2 errors of the solutions do not change more than 1% per cycle. CPU time was measured on a PC with Pentium III 800 MHz processor. As usual, work unit is defined as one relaxation sweep on the finest grid (3.06 GHz for unstructured grid cases).

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4.1. Structured grids

The first test case is the classical flow whose complex potential is given by

$$\mathscr{W} = \phi + \mathrm{i}\psi = z^n \tag{12}$$

where $\mathrm{i} = \sqrt{-1}$, ϕ is the velocity potential, ψ is the streamfunction, $z = x + \mathrm{i}y$, and n is an integer. In this work, we take $n = 7$. Taking $u = \phi$ and $v = \psi$, we solve the Cauchy–Riemann equations in a square domain $\{(x, y) | 0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant 1\}$. On the boundary, we specify the exact solution for both variables as boundary conditions, setting up a Dirichlet problem. The grid is a uniform triangular grid as shown in Figure 1. The grids used are 16×16, 32×32, 64×64, 128×128, 256×256, and 512×512 with the coarsest grid is taken to be 2×2 in every case, resulting 4, 5, 6, 7, 8, and 9 multigrid levels, respectively. As a smoother, we employ the weighted Jacobi iteration with $\omega = 4/5$ for this problem.

Figure 2 shows that the method is indeed third-order accurate, where the solid line is the line of slope 3 and the circles are the results obtained. Figure 3 shows the convergence history: the

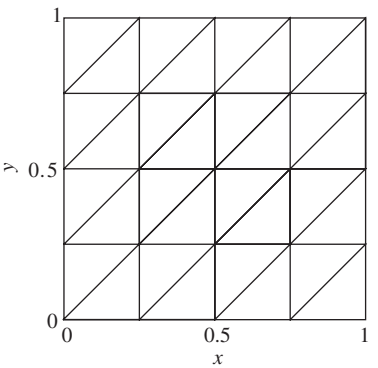

Figure 1. A 4×4 uniform triangular grid.

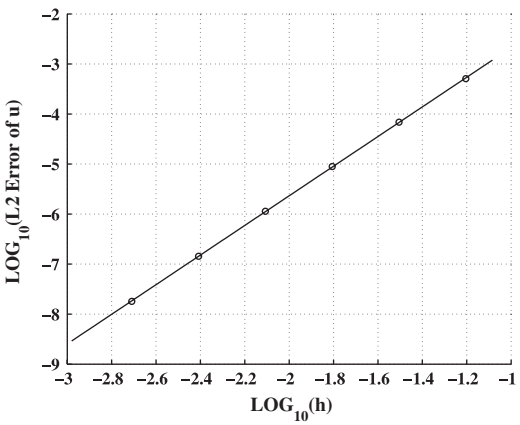

Figure 2. Error convergence.

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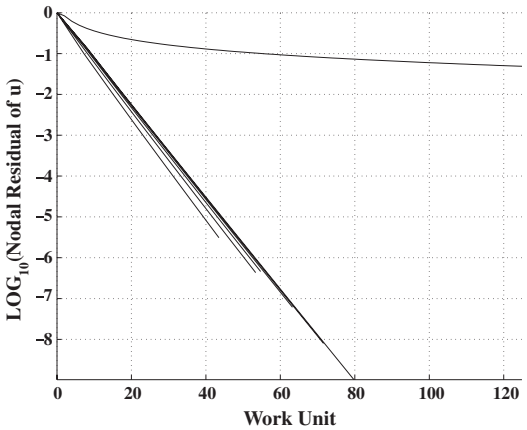

Figure 3. Convergence history.

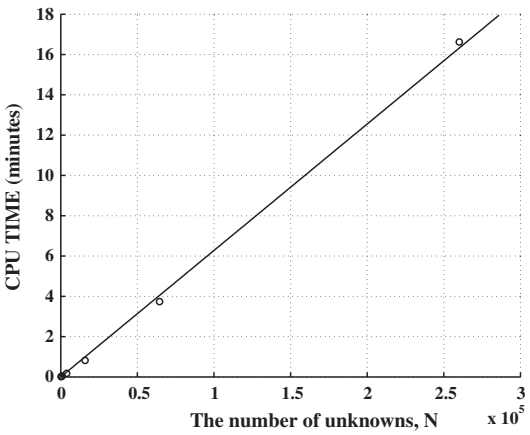

Figure 4. CPU time vs grid size.

history of the L_2 norm of the nodal residual (scaled by the initial one) of u versus work unit. The straight lines are the multigrid results. We see that the method converges for all the grids well less than 100 work units, and also that the residuals had to be reduced further as the grid gets finer to reach the level of discretization error as well known. The isolated flat curve indicates the residual history for a single grid case for 64×64 grid, which took about 2200 work units to reach the same error level as that of the multigrid method for the same grid. In terms of the number of W-cycles, the multigrid method took 7, 8, 8, 9, 10, and 11 cycles for the grid sizes 16×16, 32×32, 64×64, 128×128, 256×256, and 512×512, respectively, yielding the convergence rate of about 0.12 for all the cases. Figure 4 is the plot of the CPU time versus the number of unknown. The data fit the straight line ($y = (6.2766\text{E-}05)x$), meaning that the method achieves $O(N)$ convergence in which sense the method is indeed optimal.

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4.2. *Unstructured grids*

The algorithm was first tested for flows around a cylinder (with circulation 2π) and a Joukowsky airfoil at an angle of attack 10°. In both cases, the grids are O-type 40×20, 80×40, and 160×80 grids with unit chord length, extending to 10-chord length to the outer boundary (Figures 5 and 9). As boundary conditions, the exact solution is given and fixed on the outer boundary. On the inner boundary which is a solid surface, we simply ignore the solution updates in the normal direction. Or equivalently, we project the solution updates in the direction tangent to the surface, so that there always exists no flow across the boundary. In the airfoil case, we impose the Kutta–Joukowsky condition by setting $v = 0$ at the trailing edge. Because the airfoil is symmetric and the trailing edge is cusped, u should not be set zero and so it is computed by the method. For these problems, we employ the successive-over-relaxation scheme with $\omega = 1.6$.

Figures 6 and 10 show the error convergence. Third-order accuracy is observed for both cases, where the solid line has the slope of 3. The method converged in about 12 W-cycles. The average

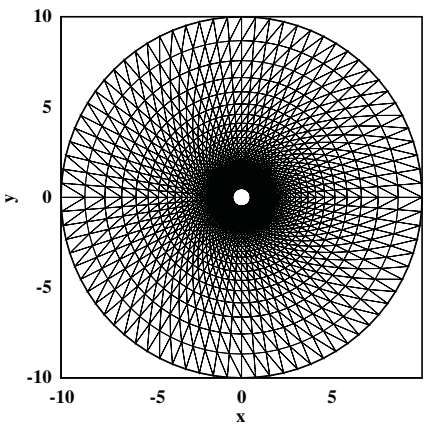

Figure 5. The 80×40 O-grid around a cylinder.

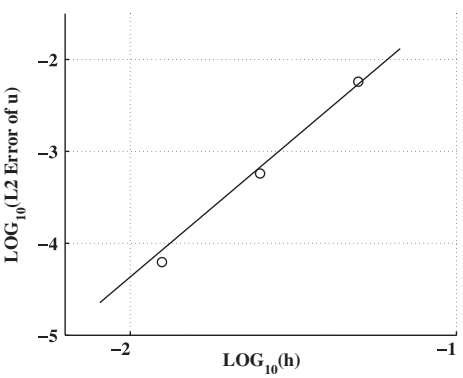

Figure 6. Error convergence: cylinder case.

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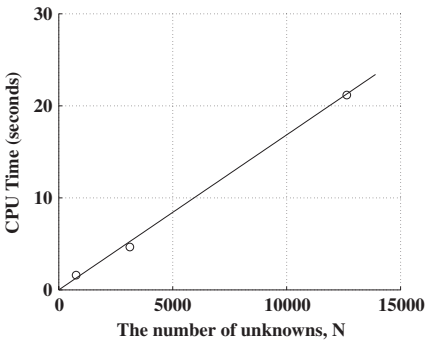

Figure 7. CPU time vs grid size: cylinder case.

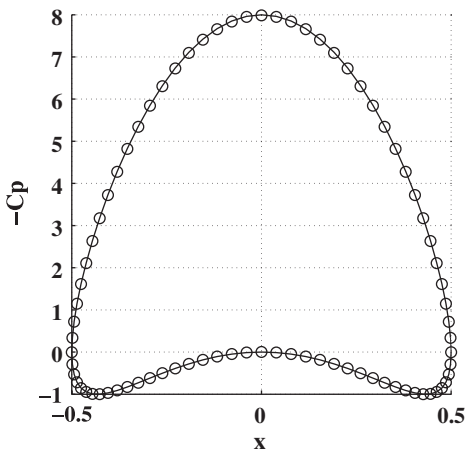

Figure 8. C_p: cylinder: 80×40 grid.

convergence rates are 0.35 for the cylinder case and 0.6 for the airfoil case, which are very small compared with that of the single grid case for which the rate is about 0.996 for both cases. Figures 7 and 11 shows that the method achieves $O(N)$ convergence in both cases as the data fit the line of slope 1, in which sense the method is optimal.

Another test case is the same flow around the same airfoil on unstructured grids. A grid was generated by 'Delaundo' developed by Muller [7] with 61 468 triangles and 320 nodes to represent the airfoil. Coarse grids were generated by removing every other points, resulting 5 multigrid levels. To determine $O(N)$ and error convergence, two other grids were used which are simply the next two grids in the 5 levels: 3658 and 15 116 triangles with 80 and 160 nodes to represent the airfoil, and 3 and 4 multigrid levels, respectively. The method converged in about 15 W-cycles. The average convergence rate is 0.5. Again, $O(N)$ convergence is observed (Figure 15). Error convergence is plotted in Figure 14 where both results of the third-order method and the second-order method are compared. The mesh size h is a simple average over the triangles of an averaged-height of a triangle, which is twice the area divided by a harmonic mean of three sides.

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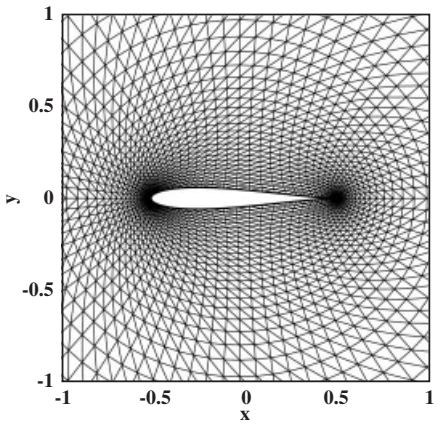

Figure 9. The 80 × 40 O-grid around an airfoil.

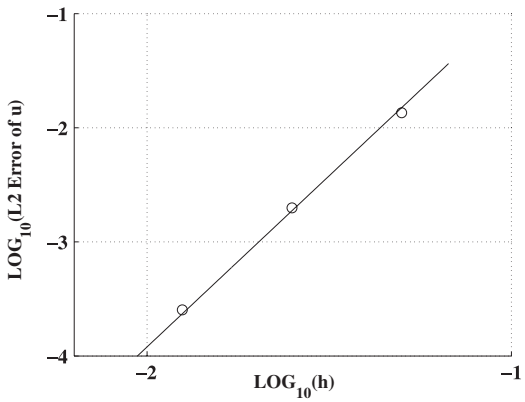

Figure 10. Error convergence: airfoil case.

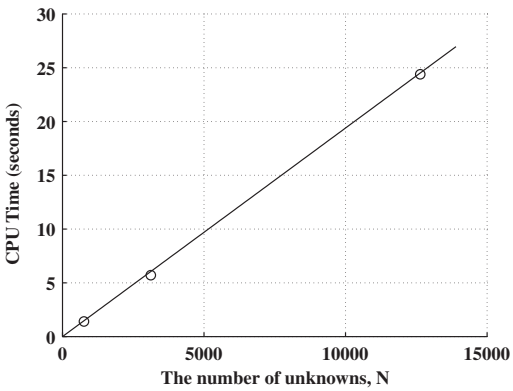

Figure 11. CPU time vs grid size: airfoil case.

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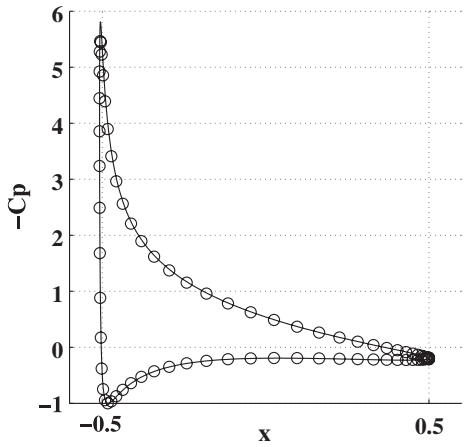

Figure 12. C_p: airfoil: 80×40 grid.

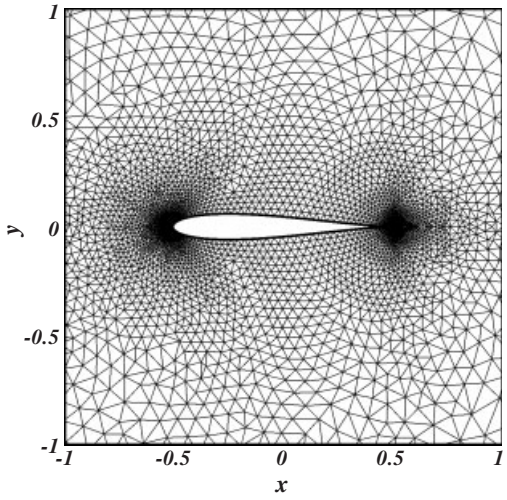

Figure 13. 80 nodes around an airfoil.

The slopes are 1.4 for the second-order method, and 2.3 for the third-order version (the slopes for the last two grids are 1.6 and 2.9), showing superior error level and convergence of the third-order method.

C_p distributions around the bodies on the 80×40 grids are plotted in Figures 8–16 show excellent agreement between the exact (solid curve) and computed values (circles). The actual solution errors are below 1%. In particular, this shows that the method preserves the circulation which is very important for accurate estimate of the lift force but is extremely difficult to achieve in the second-order version [9].

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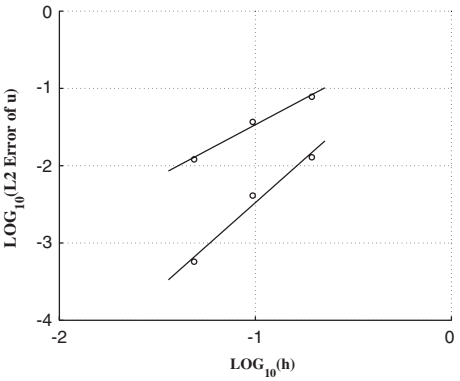

Figure 14. Error convergence: airfoil case.

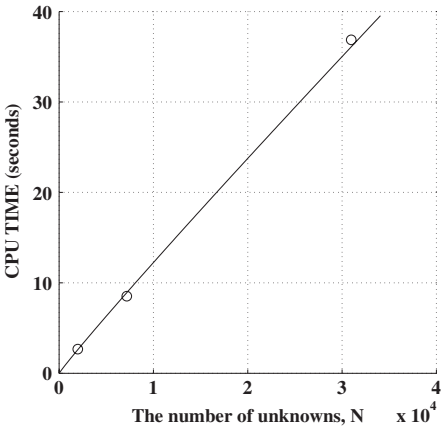

Figure 15. CPU time vs grid size: airfoil case.

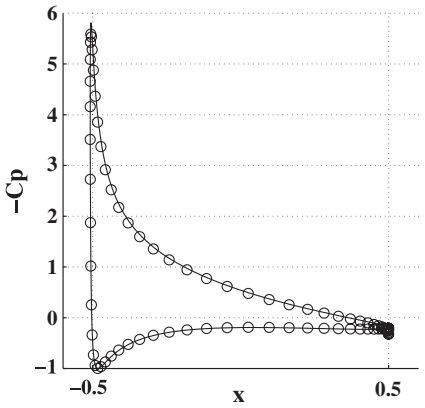

Figure 16. C_p: airfoil: 80 nodes on the airfoil.

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5. CONCLUDING REMARKS

A multigrid third-order least-squares method was developed for the Cauchy–Riemann Equations for unstructured triangular grids. It is demonstrated that the method achieves the optimal multigrid convergence rate and third-order accuracy for both structured and unstructured grids. Also, a simple fix was proposed to resolve the convergence difficulty with bad coarsening ratio that may often arises for unstructured grids.

The method can be used for solving the Laplace (or Poisson) equation by ignoring one of the variables. This can serve also as a high-order compact discretization of the viscous operator of the Navier–Stokes equations [14].

Future work includes the application of the method for the elliptic part of the two-dimensional Euler equations. Combining the method with a hyperbolic solver with semi-coarsening multigrid method, we expect to obtain a highly accurate and optimally efficient two-dimensional Euler solver.

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