p-adic L-functions for $GSp(4) \times GL(2)$

by

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To my parents

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CHAPTER I

Introduction

The interaction between arithmetic and analytic objects is one of the deepest and most fascinating themes in number theory. The general philosophy for this interaction can be summarized as follows: To any global arithmetic object X- say a number field, an elliptic curve, or a galois representation - we associate an L-function $L(X, s)$, a kind of generating function. The L-function $L(X, s)$ provides us with analytic tools for understanding the local or global arithmetic structure of X, which may not be otherwise as accessible.

One classic example of this philosophy is the analytic class number formula relating the residue (the analytic data) of the L-function (analytic object) associated to a number field K with the order of its class group (the arithmetic data). Another famous example is the celebrated Birch-Swinnerton-Dyer conjecture that predicts precise relations between the L-function $L(E, s)$ of an elliptic curve $E/\mathbf{Q}$ and various arithmetic objects like the Mordell-Weil group $E(\mathbf{Q})$(the group of rational points of E) and the Tate-Shafarevich group $Ш(E)$. Amongst other things this conjecture relates a special value of $L(E, s)$ to the size of the group of rational points of E.

One way to approach these problems on "special values" that has had significant success is Iwasawa Theory, which is a systematic analysis of the variation of the

p-parts of the class groups or their generalizations, Selmer groups, in question for a fixed odd prime p. The corresponding special L-values are packaged into a single p-adic L-function. In this framework the existence of p-adic L-functions associated to various arithmetic objects marks a starting point for the use of this theory.

In this work we are concerned with the construction of a p-adic L-function associated to a special value of the degree eight L-function for $GSp(4)\times GL(2)$ which is the convolution of the degree four spin L-function on $GSp(4)$ associated to a holomorphic Siegel cusp form and the standard degree two L-function on $GL(2)$ associated to an elliptic cusp form. Like the degree two L-function for holomorphic forms on GL_2; the spin L-function on $GSp(4)$ is the L-function of a compatible family of 4-dimensional ℓ-adic Galois representations associated to the holomorphic Siegel cusp form. The degree eight L-function is the L-function of the tensor product of these two and four dimensional representations. This makes the above p-adic L-function particularly interesting for arithmetic.

There has been no previous work constructing a p-adic L-function associated to the spin L-function. However, p-adic L-functions interpolating the special values of the standard L-function - the degree five L-function - have been constructed by Panchishkin [CP04] and Bocherer and Schmidt [BS00].

1.1 The integral representation

Let F be a holomorphic Siegel eigen cusp form on $\mathbf{H}_n$ (the Siegel upper half-space) and π the irreducible cuspidal automorphic representation of $GSp(2n,\mathbf{A})$ associated to it. Let f be an elliptic eigen cusp form on $\mathfrak{H}_2$ and σ the irreducible cuspidal representation of $GL_2(\mathbf{A})$ associated to it. In [Fur93] , Furusawa gives an integral representation of the degree eight L-function $L(s,\pi\times\sigma)$ for $GSp(4)\times GL(2)$. This

L-function is realized as a Rankin-Selberg integral. Global integrals for the spin L-function were considered first by Novodvorsky and later by Piatetski-Shapiro and Soudry, but their methods are applicable only when π has a Whittaker model or a special Bessel model. On the other hand, Furusawa's integral applies whenever π has some Bessel model. This is crucial to our case since we know that holomorphic Siegel modular forms do not have Whittaker models and need not have special Bessel models either but they do have Bessel models. Below, we outline the representation given in [Fur93].

In [Fur93], Furusawa associates a Klingen Eisenstein series $E(P, g, s, \sigma)$ on $GU(2,2)(\mathbf{A})$ to a form in the space of σ. This is done by identifying a copy of $GL_2(\mathbf{A})$ in the levi of the Klingen parabolic subgroup P of $GU(2,2;\mathcal{K})$ (Here $\mathcal{K}$ is some imaginary quadratic extension of $\mathbf{Q}$.). Then the L-function $L(\pi \times \sigma)$ is realized as a Rankin-Selberg integral wherein an automorphic from φ belonging to the space of π is integrated against the restriction of the Eisenstein series $E(P, g, s, \sigma)$ to $H := GSp(4)$ i.e.

$$Z(s) = Z(s, \sigma, \varphi) = \int_{Z_H(\mathbf{A})H(\mathbf{Q})\backslash H(\mathbf{A})} E(P, h, s, \sigma)\varphi(h)dh.$$

By making appropriate choices he ensures that

$$Z(s) = \prod_v Z_v(s)$$

where v runs over all the places of $\mathbf{Q}$ and $Z_v(s)$ is an explicitly given local integral. These local integrals are expressed in terms of a degenerate Whittaker model on $GU(2,2)$ and a Bessel model on $GSp(4)$. For *unramified* places v it is shown that

$$Z_v(s) = (normalizing factor) \times L(s - 1/2, \tilde{\pi}_v \times \tilde{\sigma}_v)$$

where $\tilde{\pi}_v$ denotes the contragredient of π_v, the local factor at v of π and similarly for $\tilde{\sigma}_v$. The normalizing factor is the inverse of a product of L-functions - the same product that shows up in the constant term of the Eisenstein series $E(P, g, s, \sigma)$.

1.2 The set-up of the problem and the solution strategy

Let p be an odd prime, $f \in S_\kappa(\Gamma_0(p^r), \chi)$ an ordinary elliptic cusp eigen form and $F \in S_{s,\kappa}(\Gamma_Q^s(p^r), \chi)$ be a Siegel cusp eigen form that is an ordinary eigenform for the Hecke operator $U_p = \Gamma_Q^s(p^r)\mathrm{diag}(p, p, 1, 1)\Gamma_Q^s(p^r)$. Let σ be the irreducible cuspidal automorphic representation of $GL_2(\mathbf{A})$ associated to f with central character $\omega_\sigma = \chi$ and let π be the irreducible cuspidal automorphic representation of $GSp(4, \mathbf{A})$ associated to F.

Our goal is to construct a one variable p-adic L-function interpolating special values of the degree eight L-function $L(s, F \times f)$ as F and f vary through families of forms. To construct this p-adic L-function we interpret the global integral of Furusawa as a Petersson inner product of a holomorphic Klingen Eisenstein series $E_f(Z)$ and the Siegel modular form $F(Z)$. So we can say that

$$(1.1) \qquad \langle\langle F, E_f \rangle\rangle_{GSp(4)} = (\text{normalizing factor})(\text{contribution from } S)L^S(F \times f)$$

where S is any finite set of places containing those places where σ or π is ramified and L^S is the L-function incomplete at S. But to be able to use this formula in general, we need to understand the contribution due to the places in S. In Furusawa's work the only ramified place where he carries out the zeta integral computations is the infinite place. Lacking a complete theory of these local zeta integrals (and strong multiplicity one for $GSp(4)$), in this work we restrict our attention to when both the Siegel and the elliptic modular form have a level a power of p (our fixed odd prime).

We now outline our interpolation argument. We vary f and F in appropriate p-adic families $\{f_\kappa\}$ and $\{F_\kappa\}$ with varying weights. As explained in the previous section, the Klingen Eisenstein series is constructed from the elliptic cusp form f. In general, an Eisenstein series is interpolated by interpolating its Fourier coefficients. Since the explicit formula for the Fourier coefficients of the Klingen Eisenstein series on the unitary group are in general considerably complicated, we do not directly interpolate them. Instead, we use a *pull-back formula* of Shimura [Shi97] to restrict a Siegel Eisenstein series on $U(3,3)$ to a Klingen Eisenstein series on $U(2,2)$. The pullback formula can be classically interpreted as follows:

Consider the embedding $\mathcal{H}_2 \times \mathcal{H}_1 \to \mathcal{H}_3$ of the Hermitian upper half spaces given by $Z \times w \mapsto \left(\begin{smallmatrix} Z & \\ & w \end{smallmatrix}\right)$. Then the pullback formula asserts that for a 'good' choice of a Siegel Eisenstein series E on $U(3,3)$,

$$\left\langle E\begin{pmatrix} Z & \\ & w \end{pmatrix}, f'(w) \right\rangle = (*)E_f(Z) \tag{1.2}$$

where E_f is the Klingen Eisenstein series on $U(2,2)$ associated to f, f' is a well understood transform of f and $(*)$ a normalizing factor. The Fourier coefficients of the Siegel Eisenstein series E can be explicitly computed and after suitable normalization we show that they are p-adically interpolated. Roughly,

$$E(z) = \sum C(B)e(\mathrm{tr}(Bz))$$

where B runs over a lattice of positive semidefinite Hermitian matrices in $M_3(\mathcal{K})$. We show that the $C(B)$'s can be p-adically interpolated into a family $\{C_\kappa(B)\}$ so that $\{E_\kappa = \sum C_\kappa(B)e(\mathrm{tr}(BZ))\}$ is a p-adic family of Siegel Eisenstein series. Roughly stated, we prove

Theorem I.1. *The Siegel Eisenstein series E can be p-adically interpolated into a one variable p-adic family $\{E_\kappa\}$ parametrized by its weight .*

We then deduce that the Klingen Eisenstein series E_f can be p-adically interpolated. To do this we express E as a Fourier-Jacobi expansion

$$E\left(\begin{smallmatrix} Z & \\ & w \end{smallmatrix}\right) = \sum_{B'} \Big(\sum_{B = \left(\begin{smallmatrix} B' & x \\ x & n \end{smallmatrix}\right)} C(B) e(\mathrm{tr}(nw)) e(\mathrm{tr}(B'Z))$$

where B' varies over all positive semi definite Hermitian matrices in $M_2(\mathcal{K})$ and $n \geq 0$. Let $C(B', w) = \sum_{B = \left(\begin{smallmatrix} B' & x \\ x & n \end{smallmatrix}\right)} C(B) e(\mathrm{tr}(nw))$ then we show that $C(B', w)$ is a modular form on $U(1,1)$. We can view it as a modular form on $GL(2)$.

Its a theorem of Hida, that if $f \in S^0_\kappa(\Gamma_0(p^r), \chi, L)$ is a primitive ordinary eigenform and $g \in M_\kappa(\Gamma_0(p^r), \chi, L)$ then

$$\ell_f(g) := a(1, g \mid_e 1_f) = \frac{\langle g, f' \rangle_{p^r}}{\langle f, f' \rangle_{p^r}}$$

where $f' = f^\rho|_{\left(\begin{smallmatrix} 0 & -1 \\ p^r & 0 \end{smallmatrix}\right)}$, e is the ordinary projector, and 1_f is the idempotent corresponding to f in the Hecke algebra. Hida has explained how this construction works for a p-adic family $\{f_\kappa\}$ as well. By an application of Hida's theorem to the Fourier Jacobi coefficients of E and relation (1.2) we get a p-adic family

$$\{E_{f_\kappa} = \sum_{B'} \ell_{f_\kappa}(C(B')) e^{(\mathrm{tr}(B'Z))}\}$$

of Klingen Eisenstein series on $U(2,2)$ with respect to a p-adic family of modular forms $\{f_\kappa\}$ on GL_2 of varying weights.

Finally, to interpolate the L-values of the degree eight L-function, we vary the Siegel modular form F in a p-adic family $\{F_\kappa\}$ and combine (1.1) with a construction for $GSp(4)$ analogous to Hida's ℓ_f. Doing this, we get a one variable (the weight) p-adic L-function.

The main theorem can be stated as

Theorem I.2. *Let $\Lambda = \mathbf{Z}_p[[T]]$ and $\mathcal{O}_L$ be the integral closure of Λ in a finite extension L of the field of fractions of Λ.*

Let $\mathcal{F}$(resp. $\mathbf{F}$) be an ordinary $\mathcal{O}_L$-adic elliptic eigenform of tame level 1 and character χ_0(resp. an ordinary $\mathcal{O}_L$-adic Siegel modular form of tame level 1 and character χ_0). Let S be a symmetric semi-integral matrix such that $\det(S) > 0$ is a fundamental discriminant. Let ξ be an unramified Hecke character of $\mathcal{K} = \mathbf{Q}(\sqrt{-\det(S)})$ of finite order. There exists $\mathcal{L} \in L$ such that if $\kappa >> 0$ and $\phi : \mathcal{O}_L \to \mathbf{Q}_p$ is a $\mathbf{Z}_p$-homomorphism such that $\phi(1+T) = \zeta(1+p)^\kappa$ for ζ a p^{r-1} root of unity, $r \geq 1$ then

$$\phi(\mathcal{L}) = a_p(\phi) \frac{L^{\{p\}}(\phi(\mathbf{F})^\rho \times \phi(\mathcal{F}), \kappa)}{\langle \phi(\mathcal{F}), \phi(\mathcal{F})^\rho \mid_\kappa \left({}_{p^r} \; {}^{-1} \right) \rangle \langle \langle \phi(\mathbf{F}), \phi(\mathbf{F})^\rho \mid_\kappa \left({}_{p^r} \; {}^{-1} \right) \rangle \rangle} B_{S,\xi,\phi(\mathbf{F})}(1_4)$$

where $a_p(\phi)$ is some normalizing factor depending on $\phi(\mathbf{F})$ and $\phi(\mathbf{f})$ and $B_{S,\xi,\phi(\mathbf{F})}(1_4)$ is the value at 1_4 of the Bessel model of $\phi(F)$ associated to S and ξ.

Remark I.3. Implicit in this theorem is a choice of an embedding $\mathbf{Q}_p \to \mathbf{C}$.

Remark I.4. For the definition of $\mathcal{O}_L$-adic forms see (9.2). The $\phi(\mathbf{F})$'s and $\phi(\mathbf{f})$'s are weight κ forms.

Remark I.5. The Bessel model is essentially a sum of Fourier coefficients associated to semi-integral symmetric matrices with determinant $= \det(S)$, weighted by values of ξ. For the precise definition see (5.3). The key point is that for any given ϕ it is possible to choose T and ξ so that $B_{S,\xi,\phi(\mathbf{F})}(1) \neq 0$, and if $B_{S,\xi,\phi(\mathbf{F})}(1) \neq 0$ for some ϕ then it is nonzero for all ϕ for $\kappa >> 0$.

Remark I.6. The factor $a_p(\phi)$ should be a ratio of partial L-factors for the representation $\pi_p \times \sigma_p$ of $GSp(\mathbf{Q}_p) \times GL_2(\mathbf{Q}_p)$; we have yet to work this out but expect to be able to do so following ideas of Sugano [Sug85].

CHAPTER II

Notation and Terminology

In this chapter we introduce some basic concepts and establish notation which we shall use throughout this thesis unless explicitly specified otherwise. Throughout this paper p is a fixed odd prime number.

2.1 Number fields and Characters

We fix once and for all algebraic closures $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}$ and $\mathbf{Q}_p$, respectively. We also fix embeddings $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$. Let $\mathcal{K} \subseteq \bar{\mathbf{Q}}$ be an imaginary quadratic extension of $\mathbf{Q}$ and denote its ring of integers by $\mathcal{O}_{\mathcal{K}}$. For a place v of $\mathcal{K}$, we denote by $\mathcal{K}_v$ the completion of $\mathcal{K}$ at v and by $\mathcal{O}_{\mathcal{K},v}$ the valuation ring of $\mathcal{K}_v$. If v is a place of $\mathbf{Q}$ then $\mathcal{K}_v = \mathcal{K} \otimes_{\mathbf{Q}} \mathbf{Q}_v$, and if v is a finite place then $\mathcal{O}_{\mathcal{K},v}$=$\mathcal{O}_{\mathcal{K}} \otimes \mathbf{Z}_v$.

We will usually denote the action of the non-trivial automorphism of $\mathcal{K}$ by $x \to \bar{x}$. This automorphism extends to $\mathcal{O}_{\mathcal{K}} \otimes A$ and $\mathcal{K} \otimes A$ by the action on the first factor for any $\mathbf{Z}$-algebra A. Let ℓ be a prime in $\mathbf{Q}$. We identify $\mathcal{O}_{\mathcal{K},\ell}$ with $\mathbf{Z}_\ell \times \mathbf{Z}_\ell$ and $\mathcal{K}_\ell$ with $\mathbf{Q}_\ell \times \mathbf{Q}_\ell$ if ℓ splits in $\mathcal{K}$ and under these identifications $\overline{(x, y)} = (y, x)$ for $(x, y) \in \mathcal{K}$.

For a number field L we let $\mathbf{A}_L$ denote the adeles of L and put $\mathbf{A} := \mathbf{A}_{\mathbf{Q}}$. We write $\mathbf{A}_{L,\infty}$ and $\mathbf{A}_{L,f}$ for the infinite part and the finite part of $\mathbf{A}_L$ respectively. When $L = \mathbf{Q}$ we will sometimes drop the L from our notation of adeles. For a place

v of $\mathbf{Q}$ and $x \in \mathbf{A}_\mathbf{Q}$ we write x_v for the v-component of x, and similarly for $\mathbf{A}_L$. For a place v of $\mathbf{Q}$, $|.|_v$ denotes the usual absolute value of $\mathbf{Q}_v$ i.e. $|p|_p = p^{-1}$. By $|.|_\mathbf{Q}$ we shall mean the absolute value on $\mathbf{A}_\mathbf{Q}$. We define an absolute $|.|_L$ on $\mathbf{A}_L$ by $|x|_L = |\mathrm{Nm}_{L/\mathbf{Q}}(x)|_\mathbf{Q}$, where $\mathrm{Nm}_{L/\mathbf{Q}}(x)$ is the norm from $\mathbf{A}_L$ to $\mathbf{A}_\mathbf{Q}$. We shall denote the usual absolute value on $\mathbf{C}$ and $\mathbf{R}$ by $|.|$.

Let N be a positive integer. A *Dirichlet character modulo* N is a group homomorphism $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}$. We can extend χ to all of $\mathbf{Z}$ by defining $\chi(n) = 0$ if $\gcd(n, N) > 1$ and $\chi(n) = \chi(n \bmod N)$ if $\gcd(n, N) = 1$. We will say that χ' modulo N' is *induced* from χ mod N if $N|N'$ and $\chi'(n) = \chi(n)$ whenever $\gcd(n, N') = 1$. If a character cannot be induced from a strictly lower level we call it *primitive*.

A *Hecke character* of $\mathbf{A}_L^\times$ is a continuous homomorphism

$$\psi : L^\times \backslash \mathbf{A}_L^\times \to \mathbf{C}^\times.$$

The character ψ factors as product of local characters $\psi = \Pi_v \psi_v$, where v runs over all the places of L. An ideal $\mathfrak{c}$ of $\mathcal{O}_L$ is called the conductor of ψ if

1. $\psi_v(x_v) = 1$ if v is a finite place of L, $x_v \in \mathcal{O}_{L,v}^\times$ and $x_v - 1 \in \mathfrak{c}\mathcal{O}_{L,v}$

2. no ideal $\mathfrak{c}'$ strictly containing $\mathfrak{c}$ has the above property.

For any ideal $\mathfrak{m}$ of $\mathcal{O}_L$, we set $\psi_\mathfrak{m} := \Pi \psi_v$, where v runs over all finite places of L such that $v|\mathfrak{m}$. We denote by ψ^* the associated ideal character.

For any commutative ring R we let $M_n(R)$ denote the set of $n \times n$ matrices with entries in R. We denote by $GL_n(R)$ the subset of $M_n(R)$ with unit determinant and by $SL_n(R)$ the subset of $GL_n(R)$ with determinant equal to 1. Let I_n denote the identity element of $GL_n(R)$. For will denote the transpose of a matrix x by ${}^t x$ and we put $x^* = {}^t\bar{x}$ and $\hat{x} = (x^*)^{-1}$.

2.2 Parabolic subgroups

We will be interested in Eisenstein series associated to various parabolic subgroups. In particular the maximal parabolics - Siegel parabolic and Klingen parabolic on some unitary similitude group G_n. Here we describe some of the parabolic subgroups and also use this chance to fix some notation.

For an integer $n \geq 1$ let

$$\omega_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

and let H_n be the group scheme over $\mathbf{Z}$ such that for any $\mathbf{Z}$-algebra R

$$H_n(R) = \{h \in GL_{2n}(\mathbf{Z} \otimes R) | {}^t h \omega_n h = \mu_n(h)\omega_n, \mu_n(h) \in R^\times\}$$

Then $H_n(\mathbf{R})$ is isomorphic to the usual symplectic similitude group. We shall refer to μ_n as the similitude factor; it is a homomorphism $\mu_n : H_n \to \mathbf{G}_m$. Let $Sp(2n) \subset H_n$ be the kernel of μ_n. Let

$$G_n(R) = \{g \in GL_{2n}(\mathcal{O}_\mathcal{K} \otimes R) | {}^t \bar{g} \omega_n g = \mu_n(g)\omega_n, \mu_n(g) \in R^\times\}$$

where $\bar{x}$ denotes the nontrivial automorphism of $\mathcal{K}$ and R is a $\mathbf{Z}$-algebra. Then G_n is the usual unitary group $GU(n, n)$. We shall refer to μ_n as the similitude factor. Let $U_n \subset G_n$ be the kernel of μ_n.

For $g \in M_{2n}$ let $A_g, B_g, C_g, D_g \in M_n$ be defined by

$$\begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$$

where we may drop the subscript g if it is understood.

Let $P_n \subset G_n$ be the subgroup of elements g in G_n such that C_g is zero and the last row in D_g is of the form $(0, \cdots, 0, *)$. This subgroup can be realized at the stabilizer of a totally isotropic line hence it is a maximal parabolic. We shall refer to it as the Klingen parabolic. By the standard theory of reductive groups, P_n has a levi decompostion $P_n = N_{P_n} M_{P_n}$ where N_{P_n} is the unipotent radical and M_{P_n} is the levi subgroup. We go into more details of the levi subgroup as it plays a crucial role in the induced representations that are used to define Eisenstein Series. The Levi subgroup M_{P_n} can be identified with $G_{n-1} \times Res_{\mathcal{O}/\mathbf{Z}}\mathbf{G}_m$ under the inclusion

$$G_{n-1} \times Res_{\mathcal{O}/\mathbf{Z}}\mathbf{G}_m \to G_n : (g,x) \mapsto m(g,x) := \begin{pmatrix} A_g & & B_g & \\ & \mu_{n-1}(g)\bar{x}^{-1} & & \\ C_g & & D_g & \\ & & & x \end{pmatrix}.$$

The unipotent radical N_{P_n} is given by

$$\left\{ \begin{pmatrix} 1 & & & \\ y & 1 & & \\ & & 1 & -{}^t\bar{y} \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 0 & p \\ & 1 & \bar{p} & q \\ & & 1 & \\ & & & 1 \end{pmatrix} | y, p \in \mathcal{K}, q \in \mathbf{Q} \right\}.$$

Moreover, we note that for any $\mathbf{Q}$-algebra R the homomorphism $(\mathcal{K} \otimes R)^\times \times GL_2(R) \to G_1(R)$ defined by $(a,g) \mapsto ag, a \in (\mathcal{K} \otimes R)^\times, g \in GL_2(R)$, induces an isomorphism

$$G_1(R) \simeq (\mathcal{K} \otimes R)^\times \times GL_2(R)/\{(a,a^{-1})|a \in R^\times\}.$$

This identification plays a crucial role in constructing an Eisenstein on $GU(2,2)$ induced from modular forms on $GL(2)$.

The Siegel parabolic $Q_n \subset G_n$ is defined by $C_g = 0$. Just as above Q_n has a levi decomposition given by $Q_n = N_{Q_n} M_{Q_n}$ where N_{Q_n} is the unipotent radical and M_{Q_n}

is the levi subgroup. More precisely N_{Q_n} and M_{Q_n} are given by

$$M_{Q_n} = \left\{ \begin{pmatrix} g & \\ & \alpha {}^t\bar{g}^{-1} \end{pmatrix} | g \in Res_{\mathcal{K}/\mathbf{Q}} GL_n, \alpha \in \mathbf{Q}^\times \right\}$$

and

$$N_{Q_n} = \left\{ \begin{pmatrix} 1_n & s \\ & 1_n \end{pmatrix} | s = {}^t\bar{s}, s \in M_n(\mathcal{K}) \right\}$$

Let B_n be the Borel subgroup containing P_n and Q_n defined by requiring A_g to be lower triangular. We shall denote by $T_n \subset B_n$ the torus of diagonal matrices in the Borel. Let $J \subset \mathcal{O}_{\mathcal{K}}$ be an ideal in $\mathcal{O}_{\mathcal{K}}$. Let R denote any of the parabolic subgroups mentioned above. Denote by $K_{R_n}(J)$ the subgroup of $G_n(\hat{\mathbf{Z}})$ such that $K_{R_n}(J) \equiv R_n$ (mod J).

We can similarly define parabolic subgroups H_n and we shall denote then also as P_n, Q_n and B_n. They are realized as the intersection of the parabolic in G_n with H_n.

CHAPTER III

Modular forms

In this thesis we deal with various types of modular forms (e.g. elliptic modular forms, Siegel modular forms and Hermitian modular forms). Each of them is dealt with in a classical setting as well as the adelic setting. The theory of elliptic modular forms is quiet standard by now and can be found in various textbooks and expository articles [Bum97], [Hid93], [Miy89]. For the theory of Siegel modular forms one can refer to [AZ95], and a basic introduction can also be found in [Kli90]. For Hermitian modular forms see [Gri90] and [Kri91].

3.1 Elliptic Modular forms

Let $\Gamma \subset SL_2(\mathbf{Z})$ be a subgroup. In particular we will be interested in the congruence subgroup $\Gamma_0(N)$ where

$$\Gamma_0(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{Z}) | c \equiv 0 \pmod N\}$$

and the subgroup $\Gamma_1(N)$ where

$$\Gamma_1(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) | a \equiv 1 \pmod N\}.$$

The group $SL_2(\mathbf{R})$ acts on $\mathbf{H}_1 = \{z \in M_2(\mathbf{C}) | \mathrm{Im}(z) > 0\}$ by fractional linear transformations given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

For ease of notation let us define the *slash operator*

$$(f|_\kappa \gamma)(z) = j(\gamma, z)^{-\kappa} (\det \gamma)^{\kappa/2} f(\frac{az+b}{cz+d}) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbf{R})$$

where $j(\gamma, z) = (c_\gamma z + d_\gamma)$.

An *elliptic modular form* of weight $\kappa \geq 0$ and level Γ is a holomorphic function $f : \mathbf{H}_1 \to \mathbf{C}$ such that

$$f|_\kappa \gamma(z) = f(z) \qquad \text{for } \gamma \in \Gamma \tag{3.1}$$

and f extends holomorphically to every cusp of Γ. We shall denote the $\mathbf{C}$-vector space of elliptic modular forms of weight κ and level $\Gamma_0(N)$ by $M_\kappa(N)$. If χ is a character modulo N and f satisfies

$$f(z)|_\kappa \gamma = \chi(d_\gamma) j(\gamma, z)^{-\kappa} f(\gamma z) \text{ where}$$

instead of 3.1 for all $\gamma \in \Gamma_0(N)$ then we call $f(z)$ a modular form of weight κ, level N and character χ. The space of all such $f(z)$ is denoted $M_\kappa(N, \chi)$.

Every elliptic modular form f possesses a Fourier expansion at the cusp at infinity given by

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

If $a_n = 0$ we say that f is a *cusp form.* We shall denote the $\mathbf{C}$-vector space of elliptic cusp forms of weight κ and level $\Gamma_0(N)$ by $S_\kappa(N)$ and cusp forms in $M_\kappa(N, \chi)$ by $S_\kappa(N, \chi)$. We now discuss the *Hecke operators* T_n which Hecke used to prove the Euler product factorization of the L-function associated certain $f(z)$'s. For any congruence subgroup Γ and $G = GL_2^+(\mathbf{Q})$ we define

$$D_\Gamma = \{\alpha \in G | \alpha\Gamma\alpha^{-1} \sim \Gamma\}$$

Then for $\alpha \in D_\Gamma$ and $f \in M_\kappa(\Gamma, \chi))$ let

$$f|_\kappa[\Gamma\alpha\Gamma] = \det(\alpha)^{\kappa/2-1} \sum_i \chi(a_{\alpha_i}) f|_\kappa \alpha_i$$

where $\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha_i$. We can define the Hecke operator T_n by

$$T_n f = \sum_{\alpha \in D_\Gamma, \det(\alpha) = n} f|[\Gamma\alpha\Gamma].$$

We sometimes write $T(n)$ for T_n.

Proposition III.1. *[Hid93] Let $\Gamma = \Gamma_0(N)$ and $\alpha = \begin{pmatrix} 1 & \\ & q \end{pmatrix}$. Then the left coset decomposition of $\Gamma\alpha\Gamma$ is given by*

$$\Gamma\alpha\Gamma = \begin{cases} \coprod_{u=1}^{q} \Gamma \begin{pmatrix} 1 & u \\ & q \end{pmatrix} \coprod \begin{pmatrix} q & \\ & 1 \end{pmatrix} & \text{if} \quad q \text{ is prime to } N, \\ \coprod_{u=1}^{q} \Gamma \begin{pmatrix} 1 & u \\ & q \end{pmatrix} & \text{if} \quad q \mid N. \end{cases}$$

Let A be a subalgebra of $\mathbf{C}$ and let

$$M_\kappa(N, \chi, A) = M_\kappa(N, \chi) \cap A[[q]]$$

and

$$S_\kappa(N, \chi, A) = S_\kappa(N, \chi) \cap A[[q]].$$

Then it can be checked that $M_\kappa(N, \chi, A)$ and $S_\kappa(N, \chi, A)$ are stable under the Hecke operators T_n if A contains $\mathbf{Z}[\chi]$. For such an A we define the Hecke Algebra $H_\kappa(N, \chi, A)$ (resp. $H_{\kappa,\mathrm{cusp}}(N, \chi, A)$) as the A-subalgebra of $\mathrm{End}_A(M_\kappa(N, \chi, A))$

(resp. $\mathrm{End}_A(S_\kappa(N,\chi,A))$) generated by T_n for all positive n. We say that f is a *Hecke eigenform* if it is an eigenform for all T_n.

For $f, g \in M_k(\Gamma)$ and if either f or g is a cusp form, we define the Petersson inner product as

$$\langle f, g\rangle := \frac{1}{[\overline{SL_2(\mathbf{Z})} : \overline{\Gamma}]} \int_{\Gamma\backslash \mathbf{H}_1} f(z)\overline{g(z)} y^{\kappa-2} dxdy$$

where $\overline{SL_2(\mathbf{Z})} := SL_2(\mathbf{Z})/ < -I_2 >$ and $\overline{\Gamma} :=$ image of Γ in $SL_2(\mathbf{Z})/ < -I_2 >$. This is independent of Γ.

As in the case of characters we have a notion of *primitive normalized eigenforms* or *newforms*. These should be thought of as forms that do not arise from a lower level. More precisely, let $S_\kappa^{\mathrm{old}}(\Gamma_0(N), \chi)$ be the subspace of $S_\kappa(\Gamma_0(N), \chi)$ generated by the set

$$\bigcup_M \bigcup_\ell \{f(\ell z) | f(z) \in S_\kappa(\Gamma_0(M), \chi)\}.$$

Here M runs over all the positive integers such that $m_\chi | M, M | N$, and $M \neq N$, ℓ runs over all the positive divisors of N/M and m_χ is conductor of χ. We shall refer to the orthogonal complement of $S_\kappa^{\mathrm{old}}(\Gamma_0(N), \chi)$ in $S_\kappa(\Gamma_0(N), \chi)$ with respect to the Petersson inner product as $S_\kappa^{new}(\Gamma_0(N), \chi)$.

Theorem III.2. *(Hecke) Suppose χ is a primitive Dirichlet character modulo N and L a number field then $H_\kappa(N, \chi; L)$ is semi-simple.*

For a modular form $f(z) = \sum_{n=0}^{n=\infty} a_n q^n \in M_\kappa(N, \chi)$, we put

$$f^\rho(z) = \sum_{n=0}^{\infty} \bar{a}_n q^n,$$

where $\bar{a}_n$ is the complex conjugate of a_n and $q = e^(2\pi i z)$. For a positive integer N, we put

$$\omega_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

Fact III.3. *[Miy89] Let χ be a Dirichlet charachter mod N.*

The correspondence "$f \mapsto f|_{\omega_N}$" induces isomorphisms:

$$M_\kappa(N,\chi) \simeq M_\kappa(N,\bar{\chi})$$

$$S_\kappa(N,\chi) \simeq S_\kappa(N,\bar{\chi})$$

If $f(z) \in M_\kappa(N,\chi)$(resp. $S_\kappa(N,\chi)$), then

$$f^\rho(z) = \overline{f(-\bar{z})}$$

and it belongs to $M_\kappa(N,\bar{\chi})$ (resp. $S_\kappa(N,\bar{\chi})$).

3.2 Siegel Modular forms

Let $H_n^+(\mathbf{R}) = \{\gamma \in H_n(\mathbf{R}) | \mu_n(\gamma) > 0\}$ and let $\mathbf{H}_n = \{Z \in M_n(\mathbf{C}) | Z^t = Z, Im(Z) > 0\}$ be the Siegel upper half space of degree n respectively. Then $H_n^+(\mathbf{R})$ acts on $\mathbf{H}_n$ via

$$\gamma(Z) = (a_\gamma Z + b_\gamma)(c_\gamma Z + d_\gamma)^{-1}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_n^+(\mathbf{R}).$$

We define the congruence subgroup

$$\Gamma_{Q_n}^s(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2n, \mathbf{Z}) | c \equiv 0 \mod N\},$$

For κ a positive integer and $\gamma \in H_n^+(\mathbf{R})$ we define the slash operator by:

$$(F|_\kappa \gamma)(z) := \mu(\gamma)^{nk/2} j(\gamma, z)^{-\kappa} F(\gamma z) \qquad \text{for} \qquad z \in \mathbf{H}_n$$

where $j(\gamma, z) = \det(c_\gamma z + d_\gamma)$. A holomorphic function $F : \mathbf{H}_n \to \mathbf{C}$ is a said to be a *Siegel modular form* of weight κ and level $\Gamma_{Q_n}^s(N)$ if

$$F|_\kappa \gamma = F \qquad \text{for all} \qquad \gamma \in \Gamma^s.$$

The $n = 1$ case is just the case of elliptic modular forms. We denote the space of Siegel modular forms of level $\Gamma_{Q_n}^s(N)$ by $M_{s,\kappa}(N)$. If χ is a Dirchlet character of conductor N then let $M_{s,\kappa}(N, \chi)$ be the space of $F \in M_{s,\kappa}(N, \chi)$ such that

$$F|_\gamma = \chi(\det(d_\gamma))F \text{ for all } \gamma \in \Gamma_{Q_n}^s(N).$$

If $F \in M_{s,\kappa}(\Gamma^s)$ for some congruence subgroup Γ^s, then there is an integer N depending only on Γ^s such that $F(z + Nh) = F(z)$ for h belonging to the semigroup of symmetric integral matrices. Thus F has a Fourier expansion given by

$$F(z) = \sum_{T \in S_n^{\geq 0}(\mathbf{Z})} a(T, F) e^{2\pi i N^{-1} \mathrm{tr}(Tz)}$$

where $S_n^{\geq 0}$ is the semigroup of positive semi-definite $n \times n$ symmetric integral matrices. We say that F is a cusp form if for all $\alpha \in H_n^+(\mathbf{R}), a(T, F\mid_\alpha) = 0$ for every T such that $\det T = 0$. We denote by $S_{s,\kappa}(\Gamma^s_{Q_n}(N), \chi)$ or $S_{s,\kappa}(N, \chi)$ the vector space of Siegel cusp forms of weight κ and level $\Gamma^s_{Q_n}(N)$ and character χ. If $\chi = \mathbf{1}$ then we omit any mention of the character. Just as in the case of elliptic modular forms we define $M_{s,\kappa}(\Gamma_{Q_n}(N), \chi, A)$(resp.$S_{s,\kappa}(\Gamma_{Q_n}(N), \chi, A)$) as the space of modular (resp. cusp) forms with Fourier coefficients in A. We can define F^ρ for F a Siegel modular form just as in the case of elliptic modular forms.

Our next goal is to recall the definition of the Petersson inner product in the Siegel modular form case. There is a $H_n^+(\mathbf{R})$ invariant measure on $\mathbf{H}_n$ given by

$$d\mu z = (\det y)^{-(n+1)} \Pi_{\alpha \leq \beta} dx_{\alpha,\beta} \Pi_{\alpha \leq \beta} dy_{\alpha,\beta}$$

where $z = x + iy$ and $z = (x_{\alpha,\beta}) + i(y_{\alpha,\beta})$ and $dx_{\alpha,\beta}$ and $dy_{\alpha,\beta}$ are the usual Euclidean measures on $\mathbf{R}$.

For F and G two Siegel modular forms of weight κ, level Γ^s and either of them being a cusp form we define for any congruence subgroup $\Gamma^s_0 \subset \Gamma^s$ the Petersson inner product

$$\langle\langle F, G\rangle\rangle = \frac{1}{[\overline{H_n(\mathbf{Z})} : \overline{\Gamma}^s_0]} \int_{\Gamma^s_0 \backslash \mathbf{H}^n} F(z)\overline{G(z)}(\det y)^\kappa d\mu z$$

where $\overline{H_n(\mathbf{Z})} := H_n(\mathbf{Z})/ < -I_n >$ and $\overline{\Gamma}^s_0 :=$ image of Γ^s_0 in $H_n(\mathbf{Z})/ < -I_n >$.

We now recall some facts about Hecke operators on Siegel modular forms. If $g \in H_n^+(\mathbf{Q})$ and $g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N$ we define the Hecke operator:

$$[\Gamma^s_{Q_n}(N) g \Gamma^s_{Q_n}(N)] : M_{s,\kappa}(\Gamma^s_{Q_n}(N), \chi) \to M_{s,\kappa}(\Gamma^s_{Q_n}(N), \chi)$$

by :

$$f \mapsto \mu(g)^{n\kappa/2} \sum_i \chi'(g) f\mid_\kappa g_i$$

where $\Gamma^s_{Q_n}(N)g\Gamma^s_{Q_n}(N) = \bigsqcup \Gamma^s_{Q_n}(N)g_i$ and $\chi'\left(\begin{smallmatrix} A & B \\ CN & D \end{smallmatrix}\right) = \chi(\det A)$ (this value is 0 if $(\det A, N) \neq 1)$.

Let $\triangle_0(N)$ be the semigroup of elements γ in $H_n^+(\mathbf{Q}) \cap M_{2n}(\mathbf{Z})$ such that $(N, \det(\gamma)) = 1$ and $\gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ mod N. We denote by $H_s^*(N, \chi, \mathbf{Z})$ the Hecke algebra spanned over $\mathbf{Z}$ by the double $\Gamma^s_{Q_n}(N)$ cosets contained in $\triangle_0(N)$. Then $H_s^*(N, \chi, \mathbf{Z})$ acts on $M_{s,\kappa}(N, \chi, \mathbf{Z})$ and this action preserves $S_{s,\kappa}(N, \chi, \mathbf{Z})$. It is also known that $S_{s,\kappa}(N, \chi)$ has a basis of eigenforms for $H_s^*(N, \chi, \mathbf{Z})$ which are orthogonal with respect to the Petersson inner product.

For n a positive integer prime to N let $T_{s,n} \in H_s^*(N, \chi, \mathbf{Z})$ denote the sum of all the $\Gamma^s_{Q_n}(N)$ double cosets in $\triangle_0(N)$ for which μ takes value n. We let $H_s(N, \chi, \mathbf{Z})$ be the subring of $H_s^*(N, \chi, \mathbf{Z})$ generated by these operators.

If $p \mid N$ we shall also consider the operators

$$U_{s,p^t} = [\Gamma^s_{Q_n}(N)\mathrm{diag}(1, 1, p^t, p^t)\Gamma^s_{Q_n}(N)]. \tag{3.2}$$

This operator is of particular interest to us. Below we list some properties of this operator.

Lemma III.4. *For $t > 0$ and $p \mid N$,*

$$U_{s,p^t} = \Gamma^s_{Q_n}(N)\,diag(1, 1, p^t, p^t)\Gamma^s_{Q_n}(N) = \bigsqcup_{X \in \mathfrak{X}} \Gamma^s_{Q_n}(N) \begin{pmatrix} 1_n & X \\ 0 & p^t 1_n \end{pmatrix}$$

where $\mathfrak{X}$ is a set of representatives of $S_n(\mathbf{Z})$ modulo p^t.

Proposition III.5. *Let $p \mid N$ and let χ be a Dirichlet character modulo N. Then $U^t_{s,p} \in End(M_{s,\kappa}(\Gamma^s_{Q_n}(N), \chi))$ and satisfies:*

1. *$a(T, F \mid_{U_{s,p^t}}) = a(p^t T, F)$*

2. *$U_{s,p^r} = U_{s,p^r}$*

3. $U_{s,p}$ commutes with the action of $H_s(N,\chi)$

The following double coset decomposition will be useful for later.

Fact III.6. *(Andrianov)*

$$\begin{aligned} &GSp(4,\mathbf{Z}_p)\,diag(p,p,1,1)GSp(4,\mathbf{Z}_p) \\ =& \coprod_{x,y,z\in\mathbf{Z}/p\mathbf{Z}} \left(\begin{smallmatrix} 1&0&x&y\\ 0&1&z&x\\ 0&0&1&0\\ 0&0&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix} p&0&0&0\\ 0&p&0&0\\ 0&0&1&0\\ 0&0&0&1\end{smallmatrix}\right) GSp(4,\mathbf{Z}_p) \\ &\coprod_{x,z\in\mathbf{Z}/p\mathbf{Z}} \left(\begin{smallmatrix} 1&x&0&z\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&-x&1\end{smallmatrix}\right)\left(\begin{smallmatrix} p&0&0&0\\ 0&1&0&0\\ 0&0&p&0\\ 0&0&0&1\end{smallmatrix}\right) GSp(4,\mathbf{Z}_p) \\ &\coprod_{x\in\mathbf{Z}/p\mathbf{Z}} \left(\begin{smallmatrix} 1&0&0&0\\ 0&1&x&0\\ 0&0&1&0\\ 0&0&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1&0&0&0\\ 0&p&0&0\\ 0&0&1&0\\ 0&0&0&p\end{smallmatrix}\right) GSp(4,\mathbf{Z}_p) \\ &\coprod \left(\begin{smallmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&p&0\\ 0&0&0&p\end{smallmatrix}\right) GSp(4,\mathbf{Z}_p) \end{aligned}$$

3.3 Hermitian Modular forms

Let $G_n^+(\mathbf{R}) = \{\gamma \in G_n(\mathbf{R}) | \mu_n(\gamma) > 0\}$ and let $\mathcal{H}_n = \{Z \in M_n(\mathbf{C}) | -i(Z - {}^t\bar{Z}) > 0\}$ be the Hermitian upper half space of degree n. Then $G_n^+(\mathbf{R})$ acts on $\mathcal{H}_n$ via

$$\gamma(Z) = (a_\gamma Z + b_\gamma)(c_\gamma Z + d_\gamma)^{-1}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n^+(\mathbf{R}).$$

We define the congruence subgroup

$$\Gamma_{Q_n}^h(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_n(\mathbf{Z}) | c \equiv 0 \mod N\},$$

Let κ be a positive integer, $\gamma \in G_n^+(\mathbf{R})$. To simplify notation we introduce the *slash operator* : for any $\gamma \in G_n^+(\mathbf{R})$, set

$$(F|\gamma)(z) := \mu(\gamma)^{n\kappa/2} j(\gamma,z)^{-\kappa} F(\gamma z) \qquad \text{for all} \qquad \gamma \in \Gamma^h \qquad \text{and} \qquad z \in \mathcal{H}_n$$

where $j(\gamma, z) = \det(c_\gamma z + d_\gamma)$. A holomorphic function $F : \mathcal{H}_n \to \mathbf{C}$ is a said to be a *Hermitian modular form* of weight κ and level $\Gamma^h_{Q_n}(N)$ if

$$F|\gamma = F \qquad \text{for all} \qquad \gamma \in \Gamma^h_{Q_n}(N).$$

One can also define Hermitian modular forms with a character. Let $\tau : \mathbf{A}^\times_\mathcal{K} \to \mathbf{C}^\times$ be a Hecke character such that for all finite ℓ, $\tau_\ell(x) = 1$ for $x \in \mathcal{O}^\times_{\mathcal{K},\ell}$ with $x-1 \in N\mathcal{O}_{\mathcal{K},\ell}$. We say that F is of level N and character τ if

$$F|_\gamma = \tau_N(\det(d_\gamma))F$$

for every $\gamma \in \Gamma^h_{Q_n}(N)$. We let $F \in M_{h,\kappa}(\Gamma^h_{Q_n}(N), \tau)$ be the space of such F.

If $F \in M_{h,\kappa}(\Gamma^h)$ for some congruence subgroup Γ^n then there is an integer N depending only on Γ^h such that $F(z + Nh) = F(z)$ for $h \in \mathcal{N}_n = \{h \in M_n(\mathcal{K}) | {}^t\bar{h} = h\}$. Thus F has a Fourier expansion given by

$$F(z) = \sum_{T \in \mathcal{N}^*_n(\mathbf{Z})^{\geq 0}} a(T, F) e^{2\pi i N^{-1} \mathrm{tr}(Tz)}.$$

Remark III.7. When $n = 1$, $\mathbf{H}_1 = \mathcal{H}_1$ and the theory of Hermitian modular forms is the essentially the same as the theory of elliptic modular forms.

3.4 Automorphic forms

For a reductive group G over a number field L we will write $\mathcal{A}(G)$ for the space of automorphic forms on $G(\mathbf{A}_L)$ and $\mathcal{A}^0(G)$ for the space of cuspforms. Sometimes we also use $\mathcal{A}_G$ to mean $\mathcal{A}(G)$.

3.4.1 Automorphic forms on $GL(2)$

We give the adelic picture associated to the classical theory of elliptic modular forms. Let f be a classical holomorphic cuspidal eigenform of weight κ level $\Gamma_0(N)$

and χ a character of $(\mathbf{Z}\backslash N\mathbf{Z})^\times$. By the strong approximation we have:

$$GL_2(\mathbf{A}_\mathbf{Q}) = GL_2(\mathbf{Q})GL_2^+(\mathbf{R})K(N),$$

where $K(N) = \prod_{\ell\nmid N} GL_2(\mathbf{Z}_\ell)\prod_{\ell|N} K_{\ell,N}$, and

$$K_{\ell,N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{Z}_\ell) : c \equiv 0 \pmod N \right\}.$$

Moreover, $GL_2(\mathbf{Q}) \cap GL_2^+(\mathbf{R})K(N) = \Gamma_0(N)$. To each Dirichlet character

$$\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$$

we can associate an idele class character as follows: The Dirichlet character χ determines a character χ_ℓ of $\mathbf{Z}_\ell^\times$ by composition with the natural homomorphism from $\mathbf{Z}_\ell^\times$ to $(\mathbf{Z}/N\mathbf{Z})^\times$. The product $\prod_\ell \chi_\ell$ then defines a character of $\hat{\mathbf{Z}}^\times$. Since $\mathbf{Z}$ has class number one,

$$\hat{\mathbf{Z}}^\times \to \mathbf{A}^\times/\mathbf{Q}^\times\mathbf{R}^\times$$

hence we get an idele class character. We will also denote this idele class character by χ and we write $\chi = \otimes\chi_\ell$. Each χ_ℓ defines a character of $K_{\ell,N} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi_\ell(a)$ giving a character of $K(N)$. We now define a function ϕ_f in $\mathcal{A}(GL_2)$ by

$$\phi_f(\gamma g_\infty k) = \chi(a_k)j(g_\infty, i)^{-\kappa}\mu(g_\infty)^{\kappa/2}f(g_\infty(i)),$$

where $\gamma \in GL_2(\mathbf{Q}), g_\infty \in GL_2^+(\mathbf{R}),$ and $k \in K(N)$. We call ϕ_f the *automorphic form corresponding to* f. Suppose f is a newform and let $\sigma_f = \otimes_\ell \sigma_{f,\ell}$ denote the automorphic representation generated by ϕ_f. Then we call σ_f the automorphic representation corresponding to f.

3.4.2 Automorphic forms on $GSp(2n)$

For any finite prime ℓ let

$$K^s_{Q_n,\ell}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GSp(2n)(\mathbf{Z}_\ell) : c \equiv 0 \pmod{N} \right\}.$$

Now let $K^s_{Q_n}(N) = \prod_{\ell \nmid N} GSp(2n)(\mathbf{Z}_\ell) \prod_{\ell | N} K^s_{Q_n,\ell}(N)$. Given $F \in M_{s,\kappa}(\Gamma^s, \chi)$, define a function ϕ_F by

$$\phi_F(g) = \chi(\det(a_k)) j(g_\infty, i)^{-\kappa} \mu(g_\infty)^{n\kappa/2} F(g_\infty(i))$$

where $g = \gamma g_\infty k \in GSp(2n)(\mathbf{Q}) GSp(2n)^+(\mathbf{R}) K^s_{Q_n}(N)$. Here, as in the previous section, χ denotes the idele class character on $\mathbf{A}^\times/\mathbf{Q}^\times$ associated to the Dirichlet character χ. Then ϕ_F is the *automorphic form on $GSp(2n)(\mathbf{A})$ associated to F.* We shall denote by π_F the automorphic representation generated by ϕ_F. Usually we will assume that any eigenform F we are working with has the property that π_F is irreducible.

3.4.3 Automorphic forms on $GU(n, n)$

Just as in the earlier two situations there is an adelic analogue of Hermitian modular forms.

For any finite prime ℓ let $K^h_{Q_n}(N) = \prod_{\ell \nmid N} GU(n, n)(\mathbf{Z}_\ell) \prod_{\ell | N} K^h_{n,\ell}(N)$, where

$$K^h_{n,\ell}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GU(n, n)(\mathbf{Z}_\ell) : c \equiv 0 \pmod{N} \right\}.$$

Let

$$K^{+,h}_{n,\infty} := \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in G_n(\mathbf{R}) | x, y \in GL(n, \mathbf{C}), x^t \bar{x} + y^t \bar{y} = 1_n, x^t \bar{y} = y^t \bar{x} \right\}$$

and $K^h_{n,\infty}$ be the subgroup generated by $K^{+,h}_{n,\infty}$ and $\mathrm{diag}(1_n, -1_n)$. Then $K^h_{n,\infty}$ is a maximal compact in $G_n(\mathbf{R})$.

Also there exists $t_i \in G_n(\mathbf{A}_{\mathcal{K},f})$ such

$$G_n(\mathbf{A}) = \bigsqcup_{i=1}^{h_{\mathcal{K}}} G_n(\mathbf{Q}) G_n^+(\mathbf{R}) t_i K^h_{Q_n}(N)$$

where $h_{\mathcal{K}}$ = class number of $\mathcal{K}$. One can take $t_i = \left(\begin{smallmatrix} u_i & \\ & u_i^* \end{smallmatrix}\right)$, $u_i \in GL_n(\mathbf{A}_{\mathcal{K},f})$ such that $\{\det(u_i)\}$ represents the class group of $\mathcal{K}$. Given an automorphic form $\phi \in \mathcal{A}(G_n)$ such that

- $\phi(gk) = \tau(a_k)\phi(g)$ for $k \in K^h_{Q_n}(N)$
- $\phi(gk) = j(k,i)^{-\kappa}\phi(g)$ for $k \in K^{+,h}_{Q_n,\infty}$
- $\phi(ag) = (a/|a|)^{-\kappa}\, \phi(g)$ for $a \in \mathbf{C}^\times \subset G_n^+(\mathbf{R})$

we put

$$F_\phi(Z) = j(g_\infty, i)^\kappa \mu(g_\infty)^{-n\kappa/2}\phi(g_\infty)$$

where $Z = g_\infty(i)$. Then

$$F_\phi(Z) \mid_\kappa \gamma = \tau(a_\gamma^{-1}) F_\phi(Z) = \tau^{\mathrm{c}}(d_\gamma) F_\phi(Z)$$

for $\gamma \in \Gamma^h_{Q_n}(N)$. So $F_\phi(Z)$ if homomorphic belongs to $M_{n,\kappa}(\Gamma^h_{Q_n}(N), \tau^{\mathrm{c}})$.

3.5 Relation between Adelic Hecke operators and Classical operators

In this section we define Hecke operators on the space of automorphic forms and then relate them to the classical Hecke operators defined earlier. This will allow us to move between the two different set ups easily.

3.5.1 Adelic Hecke operators

GL_2 local theory

Let ℓ be a finite prime of $\mathbf{Q}$ and (V, σ) an irreducible admissible representation of $GL_2(\mathbf{Q}_\ell)$. Let K_ℓ be the maximal open compact subgroup $GL_2(\mathbf{Z}_\ell)$ of $GL_2(\mathbf{Q}_\ell)$. Let $\mathcal{H}_{K_\ell}$ be the algebra of compactly supported, bi-K_ℓ invariant smooth functions from $GL_2(\mathbf{Q}_\ell)$ to $\mathbf{C}$. The multiplication in $\mathcal{H}_{K_\ell}$ is defined by the convolution

$$(\phi_1 * \phi_2)(g) = \int_{GL_2(\mathbf{Q}_\ell)} \phi_1(gh^{-1})\phi_2(h)dh.$$

Then $\mathcal{H}_{K_\ell}$ is commutative.

The representation σ defines an action of $\mathcal{H}_{K_\ell}$ on V^{K_ℓ} (the vectors of V fixed by K_ℓ) given by $v \mapsto \sigma(\phi)v$ where

$$\sigma(\phi)v = \int_{GL_2(\mathbf{Q}_\ell)} \phi(g)\sigma(g)vdg.$$

Then for $a \in GL_2(\mathbf{Q}_\ell)$ and $v \in V$ it is easily checked that

$$\sigma([K_\ell a K_\ell])v = \sum_{i=1}^{n} \sigma(a_i)v$$

where $[H]$ denotes the characteristic function of H and $K_\ell a K_\ell$ have a coset decomposition given by $K_\ell a K_\ell = \coprod_{i=1}^{n} a_i K_\ell$. Let

$$H_\ell = K_\ell \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} K_\ell = \bigsqcup_{b=0}^{\ell-1} \begin{pmatrix} \ell & -b \\ 0 & 1 \end{pmatrix} K_\ell \bigsqcup \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} K_\ell.$$

If $\sigma = \sigma(\alpha, \beta)$ is an unramified principle series representation and $v \in V^{K_\ell}$ is a new vector then

$$[H_\ell]v = \ell^{1/2}(\alpha(\ell) + \beta(\ell))v$$

GL_2 global theory

Let $\Gamma = \Gamma_0(N)$ and ℓ be a finite rational prime not dividing N and $K_\ell = GL_2(\mathbf{Z}_\ell)$. We can define the action of $[H_\ell]$ on the space of functions in $\mathcal{A}(GL_2)$ fixed by K_ℓ by the same formula as above. The adelic and classical picture can be now be tied up together by the following lemma.

Lemma III.8. *Let $f \in S_\kappa(\Gamma_0(N), \chi)$ and let ϕ_f be the associated automorphic form. Then*

$$\ell^{\kappa/2-1}[H_\ell]\phi_f = \chi(\ell)^{-1}\phi_{T_\ell f}.$$

Proof. If $K_\ell \left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right) K_\ell = \bigsqcup_{i=1}^n a_i K_\ell$ then given $k_\ell \in K_\ell$, $k_\ell a_i = a_{i'} k_{\ell,i}$ where $i \mapsto i'$ is a permutation. Let $g = \gamma g_\infty k$ and $k = k_\ell k^\ell$, $k_\ell \in K_\ell$ and k^ℓ has the ℓ^{th} entry equal to 1. Then

$$\begin{aligned}
[H_\ell]\phi_f(g) & \sum_{i=1}^n \phi_f(\gamma g_\infty k a_i) \\
&= \sum_{i=1}^n \phi_f(\gamma g_\infty a_{i'} k^\ell k_{\ell,i}) \\
&= \sum_{i=1}^n \phi_f(g_\infty a_i)\chi(a_{k^\ell})
\end{aligned}$$

Now observe that where $\gamma \in GL_2(\mathbf{Q}), g_\infty \in GL_2^+(\mathbf{R})$, and $k \in K(N)$. Now observe that

$$g_\infty \begin{pmatrix} \ell & -b \\ 0 & 1 \end{pmatrix}_\ell = \gamma' \begin{pmatrix} \ell^{-1} & b\ell^{-1} \\ 0 & 1 \end{pmatrix}_\infty g_\infty k'$$

where

$$\gamma' = \left(\begin{pmatrix} \ell & -b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \ell & -b \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} \ell & -b \\ 0 & 1 \end{pmatrix}, \cdots \right)$$

and

$$k' = \left(1, \begin{pmatrix} \ell & -b \\ 0 & 1 \end{pmatrix}^{-1}, \begin{pmatrix} \ell & -b \\ 0 & 1 \end{pmatrix}^{-1}, \dots, 1, \begin{pmatrix} \ell & -b \\ 0 & 1 \end{pmatrix}^{-1}, \cdots \right)$$

where 1 is at the ℓ-th place. Hence

$$\phi\left(g \begin{pmatrix} \ell & -b \\ 0 & 1 \end{pmatrix}_\ell \right) = f\left(\begin{pmatrix} \ell & -b \\ 0 & 1 \end{pmatrix}^{-1} z \right) \ell^{-\kappa/2}\chi_\ell(\ell) = f\left(\frac{z+b}{\ell}\right) \ell^{-\kappa/2}\chi_\ell(\ell)$$

and likewise we can show that

$$\phi\left(g\begin{pmatrix}1 & 0\\ 0 & \ell\end{pmatrix}_{\ell}\right) = f\left(\begin{pmatrix}1 & 0\\ 0 & \ell\end{pmatrix}^{-1} z\right) = f(\ell z)\ell^{\kappa/2}. \tag{3.3}$$

So finally we have

$$\begin{aligned}
\ell^{\kappa/2-1}[H_\ell]\phi_f &= \ell^{-1}\sum_{b=0}^{\ell-1} f\left(\frac{z+b}{\ell}\right)\chi_\ell(\ell) + \ell^{\kappa-1}f(\ell z)\\
&= \ell^{\kappa-1}\{\sum_{b=0}^{\ell-1} f\left(\frac{z+b}{\ell}\right)\ell^{-\kappa}\chi_\ell(\ell) + f(\ell z)\}\\
&= \chi(\ell)^{-1}\phi_{T_\ell f}.
\end{aligned}$$

$\square$

In a similar manner one defines the adelic Hecke operators for $GSp(4)$ and relates them to the classical Hecke operators. For me details we refer the reader to Schmidt-Asgari, [AS01].

CHAPTER IV

Eisenstein series

This thesis makes extensive use of Eisenstein Series on various groups. In this chapter we shall define an Eisenstein series and state some basic facts about them. We also give some examples which will be useful later.

Generally speaking, Eisenstein series compliment the space of cusp forms in the space of modular forms. They are automorphic forms which are also functions of several complex variables, often realized as a series which converges on some subset of $\mathbf{C}^n$. Langlands [Lan76] established that these series have a meromorphic continuation on $\mathbf{C}^n$ and satisfy nice functional equations. They have played a crucial role in the study of automorphic L-functions via the Rankin-Selberg method. The Rankin-Selberg method realizes some L-functions as inner products of cusp forms and Eisenstein Series (as exploited in this thesis). In such cases the meromorphic continuation and functional equation of the L-function can be derived from those of the Eisenstein series. For a survey of the Rankin-Selberg method one can refer to Bump, [Bum05]. Some L-functions also arise as the constant terms of Eisenstein series. For general overview of automorphic forms and Eisenstein series we refer the reader to [CKM04].

4.1 Induced representations

Let G be a quasi-split reductive group and let $P = MN$ be the Levi decomposition of a parabolic subgroup P defined over a field $\mathbf{Q}$. Let A be the connected component of the center of M. Let $X^*(M)$ and $X^*(A)$ be the group of $\mathbf{Q}$-rational characters of M and A respectively. Then $X^*(M)$ is of finite index in $X^*(A)$ by [CKM04]. Hence $X^*(M) \otimes_{\mathbf{Z}} \mathbf{R} = X^*(A) \otimes_{\mathbf{Z}} \mathbf{R}$. Set $\mathfrak{a}^* = X^*(M) \otimes_{\mathbf{Z}} \mathbf{R}$ and $\mathfrak{a}^*_{\mathbf{C}} = \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C}$. Let $\mathfrak{a} = \mathrm{Hom}(X^*(M), \mathbf{R}) = \mathrm{Hom}(X^*(A), \mathbf{R})$ be the dual space. In fact, there is a canonical pairing $\langle \cdot, \cdot \rangle : \mathfrak{a} \times \mathfrak{a}^* \to \mathbf{R}$ given by $\langle \psi, \chi \otimes r \rangle := r\psi(\chi)$ where $\chi \otimes r \in \mathfrak{a}^*, r \in \mathbf{R}$ and $\psi \in \mathfrak{a}$. The above pairing extends to a $\mathbf{C}$- bilinear pairing $\langle \cdot, \cdot \rangle_{\mathbf{C}} : \mathfrak{a}_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}} \to \mathbf{C}$. To ease notation we will denote both by $\langle \cdot, \cdot \rangle$. We now define a homomorphism $H_M : M(\mathbf{A}) \to \mathfrak{a}$ by

$$\langle \psi, H_M(m) \rangle = \log |\psi(m)|_{\mathbf{A}}$$

for $\psi \in X^*(M)$. This homomorphism factors into a product of local factors $H_M = \prod_v H_{M,v}$ where $H_{M,v} : M(\mathbf{Q}_v) \to \mathfrak{a}$ is defined by

$$\langle \psi, H_M(m_v) \rangle = \log |\psi(m_v)|_{\mathbf{Q}_v}$$

where v is a place of $\mathbf{Q}$.

Let K be a maximal compact subgroup of $G(\mathbf{A})$. Then Iwasawa decomposition allows us to write

$$G(\mathbf{A}) = P(\mathbf{A})K = M(\mathbf{A})N(\mathbf{A})K. \tag{4.1}$$

Using 4.1 we can extend H_M to $G(\mathbf{A})$; we denote this extension by H_P. We call H_P the Harish-Chandra homomorphism. It is closely related to the modulus character δ_P of P. Note that P is not unimodular. The modulus character is the ratio of the right and the left invariant Haar measures on P.

Let $P_0 = M_0 N_0$ be a minimal $\mathbf{Q}$-parabolic of G. As earlier we denote by A_0 the connected component of the center of M_0. Let R be the set of positive roots of G relative to A_0. Then the choice of P_0 determines a subset of positive roots R^+ of R. We denote by $\triangle \subset R^+$ the simple roots. Then by standard results there is a one-to-one correspondence between the set of standard parabolics containing P_0 and the set of subsets $\theta \in \triangle$. Under this correspondence the maximal parabolic subgroups correspond to $\theta = \triangle - \{\alpha\}$ for some $\alpha \in \triangle$ and P_0 corresponds to the empty set. Let ρ_P be half the sum of elements of R^+ which occur in N.

Let $Ad : M \to End(\mathfrak{n})$ be the adjoint representation, where $\mathfrak{n}$ is the Lie Algebra of N. Then we can show that $\delta_P(m) = |\det Ad(m)|$ for $m \in M$. One can also check that $|(2\rho_P)(m)| = \delta_M(m)$. Hence

$$\exp(\langle t\rho_P, H_P(m)\rangle) = \exp(\frac{t}{2}\langle 2\rho_P, H_P(m)\rangle) = |(2\rho_P)(m)|^{\frac{t}{2}} = \delta_P(m)^{\frac{t}{2}}.$$

Let (π, W) be an irreducible automorphic representation of $M(\mathbf{A})$. Then π factors into a restricted product $\otimes' \pi_v$ of irreducible representations of $M(\mathbf{Q}_v)$. For each place v of $\mathbf{Q}$ and each $\nu \in \mathfrak{a}^*_{\mathbf{C}}$ we define

$$I'_P(\nu, \pi_v) = \mathrm{Ind}_{P(\mathbf{Q}_v)}^{G(\mathbf{Q}_v)} \pi_v \otimes \exp(\langle \nu, H_{P,v}(\cdot)\rangle)$$

as the space of functions $f : G(\mathbf{Q}_v) \to \mathbf{C}$ such that

$$f(mng) = \pi_v(m)\exp(\langle \nu + \rho_P, H_P(m)\rangle) f(g)$$

where $m \in M(\mathbf{Q}_v), n \in N(\mathbf{Q}_v)$ and $g \in G(\mathbf{Q}_v)$. We let $G(\mathbf{Q}_v)$ act on $I'(\nu, \pi_v)$ by the right regular action. Now we define

$$I'_P(\nu, \pi) := \bigotimes_v I'_P(\nu, \pi_v)$$

a restricted tensor product, i.e. given $f \in I'(\nu, \pi)$ there exists a finite set S of places of $\mathbf{Q}$ containing infinity such that

$$f \in \bigotimes_{v \in S} I'_P(\nu, \pi_v) \otimes \bigotimes_{v \notin S} f_v^0$$

where $f_v^0(k_v) = 1$ for all $k_v \in K_v$- the maximal compact of $G(\mathbf{Q}_v)$.

Definition IV.1. We call the functions f_v^0 *spherical vectors.*

Definition IV.2. We call f a K-finite vector if the space of functions on $G(\mathbf{A})$ spanned by the right regular translates of f by $k \in K$ is finite-dimensional.

Let $I_P(\nu, \pi)$ be the subspace of K-finite vectors of $I'_P(\nu, \pi)$. By the Iwasawa decomposition we have

$$M(\mathbf{Q})N(\mathbf{A})\backslash G(\mathbf{A}) = (M(\mathbf{Q})\backslash M(\mathbf{A})) \cdot K.$$

Using this decomposition we can give a more convenient description of the space $I_P(\nu, \pi)$.

Definition IV.3. A smooth function $f : M(\mathbf{Q})N(\mathbf{A})\backslash G(\mathbf{A}) \to \mathbf{C}$ is called a *P-automorphic form* if f is right K-finite and for every $k \in K, m \in M(\mathbf{A}), m \mapsto f(mk)$ is an automorphic form on $M(\mathbf{A})$. We shall denote this space of P-automorphic forms by $\mathcal{A}_P$.

Proposition IV.4. *[CKM04] The representation $I_P(\nu, \pi)$ is equivalent to the right regular representation of $G(\mathbf{A})$ on the space of functions*

$$\{f \exp(\langle \nu + \rho_P, H_P(\cdot)\rangle) | f \in \mathcal{A}_{P,\pi}\}$$

where $\mathcal{A}_{P,\pi} = \{f \in \mathcal{A}_P \mid m \mapsto f(mk) \in W_\pi\}$

In this thesis our primary interest will be in Eisenstein series associated to maximal parabolics.

Definition IV.5. Let $f \in I_P(\nu, \pi)$. Define

$$f_\nu := f \exp(\langle \nu + \rho_P, H_M(\cdot) \rangle).$$

The function f_ν is not an automorphic form on the whole group $G(\mathbf{A})$ as it is not invariant under $G(\mathbf{Q})$ on the left. But is it is invariant under left translation by $P(\mathbf{Q})$. To get a function which is $G(\mathbf{Q})$ invariant on the left we average the function over all coset representatives of $P(\mathbf{Q})\backslash G(\mathbf{Q})$.

Definition IV.6. We call

$$E(P, \nu, \pi, f, g) := \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} f_\nu(\gamma g) \tag{4.2}$$

an *Eisenstein series* associated to the datum (P, ν, π, f, g).

The Eisenstein series converges absolutely for ν in a certain cone of $\mathfrak{a}^*_{M,\mathbf{C}}$ and on identification of $\mathfrak{a}^*_{\mathbf{C}}$ with $\mathbf{C}^n$ by choosing a basis we can get a holomorphic function of n complex variables on the corresponding convex subset of $\mathbf{C}^n$. Langlands [Lan76] showed that the series (4.2) has a meromorphic continuation to all of $\mathbf{C}^n$.

4.2 Examples

Now we give examples of some Eisenstein series that we will encounter in this thesis. The Eisenstein series that we are interested in are induced from a character or from a modular form on a lower rank group.

4.2.1 Siegel Eisenstein series on Unitary Groups

Let δ_{Q_n} be the modulus character of the Siegel parabolic subgroup

$$Q_n = \left\{ \begin{pmatrix} A & \\ & \alpha^t A^{-1} \end{pmatrix} \begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix} | A \in GL_n, S \in M_n, {}^t\bar{S} = S \right\}$$

of G_n. Then

$$\delta_{Q_n}\left(\begin{pmatrix} A & \\ & D \end{pmatrix}\begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix}\right) = |\det A \det D^{-1}|_{\mathbf{A}}^{2n}.$$

Let $\mathcal{K}$ be an imaginary quadratic extension of $\mathbf{Q}$, v be a place of $\mathbf{Q}$, χ_v be a character of $\mathcal{K}_v^\times$ and let $s \in \mathbf{C}$. Let $K_{n,\mathbf{A}}^h$ be a compact subgroup in G_n. We let $I_n(\chi_v)$ be the space of functions $f_v : K_{n,v}^h \to \mathbf{C}$ such that $f_v(qk) = \chi_v(\det D_q) f_v(k)$ for all $q \in Q_n(\mathbf{Q}_v) \cap K_{n,v}^h$. Given $s \in \mathbf{C}$ and $f_v \in I(\chi_v)$ we define $f_{v,s} : G_n(\mathbf{Q}_v) \to \mathbf{C}$ by

$$f_{v,s}(qk) = \chi_v(\det D_q)|\det A_q D_q^{-1}|_v^s f_v(k) \qquad q \in Q_n(\mathbf{Q}_v) \text{ and } k \in K_{n,v}^h.$$

Remark IV.7. Note here we work with s instead of $ns + n/2$ so that our notation is consistent with Shimura's [Shi97].

Let $\chi = \otimes \chi_v$ be an idele class character of $\mathbf{A}_{\mathcal{K}}^\times$. We similarly define a space $I_n(\chi)$ of smooth functions on $K_{n,\mathbf{A}}^h$ and f_s associated to $f \in I_n(\chi)$. We can then make the following identification

$$I_n(\chi) = \bigotimes{}' I_n(\chi_v),$$

by noting

$$f \in \bigotimes_{v \in S} I_n(\chi_v) \otimes \bigotimes_{p \notin S} f_v^0$$

for some finite set S containing the infinite place.

For $\chi = \otimes \chi_v$ a unitary idele class character of $\mathbf{A}_{\mathcal{K}}^\times$ and $f \in I_n(\chi)$ we define the *Siegel Eisenstein series on* G_n as

$$E(Q_n, s, \chi, f, g) := \sum_{\gamma \in Q_n(\mathbf{Q}) \backslash G_n(\mathbf{Q})} f_s(\gamma g).$$

This series converges absolutely and uniformly for (s, g) in a compact subset of $\{\mathrm{Re}(s) > 0\} \times G_n(\mathbf{A})$ and defines an automorphic form on G_n and a holomorphic function on $\{\mathrm{Re}(s) > 0\}$. The Eisenstein series has a meromorphic continuation in s to all of $\mathbf{C}$ with at most finitely many simple poles.

4.2.2 Klingen Eisenstein series

In this section we discuss a particular Eisenstein series on $G = GU(2,2)$ that we will be interested in. Here we drop the subscript 2 in order to avoid any confusion with the exceptional group G_2. This Eisenstein series is induced from a modular form on $GL(2)$ and a character. It gets its name - Klingen Eisenstein series - from the fact that the sections are supported on the Klingen parabolic of G. Let κ be a positive integer and χ be a Dirichlet character of conductor N. Let $f' \in S_\kappa(N,\chi)$ be a cuspidal eigenform on GL_2 and $(\sigma_{f'}, V)$ be the irreducible cuspidal automorphic representation associated to it on $GL_2(\mathbf{A})$. Using the canonical inclusion of $GL_2(\mathbf{Q})$ into $G_1(\mathbf{Q})$ for $m_1 \in G_1(\mathbf{A})$ we can write $m_1 = xa$ with $x \in GL_2(\mathbf{A})$ and $a \in \mathbf{A}_{\mathcal{K}}^\times$. Let P be the Klingen parabolic subgroup of G. Then the Levi subgroup of P can be identified with $G_1 \times Res_{\mathcal{K}/\mathbf{Q}}\mathbf{G}_m$. Let $p = mn \in P(\mathbf{A})$ with $m = m(ax, b) \in M_P, b, a \in \mathbf{A}_{\mathcal{K}}^\times, x \in GL_2(\mathbf{A})$ and $n \in N_P$.

Let $\omega_\sigma = \chi$ be the central character of σ and τ, ψ: $\mathbf{A}_{\mathcal{K}}^\times \to \mathbf{C}^\times$ be Hecke characters such that $\psi|_{\mathbf{A}_{\mathbf{Q}}^\times} = \omega_\sigma$. Then we extend σ to a representation ρ of $M(\mathbf{A})$ on V by

$$\rho(g)v = \tau(b)\psi(a)\sigma(x)v, v \in V \tag{4.3}$$

and then trivially extend it to $P(\mathbf{A})$. Let $I(\rho)$ be the space of $K_{\mathbf{A}}^h = K_{2,\mathbf{A}}^h$ finite functions $f : G(\mathbf{A}) \to V$ such that

$$f(g)v = \rho(p)f(k) \qquad \text{where} \qquad g = pk \in P(\mathbf{A})K_{\mathbf{A}}^h$$

For each $f \in I(\rho)$ and each $s \in \mathbf{C}$ we define a function f_s on $G(\mathbf{A})$ by

$$f_s(g) = f_s(pk) = \delta_P(p)^s \rho(p) f(k), g = pk \in P(\mathbf{A})K_{\mathbf{A}}^h$$

where δ_P is the modulus character of P_2. Here one can easily check that the modulus character is given by $\delta_P(p) = |\mathrm{Nm}_{\mathcal{K}/\mathbf{Q}}(b)|^3|\mu(xa)|^{-3}$. For each $s \in \mathbf{C}$ there is a map

$$I(\rho) \hookrightarrow \mathcal{A}^0(M(\mathbf{Q})N(\mathbf{A})\backslash P(\mathbf{A}))$$

given by

$$f \mapsto (g \mapsto f_s(g)(1))$$

which we used to identify each f_s with a function on $G(\mathbf{A})$.

We define *Klingen Eisenstein series on G* to be

$$E(P, s, f, g) = \sum_{\gamma \in P_2(\mathbf{Q})\backslash G(\mathbf{Q})} f_s(\gamma g) \tag{4.4}$$

This is known to converge absolutely and uniformly for (s, g) in compact subsets of $\{s \in \mathbf{C} | \mathrm{Re}(s) > 0\} \times G(\mathbf{A})$ and defines an automorphic form on G.

Let τ, ψ, χ, σ and V be as above such that $\tau = \otimes\tau_v$, $\psi = \otimes\psi_v$, $\chi = \otimes\chi_v$, $\sigma = \otimes\sigma_v$ and $V = \otimes V_v$ where v runs over all the places of $\mathbf{Q}$. To (τ, σ, ψ) we can associate a representation of $(P(\mathbf{R}) \cap K_\infty^h) \times (P(\mathbf{A}_f))$ where K_∞^h is the maximal compact subgroup of $GU(2,2)$. For $m \in P(\mathbf{R}) \cap K_\infty^h) \times P(\mathbf{A}_f)$ and $w = \otimes w_v \in \otimes V_v$ we assume ρ decomposes as

$$\rho(m)w = \otimes_v(\rho_v(m_v)w_v)$$

and

$$f_s = \otimes f_{v,s}$$

where ρ_v and $f_{v,s}$ are defined as follows: For $p = mn, n \in N(\mathbf{R}), m = m(a, bx) \in M(\mathbf{R})$ with $a, b \in \mathbf{C}^\times, x \in GL_2(\mathbf{R})$, put

$$\rho_\infty(p)w_\infty = \tau_\infty(a)\psi_\infty(b)\sigma_\infty(x)w_\infty, w_\infty \in V_\infty. \tag{4.5}$$

Let $I(\rho_\infty)$ be the space of K_∞^h- finite functions $f_\infty \in \mathbf{C}^\infty(K_\infty^h, V_\infty)$ such that $f_\infty(k_1k_2) = \rho_\infty(k_1)f_\infty(k_2)$ for $k_1 \in P(\mathbf{R}) \cap K_\infty^h$. For $s \in \mathbf{C}$ and $f_\infty \in I(V_\infty)$ let

$$f_{\infty,s}(p) = \delta_P(m)^s \rho_\infty(m) f_\infty(k), p = mk \in P(\mathbf{R})K_\infty^h$$

and $g \in G(\mathbf{R})$ act on $f_{\infty,s}$ by the right regular action.

For $p = mn, n \in N(\mathbf{Q}_\ell), m = m(a, bx) \in M(\mathbf{Q}_\ell)$ with $a, b \in \mathcal{K}_\ell^\times, x \in GL_2(\mathbf{Q}_\ell)$, put

$$\rho_\ell(p)w_\ell = \tau_\ell(a)\psi_\ell(b)\sigma_\ell(x)w_\ell, w_\ell \in V_\ell. \tag{4.6}$$

Let $I(\rho_\ell)$ be the space of functions f_ℓ such that $f_\ell(gu) = f_\ell(g)$ for some open subgroup $U \subset K_\ell$ and $f_\ell(k_1k_2) = \rho_\ell(k_1)f_\ell(k_2)$ for $k_1 \in P(\mathbf{R}) \cap K_\ell^h$. For $s \in \mathbf{C}$ and $f_\ell \in I(\rho_\ell)$ let

$$f_{\ell,s}(p) = \delta_P(m)^s \rho_\ell(m) f_\ell(k), p = mk \in P(\mathbf{Q}_\ell)K_\ell^h$$

and $g \in G(\mathbf{Q}_\ell)$ act on $f_{\ell,s}$ by the right regular action.

If $\sigma_\ell, \tau_\ell, \psi_\ell$ are unramified then

$$\dim_{\mathbf{C}} I(\rho_\ell)^{K_\ell^h} = 1.$$

In this case $f_{\ell,s}$ is the unique K_ℓ^h-spherical vector.

CHAPTER V

Whittaker and Bessel models

In this chapter we discuss the notion of Whittaker and Bessel models. We first discuss the theory of Whittaker models for GL_n though our primary interest is in the Whittaker models on GL_2 which we later extend to get a degenerate Whittaker model on $GU(2,2)$. Then we discuss the notion of Whittaker model for $GSp(4)$, partly to indicate a well known fact that a holomorphic Siegel modular form is not generic. This leads to the study of Bessel models a kind of generalized Whittaker model more suitable for our context. Everything discussed in this chapter is available in the literature, for example Bump [Bum97] , Garrett [Gar84] and Novodvorsky and Piatetski-Shapiro [NPŠ73].

5.1 Whittaker Models

Let (σ, V_σ) be a smooth cuspidal representation of $GL_n(\mathbf{A})$, so $V_\sigma \subset \mathcal{A}_0$. Let $\phi \in V_\sigma$ be a cusp form on GL_n and let $\psi : \mathbf{Q}\backslash\mathbf{A} \to \mathbf{C}$ be a character. Let

$$N = N_n = \left\{ n = \begin{pmatrix} 1 & x_{1,2} & & * \\ & \ddots & \ddots & \\ & & \ddots & x_{n-1,n} \\ 0 & & & 1 \end{pmatrix} \right\}$$

be the maximal unipotent subgroup of GL_n. Then ψ defines a character of $N(\mathbf{Q})\backslash N(\mathbf{A})$ by

$$\psi(n) = \psi\Bigg(\begin{pmatrix} 1 & x_{1,2} & & & * \\ & \ddots & \ddots & & \\ & & & \ddots & x_{n-1,n} \\ 0 & & & & 1 \end{pmatrix}\Bigg) = \psi(x_{1,2} + \cdots + x_{n-1,n}).$$

The *Whittaker function* associated to ϕ and ψ is given by

$$W_\phi(g) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \phi(ng)\psi^{-1}(n)dn.$$

It is easy to check that W_ϕ is a smooth function on $GL_n(\mathbf{A})$ such that $W_\phi(ng) = \psi(n)W_\phi(g)$ for all $n \in N(\mathbf{A})$. The Fourier expansion of ϕ then can be given as

$$\phi(g) = \sum_{\gamma \in N_{n-1}(\mathbf{Q})\backslash GL_{n-1}(\mathbf{Q})} W_\phi\Bigg(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\Bigg).$$

In particular, for GL_2

$$\phi(g) = \sum_{\gamma \in \mathbf{Q}^\times} W_\phi\Bigg(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\Bigg).$$

Let

$$\mathcal{W}(\sigma, \psi) = \{W_\phi | \phi \in V_\sigma\}.$$

The group $GL_n(\mathbf{A})$ acts on $\mathcal{W}(\sigma, \psi)$ by right translations and the map

$$\phi \mapsto W_\phi \qquad \text{intertwines} \qquad V_\sigma \simeq \mathcal{W}(\sigma, \psi).$$

As we have seen, in the case of GL_n we can recover ϕ from W_ϕ through its Fourier expansion. So we know that $W_\phi \neq 0$ for all $\phi \neq 0$. The space $\mathcal{W}(\sigma, \psi)$ is called the *ψ-Whittaker model of σ*. More generally a $\psi-$Whittaker model of (σ, V_σ) is a $GL_n(\mathbf{A})$ embedding

$$V_\sigma \hookrightarrow \{ \text{ smooth functions } W : GL_n(\mathbf{A}) \to \mathbf{C} \mid W(ng) = \psi(n)W(g) \text{ for all } n \in N(\mathbf{A})\}$$

The representation (σ, V_σ) has a Whittaker model if and only if there exists a non-zero *Whittaker functional* $\Lambda : V_\sigma \to \mathbf{C}$ such that

$$\Lambda(\sigma(n)v) = \psi(n)\Lambda(v) \text{ for all} n \in N(\mathbf{A}), v \in V_\sigma\}.$$

A model $v \mapsto W_v$ gives a functional $\Lambda(v) = W_v(1)$ and a functional Λ gives a model $v \mapsto (g \mapsto \Lambda(\sigma(g)v))$.

The notion of Whittaker models and Whittaker functionals also make sense for $GL_2(\mathbf{Q}_v)$. We have the following fundamental result of Gelfand, Kazhdan and Shalika, [GK75], [Sha74].

Theorem V.1. *(Local Uniqueness) For any place v, given (σ_v, V_{σ_v}) an irreducible admissible smooth representation of $GL_n(\mathbf{Q}_v)$, the space of Whittaker functionals is at most one dimensional, that is, σ_v has at most one Whittaker model.*

Definition V.2. A representation (σ_v, V_{σ_v}) of a reductive group, having a Whittaker model (with respect to the maximal unipotent subgroup) is called *generic*.

The local uniqueness has a global consequence and can be stated as

Theorem V.3. *(Global uniqueness)If $\sigma = \otimes' \sigma_v$ is an irreducible admissible smooth representation of $GL_n(\mathbf{A})$ then the space of Whittaker functionals is at most one dimensional, that is, σ has at most one Whittaker model.*

As an easy corollary of the above theorems we get

Corollary V.4. *If (σ, V_σ) is a cuspidal representation such that $\sigma = \otimes' \sigma_v$ then σ and σ_v's are generic.*

In many application of the Rankin-Selberg method to construct L-functions, the existence of Whittaker models plays a crucial role in establishing the Euler product criterion. This is essentially due to the following corollary:

Corollary V.5. *(Factorization of Whittaker functions) If (σ, V_σ) is a cuspidal representation with $\sigma = \otimes' \sigma_v$ and $\phi \in V_\sigma$ such that $V_\sigma \simeq \otimes' V_{\sigma_v}$ where $\phi \mapsto \otimes \xi_v$ then*

$$W_\phi(g) = \prod_v W_{\xi_v}(g_v)$$

Another useful interpretation of the Whittaker models in the context of classical modular forms comes from realizing them as Fourier coefficients. In fact, let $f = \sum a_n e^{(2\pi i n z)}$ be a classical eigenform of weight κ and let ϕ be the associated automorphic form. Then by multiplicity one both ϕ and its Whittaker function W_ϕ are decomposable. Now if we decompose W_ϕ as $W_\phi = W_\infty W_f$ then using the transition between the classical and the adelic setting one can show that

$$W_f \begin{pmatrix} n & \\ & 1 \end{pmatrix} = a_n.$$

For more details we refer the reader to [Gel75] and [CKM04].

5.2 Whittaker models for $GSp(4)$

Let F be a holomorphic Siegel eigen cusp form on $GSp(4)$ such that ϕ_F generates an irreducible cuspidal automorphic representation π_F of $GSp(4, \mathbf{A})$.

In this section we show that π_F is not generic. For this it is enough to show that for any choice of additive character ψ, $W_\phi(g) = 0$ for all $g \in GSp(4, \mathbf{R})^+$. Let $\psi_1 : \mathbf{Q} \backslash \mathbf{A} \to \mathbf{C}, \psi_2 : \mathbf{Q} \backslash \mathbf{A} \to \mathbf{C}$ be continuous characters. Then the value on g of the Whittaker function of $\phi := \phi_F$ with respect to ψ_1 and ψ_2 is a non-zero multiple of

$$W_\phi(g) \quad = \quad \int_{(\mathbf{R}/M\mathbf{Z})^4} \phi\left(\left(\begin{smallmatrix} 1 & & a & b \\ & 1 & b & c \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right) g_x\right) \psi_1(x) \psi_2(a) dx \;\; da \;\; db \;\; dc \tag{5.1}$$

where $g_x = \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} g$ and M is a sufficiently divisible integer. From 3.4.2 we have

$$\begin{aligned} \phi\left(\left(\begin{smallmatrix} 1 & & a & b \\ & 1 & b & c \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right) g_x\right) &= F\left(\left(\begin{smallmatrix} 1 & & a & b \\ & 1 & b & c \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right) g_x(i)\right) j\left(\left(\begin{smallmatrix} 1 & & a & b \\ & 1 & b & c \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right) g_x, i\right)^{-\kappa} \\ &= F\left(\left(\begin{smallmatrix} 1 & & a & b \\ & 1 & b & c \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right) g_x(i)\right) j\left(g_x, i\right)^{-\kappa} \\ &= F\left(Z_x + \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\right) j\left(g_x, i\right)^{-\kappa} \end{aligned}$$

So we have

$$W_\phi(g) = \int_{(\mathbf{R}/M\mathbf{Z})} \left(\int_{(\mathbf{R}/M\mathbf{Z})^3} F\left(Z_x + \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\right) \psi_2(a) da \;\; db \;\; dc \right) j\left(g_x, i\right)^{-\kappa} \psi_1(x) dx.$$

But F has a Fourier expansion given by

$$F(Z) = \sum_{T>0} a(T) e(\mathrm{tr}(TZ)) \text{ where } e(Z) = \exp(2\pi i Z)$$

for T is in some lattice. Hence (5.1) equals

$$\int_{(\mathbf{R}/M\mathbf{Z})} \sum_{T>0} e(\mathrm{tr}(TZ_x)) a(T) \left(\int_{(\mathbf{R}/M\mathbf{Z})^3} e\left(\mathrm{tr}\left(T\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\right)\right) \psi_2(a) da \;\; db \;\; dc \right) j\left(g_x, i\right)^{-\kappa} \psi_1(x) dx.$$

Let

$$I_T = \int_{(\mathbf{R}/M\mathbf{Z})^3} e\left(\mathrm{tr}\left(T\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\right)\right) \psi_2(a) da \;\; db \;\; dc$$

For $T = \left(\begin{smallmatrix} \alpha & \beta \\ \beta & \gamma \end{smallmatrix}\right)$

$$I_T = \int_{(\mathbf{R}/M\mathbf{Z})} e(\alpha a) \psi_2(a) da \int_{(\mathbf{R}/M\mathbf{Z})} e(2\beta b) db \int_{(\mathbf{R}/M\mathbf{Z})} e(\gamma c) dc$$

Hence I_T is non zero only if $\beta = 0, \gamma = 0$. But then T is not positive definite and hence F cannot be a holomorphic eigen cusp form. So we have shown that

Proposition V.6. *A holomorphic Siegel eigen cusp form is not generic.*

5.3 Bessel models

In this section we closely follow the exposition given by Novodvorsky and Piatetski-Shapiro, [NPŠ73] and Furusawa, [Fur93]. As noted in the previous section, automorphic representations of $GSp(4, \mathbf{Q})$ do not always have a Whittaker model. So in this section we shall consider a generalized Whittaker model - the Bessel Model - defined with respect to a subgroup R introduced in the section below.

Let $S \in M_2(\mathbf{Q})$ such that $S = S^t$. We define the discrimnant $d = d(S)$ by $d(S) = -4\det S$ and assume that S is anisotropic over $\mathbf{Q}$. Under this assumption d is not a square in $\mathbf{Q}$. Let $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$. Then we define an element $\xi = \xi_S$ in $M_2(\mathbf{Q})$ by $\xi_S = \begin{pmatrix} b/2 & c \\ -a & -b/2 \end{pmatrix}$. Since $4\xi^2 = d, Q(\xi) = \{x + y\xi | x, y \in \mathbf{Q}\}$ is a quadratic extension of $\mathbf{Q}$ in $M_2(\mathbf{Q})$. We identify $Q(\xi)$ with $Q(\sqrt{d})$ via

$$F(\xi) \ni x + y\xi \mapsto x + \frac{y}{2}\sqrt{d} \in \mathbf{Q}(\sqrt{d}), x, y \in \mathbf{Q}$$

Now define a subgroup T of GL_2 by

$$T = \{g \in GL_2(\mathbf{Q}) | {}^t gSg = (\det g)S\}.$$

Then

$$T(\mathbf{Q}) \simeq \mathbf{Q}(\sqrt{d})^\times.$$

and we identify $T(\mathbf{Q})$ with $\mathbf{Q}(\sqrt{d})$ using the above identifications. We consider T as a subgroup of $H := H_2$ via

$$g \to \begin{pmatrix} g & 0 \\ 0 & \det g \cdot {}^t g^{-1} \end{pmatrix} \in H \text{ for } g \in T.$$

Let us denote by U the subgroup of H defined by

$$U = \Big\{ u(X) = \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix} | X^t = X \Big\}.$$

Then U is an abelian group and is equal to the unipotent radical of the Siegel parabolic in H. Finally we define a subgroup R of H by $R = TU$. Let ϑ be a non-trivial character of $\mathbf{Q}\backslash\mathbf{A}$. The we define a character ϑ_S on $U(\mathbf{A})$ by $\vartheta_S(u(X)) = \vartheta(tr(SX))$ for $X = {}^t X \in M_2(\mathbf{A})$. We will sometimes write ϑ_S as ϑ. Let Λ be a character of $T(\mathbf{Q})\backslash T(\mathbf{A})$. We will denote by $\Lambda \otimes \vartheta_S$ the character on $R(\mathbf{A})$ defined by $(\Lambda \otimes \vartheta)(tu) = \Lambda(t)\vartheta_S(u)$ for $t \in T(\mathbf{A})$ and $u \in U(\mathbf{A})$. Here one should note that T as defined above is the connected component of the stabilizer of ϑ_S in the levi of the Siegel parabolic.

Let π be an irreducible automorphic representation of $H(\mathbf{A}) = GSp_4(\mathbf{A})$ and V_π be its underlying space of automorphic functions. We assume that

$$\Lambda|_{\mathbf{A}^\times} = \omega_{\pi.}$$

Then for $\varphi \in V_\pi$, we define a function B_φ on $H(\mathbf{A})$ by

$$B_\varphi(h) = \int_{Z_H(\mathbf{A})R(\mathbf{Q})\backslash(\mathbf{A})} (\Lambda \otimes \vartheta_S)(r)^{-1} \cdot \varphi(rh)dh$$

We say that π has a global *Bessel model* of type (S, Λ, ϑ) if for some $\varphi \in V_\pi$, the function B_φ is non-zero.

Just as in the case of Whittaker models it is desirable to have a theory of local Bessel models. For this we fix a local field $\mathbf{Q}_v$ and local characters Λ_v, ϑ_v and ϑ_{Sv} analogous to the characters Λ, ϑ and $\vartheta_S above$. Let (π_v, V_{π_v}) be an irreducible admissible representation of the $H(\mathbf{Q}_v)$, when v is finite, or $(\mathfrak{g}, \mathcal{L})$ when v is archimedean. Then we say that the representation π has a local Bessel model of type $(S_v, \Lambda_v, \vartheta_v)$ if there is a non-zero map in

$$\mathrm{Hom}(\pi_v, Ind_{R(\mathbf{Q}_v)}^{H(\mathbf{Q}_v)}(\Lambda_v \otimes \vartheta_v)).$$

Here the Hom-space is the collection of $H(\mathbf{Q}_v)$-intertwining maps when v is finite, and the collection of all $(\mathfrak{g}, \mathcal{L})$-maps when v is archimedean. In fact we can define the *Bessel functionals* just like the Whittaker functionals as $l_v : V_{\pi_v} \to \mathbf{C}$ such that

$$l_v(\pi_v(r)\xi_v) = (\Lambda_v \otimes \vartheta_v)(r)l(\xi_v) \qquad \text{for all} \qquad r \in R(\mathbf{Q}_v) \qquad \text{and} \qquad \xi_v \in V_{\pi_v}.$$

Then we have a local uniqueness result

Theorem V.7. *[NPŠ73] Let (π, V_π) be an irreducible smooth admissible representation of $H(\mathbf{Q}_v)$ and let $\Lambda_v \otimes \vartheta_v$ be a character of $R(\mathbf{Q}_v)$ as above. Then the space of Bessel functionals is at most one dimensional, that is , there is at most one Bessel model.*

The local uniqueness has a global consequence

Theorem V.8. *(Global uniqueness)If $\pi = \otimes' \pi_v$ is an irreducible admissible smooth representation of $H(\mathbf{Q}_v)$ then the space of Bessel functionals is at most one dimensional, that is, π has at most one Bessel model.*

Just as in the case of Whittaker models we have

Corollary V.9. *(Factorization of Bessel functions) If (π, V_π) is a cuspidal representation with $\pi = \otimes' \pi_v$ and $\varphi \in V_\pi$ such that $V_\pi \simeq \otimes' V_{\pi_v}$ where $\varphi \mapsto \otimes \xi_v$ then*

$$W_\varphi(g) = \prod_v W_{\xi_v}(g_v)$$

Suppose $F \in S_{s,\kappa}(\Gamma^s_{Q_n}(N), \chi)$ and the Fourier coefficient $a(F, S) \neq 0$ for some $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ such that $-D = b^2 - 4ac$ is the discriminant of the imaginary quadratic extension $\mathbf{Q}(\sqrt{-D})$.

Now let us define a two-by-two symmetric matrix $S(-D)$ by

$$S(-D) = \begin{cases} \begin{pmatrix} D/4 & 0 \\ 0 & -1 \end{pmatrix}, & \text{if } D \equiv 0 \pmod 4 \\ \begin{pmatrix} (1+D)/4 & 1/2 \\ 1/2 & 1 \end{pmatrix}, & \text{if } D \equiv 3 \pmod 4 \end{cases} \tag{5.2}$$

Let $T(\mathbf{Q}) = T_{S(-D)} = \{g \in GL_2(\mathbf{Q}) | {}^t g S(-D) g = (\det g) \cdot S(-D)\} \simeq \mathbf{Q}(\sqrt{-D})^\times$.

Let ξ be any unramified ideal class character of $\mathcal{K} = \mathbf{Q}(\sqrt{-D})$ and let

$$\Lambda = (\chi \circ \mathrm{Nm}_{\mathcal{K}/\mathbf{Q}}) \circ \xi$$

which we view as a character of $\mathbf{A}^\times$ using the identification $T(\mathbf{A}) \simeq \mathbf{A}_{\mathcal{K}}^\times$. Let $\vartheta = e_{\mathbf{A}}(\cdot)$ be the standard additive character. We consider the Bessel model of type $(S(-D), \Lambda, \vartheta)$ associated to $\bar{\varphi}$

$$B_{\bar{\varphi}}(h) = \int_{Z_H(\mathbf{A})R(\mathbf{Q})\backslash R(\mathbf{A})} (\Lambda \otimes \vartheta_S)(r)^{-1} \cdot \bar{\varphi}(rh) dh$$

where $\bar{\varphi}(h) = \overline{\varphi(h)}$ and $\varphi = \varphi_F$ is the automorphic form associated to F. Then we have the following important proposition that relates this global Bessel model to the Fourier coefficients of F.

Proposition V.10. *Let $h_\infty \in H_\infty^+$, then*

$$B_{\bar{\varphi}}(h_\infty) = \mu_2(h_\infty)^\kappa \overline{j(h_\infty, i)^{-\kappa}} e(-tr(S(-D)\overline{h_\infty(i)})) \sum_{i=1}^{h(-D)} \Lambda(t_j)^{-1} \overline{a(S_j, F)}$$

where $h(-D)$ is the class number of $\mathbf{Q}(\sqrt{-D})$, t_j $(j = 1, \cdots, h(-D))$ are the representatives of $T(\mathbf{Q})\backslash T(\mathbf{A})/T(\mathbf{R}) \prod_{p<\infty}(T(\mathbf{Q}_p) \cap GL_2(\mathbf{Z}_p)) \simeq$ class group of $Q(\sqrt{-D})$ such that $t_j \in \prod_{p<\infty} T(\mathbf{Q}_p)$, $S_j = \det \gamma_j^{-1} \cdot {}^t\gamma_j S(-D) \gamma_j$ where

$$t_j = \gamma_j m_j k_j, \gamma_j \in GL_2(\mathbf{Q}), m_j \in GL_2(\mathbf{R})^+, k_j \in \prod_{p<\infty} GL_2(\mathbf{Z}_p).$$

Proof. See Sugano,[Sug85] □

Here we note that S_j $(j = 1, \cdots, h(\sqrt{-D}))$ are the representatives of the $SL_2(\mathbf{Z})$ equivalence classes of primitive semi-integral two-by-two matrices of discriminant $-D/4$.

Remark V.11. By our assumption on F there exists a ξ such that $B_{F,S,\xi}(1_4) \neq 0$.

CHAPTER VI

L-functions

In this chapter we discuss the various L-functions that we will encounter in this work. For L-functions associated to elliptic cusp forms one can see Miyake, [Miy89].

6.1 Standard L-function on $GL(2)$

Let $f \in S_\kappa(\Gamma_0(N), \chi)$ be an elliptic cusp form with a Fourier expansion of the form

$$f(z) = \sum_{i=1}^{\infty} a_n e^{2\pi i n z}.$$

To f we can associate an L-function given by

$$L(s, f) = \sum_{i=1}^{\infty} a_n n^{-s}.$$

Then $L(s, f)$ converges absolutely and uniformly on any compact subset of $\mathrm{Re}(s) > 1 + \kappa/2$. For $N > 0$ we put

$$\Lambda_N(s, f) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) L(s, f).$$

Then $\Lambda_N(s, f)$ can be analytically continued to the whole s-plane, satisfying the functional equation

$$\Lambda_N(s, f) = i^\kappa \Lambda_N(\kappa - s, f \mid_\kappa \left(\begin{smallmatrix} & -1 \\ N & \end{smallmatrix}\right)).$$

If $f \in S_\kappa(\Gamma_0(N), \chi)$ is an eigenform then $L(s,f)$ has an Euler product expansion given by

$$L(s,f) = \prod_{p|N}(1 - a_p p^{-s} + \chi(p)p^{\kappa-1-2s})^{-1} \times \prod_{p|N}(1 - a_p p^{-s})^{-1}$$

Now suppose (σ, V_σ) is an irreducible cuspidal automorphic representation of $GL(2)$ and $\sigma = \otimes\sigma_v$. Let

$$L(s,\sigma) = \prod_{v\nmid\infty} L(s,\sigma_v).$$

For an unramified place v and $\sigma_v = \sigma_v(\alpha_1, \alpha_2)$ define

$$L(s,\sigma_v) = L(s,\alpha_1)L(s,\alpha_2).$$

Then for f an eigenform and $\sigma = \sigma_f$,

$$L^\Sigma(s,f) = L^\Sigma(s - (\kappa-1)/2, \tilde{\sigma_f})$$

where $\tilde{\sigma_f}$ is the contragredient representation associated to σ_f and Σ is the set ramified places of σ_f. If f is a new form then

$$L(s,f) = L(s - (\kappa-1)/2, \tilde{\sigma_f})$$

6.2 Spin L-function on $GSp(4)$

Let (π, V_π) be an irreducible cuspidal automorphic representation of $GSp(4)$ and suppose $\pi = \otimes\pi_v$. Let S be the set of places where π is ramified and the infinite place. Let B be the Borel of $GSp(4)$ and μ_1, μ_2 and λ be characters of $\mathbf{Q}_\ell^\times$. Consider the character of $B(\mathbf{Q}_\ell)$ given by

$$\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & ta^{-1} & * \\ & & & tb^{-1} \end{pmatrix} \mapsto \mu_1(a)\mu_2(b)\lambda(t).$$

One can check that the modulas character is given by $\delta_B(h) = |\frac{a^4b^2}{t^3}|$. Then for each unramified place ℓ, π_ℓ is realized as the right regular representation on the space of locally constant complex valued functions ϕ on $GSp(4, \mathbf{Q}_\ell)$ satisfying

$$\phi(hg) = \delta_B(h)^{1/2}\mu_1(a)\mu_2(b)\lambda(t)\phi(g) \text{ for all } h = \begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & ta^{-1} & * \\ & & & tb^{-1} \end{pmatrix}$$

For such ℓ we write $\pi_\ell = \pi_\ell(\mu_1, \mu_2, \lambda)$. Now we define

$$L^S(\text{spin}, s, \pi) = L^S(s, \pi) = \prod_{\ell \notin S} L(s, \pi_\ell)$$

where for $\ell \notin S$ and

$$L(s, \pi_\ell) = L(s, \lambda)L(s, \mu_1\lambda)L(s, \mu_2\lambda)L(s, \mu_1\mu_2\lambda).$$

Then by Piatetski-Shapiro [PS97], $L^S(\text{spin}, s, \pi)$ converges in some real half plane $\text{Re}(s) > s_0$, satisfies a functional equation and has a meromorphic continuation to the whole s-plane.

Suppose F is a Siegel eigenform and ϕ_F is the associated automorphic form on $GSp(4)$. Further assume that the automorphic representation π_F generated by ϕ_F is irreducible. Then we define the spin L-function associated to F as

$$L^S(\text{spin}, s, F) = L^S(s, F) = L^S(\text{spin}, s - \kappa + 3/2, \tilde{\pi}_F)$$

6.3 L-function for $GSp(4) \times GL(2)$

Using the notation in the previous section. Let S' also denote the union of all the ramified places of σ and π and infinity. Then we define the degree eight L-function $L^S(s, \pi \times \sigma)$ as the convolution of $L^{S'}(s, \pi)$ and $L^{S'}(s, \sigma)$. For F and f as above we can define a degree eight L-function $L^S(s, F \times f)$ as the convolution of $L^S(s, F)$ and $L^S(s, f)$. Then it follows from the discussion above that $L^S(s, F \times f)$ has the property:

$$L^S(s, F \times f) = L^S(s - 3\kappa/2 + 2, \tilde{\pi}_F \times \tilde{\sigma}_f)$$

CHAPTER VII

Global Integral

7.1 The Global Integral

In this section we discuss the integral representation of the degree eight L-function associated to a holomorphic Siegel cusp eigenform on $GSp(4)$ and a cusp eigenform on GL_2.

Let (π, V_π) be an irreducible cuspidal automorphic representation of $GSp(4, \mathbf{A})$. Suppose (π, V_π) has a Bessel model (as in section 5.3) of type (S, Λ, ϑ) denoted by

$$\varphi \mapsto B_\varphi = \int_{Z_H(\mathbf{A})R(\mathbf{Q})\backslash R(\mathbf{A})} (\Lambda \otimes \vartheta_S)^{-1}(r)\varphi(rh)dh$$

Let

- (σ, V_σ) be an irreducible automorphic cuspidal representation of $GL_2(\mathbf{A})$
- $\mathcal{K} = \mathbf{Q}(\sqrt{d(S)})$ be an imaginary quadratic extension of $\mathbf{Q}$
- τ, ψ be characters of $\mathbf{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ such that $\psi \mid_{\mathbf{A}^\times} = w_\sigma$ and $\Lambda = \tau^c\psi$.

Let $I(\rho)$ be the space of functions associated to σ, τ, ψ in the discussion of Klingen Eisenstein series. For $f \in I(\rho)$ let $E(P, s, f, g)$ be the Klingen Eisenstein series associated to it as in (4.4).

Now consider the global integral

$$Z(s) = Z(s, f, \varphi) = \int_{H(\mathbf{Q})\backslash H(\mathbf{A})} E(P, s, f, h)\varphi(h)dh.$$

Remark VII.1. In our normalization we write s for $3s+3/2$ in Furusawa [Fur93].

Proposition VII.2. *[Fur93](Basic Identity)*

$$Z(s) = \int_{R(\mathbf{A})\backslash H(\mathbf{A})} W(f;\theta h, s) B_\varphi(h) dh$$

where

$$W(f;g,s) = \int_{\mathbf{Q}\backslash\mathbb{A}} f\left(\left(\begin{smallmatrix} 1 & & x & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right) g,\ s\right) \vartheta_S(ax)dx,$$

ϑ_S is the additive character associated the Bessel model $B(S,\Lambda,\vartheta)$ where $S = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$ and θ is the nontrivial representative of the coset $Q(\mathbf{Q})\backslash G(\mathbf{Q})/H(\mathbf{Q})$ given by $\left(\begin{smallmatrix} 1 & \alpha & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{\alpha} & 1 \end{smallmatrix}\right)$ where $\alpha = \frac{b+\sqrt{d}}{2a}$.

By the uniqueness of the Bessel model B_φ and its local analogs we know that there are local Bessel models $\varphi_v \mapsto B_{\varphi_v}(h_v)$ of π_v of the type $(S, \Lambda_v, \vartheta_v)$, one for each finite place v of $\mathbf{Q}$ such that

- for all but finitely many places v at which π_v is unramified $B_{\varphi_v^{\mathrm{sph}}}(1_4) = 1$, $\varphi_v^{\mathrm{sph}} \in V_{\pi_v}$ being the distinguished unramified vector implied in the identification $V_\pi \simeq \otimes_v V_{\pi_v}$;

- if $\varphi = \otimes_v \varphi_v$ is a pure tensor then

$$B_\varphi(h) = \prod B_{\varphi_v}(h_v).$$

The function $W(f;g,s)$ is essentially a degenerate Whittaker model and has a similar product decomposition.

Recalling that $y \mapsto f_s(m(y,1)g)$ is a cuspform in V_σ, we see that if $\lambda : V_\sigma \to \mathbf{C}$ is the Whittaker functional for the character $\vartheta_S(a(\cdot))$ such that

$$\lambda(\phi) = \int \phi\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) \vartheta_S(ax)dx,$$

then

$$W(f;g,s) = \lambda(f_s(g)).$$

The uniqueness of the Whittaker functionals implies that there are local Whittaker functionals $\lambda_v : V_{\sigma_v} \to \mathbf{C}$ for $\vartheta_v(a(\cdot))$, one for each place v of $\mathbf{Q}$ such that

- for all but finitely many places v at which σ_v is unramified $\lambda_v(\phi_v^{\mathrm{sph}}) = 1$, $\phi_v^{\mathrm{sph}} \in V_{\sigma_v}$ being the distinguished unramified vector implied in the identification $V_\sigma \simeq \otimes_v V_{\sigma_v}$;

- if $f = \otimes_v f_v$ then
$$W(f,g,s) = \prod_v W(f_v, g_v, s)$$
where $W(f_v; g_v, s) = \lambda_v(f_{v,s}(g_v))$.

Theorem VII.3. *If $f = \otimes_v f_v$ and $\varphi = \otimes_v \varphi_v$, $Re(s) >> 0$ then*

$$Z(s) = \prod_v Z_v(s)$$

where

$$Z_v(s) = \int_{R(Q_v)\backslash H(Q_v)} W(f_v; \theta h_v, s) B_{\varphi_v}(h_v) dh_v$$

7.1.1 Unramified calculations

Let Ω be a finite set of places of $\mathbf{Q}$ such that Ω contains the infinite place and for a finite ℓ, $\ell \notin \Omega$ implies

1. the local components of π, σ, τ, ψ at ℓ are all unramified;

2. the conductor of ϑ_ℓ is $\mathbf{Z}_\ell$;

3. $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbf{Z}_\ell)$ and $a \in \mathbf{Z}_\ell^\times$;

4. $d = d(S) = b^2 - 4ac$ is the generator of the discriminant of $\mathcal{K}_\ell/\mathbf{Q}_\ell$.

Under the above assumptions the local integral $Z_\ell(s)$ for $\ell \notin \Omega$ can be computed by explicitly understanding the contribution of the Whittaker model and the Bessel model using a theorem of Sugano [Sug85]. After a lengthy computation Furusawa proves that

Theorem VII.4. *[Fur93] For $\ell \notin \Omega$, $\psi_\ell = \psi_\ell^{sph}$ and $f_\ell = f_\ell^{sph}$*

$$Z_\ell(s) = \frac{L(s-1, \tilde{\pi}_\ell \times \tilde{\sigma}_\ell)}{L(2s-2, \tau_\ell \mid \mathbf{Q}_\ell^\times) L(s-3/2, \sigma_\ell \times \sigma_\ell(\Lambda_\ell) \times \tau_\ell \mid \mathbf{Q}_\ell^\times)} W_{f_\ell}(1)$$

where $\tilde{\pi}_\ell$ (resp. $\tilde{\sigma}_\ell$) denotes the contragredient of π_ℓ (resp. σ_ℓ).

7.1.2 Archimedian place calculations

Let F be a holomorphic Siegel eigenform of weight κ such that φ_F generates an irreducible cuspidal automorphic representation π_F. Suppose that $\varphi = \bar{\varphi}_F \in \tilde{\pi}_F$. Then we can decompose φ as

$$\varphi = \varphi_\infty \otimes \varphi^\Omega \otimes (\otimes_{v \in \Omega, v \neq \infty} \varphi_v^{\mathrm{sph}}) \in \otimes \tilde{\pi}_{F,v}$$

for some finite set of finite places Ω. Suppose f' is a holomorphic modular eigenform of weight κ such that $\phi_{f'}$ generates an irreducible cuspidal representation σ and $\sigma = \otimes \sigma_v$.

Let

- $\mathcal{K} = \mathbf{Q}(\sqrt{d(S)})$ be an imaginary quadratic extension of $\mathbf{Q}$;
- τ, ψ be characters of $\mathbf{A}_\mathcal{K}^\times$ such that $\tau_\infty(z) = \psi_\infty(z) = (\frac{z}{|z|})^{-\kappa}$.

Let $I(\rho)$ be the space associated to σ, τ, ψ as in the discussion of Klingen Eisenstein series. Suppose $f_\infty \in I(\rho)$ such that

$$f_{\infty,s}(pk) = \delta_P(s)^{s/3} \rho(p) j(k,i)^{-\kappa}$$

where $p \in P(\mathbf{R}), k \in K_\infty^{h,+}$ - the maximal compact of $U(2,2)$.

Now consider the local integral at infinity

$$Z_\infty(s) = Z(s, f_\infty, \varphi_\infty) = \int_{R(\mathbf{R})\backslash H(\mathbf{R})} W(f_\infty, \theta h, s) B_{\varphi_\infty}(h) dh$$

and we further assume that:

For F the Siegel cusp form of degree two and weight κ

1. $a(S, F) \neq 0$
2. the weight κ is divisible by $w(-D)$, the number of roots of unity in $\mathbf{Q}(\sqrt{-D})$

then

Theorem VII.5. *[Fur93]*

$$Z_\infty(s) = \pi(4\pi)^{-s-3\kappa/2+3}(D^{1/2})^{-2s+3-\kappa}\frac{\Gamma(s+3\kappa/2-3)}{2s+\kappa-4}W_\infty(f_\infty; 1, s)a_F(1)$$

where $\tilde{B}_{S,F}(1) = \sum_{i=1}^{h(-D)} \Lambda(t_j)^{-1}\overline{a(S_j, F)}$ *as in (V.10).*

CHAPTER VIII

Special Eisenstein series and Pull back formula

In this chapter we choose a special Siegel Eisenstein series on $GU(n,n)$. The support for the sections defining this Eisenstein series are chosen so that the Eisenstein series is suitable for the pullback formula and has Fourier coefficients that can be easily interpolated. In the first section we recall the doubling method of Piatetski-Shapiro and Rallis. In their work [GPSR87], they use this method to construct L-functions on classical groups. We are interested in a generalization of the doubling method to construct Eisenstein series. This generalization gives an Eisenstein series on a lower rank group as restriction ('pullback') of an Eisenstein series on a higher rank group. We refer the reader to Garrett [Gar84] and Shimura [Shi97] for additional discussions.

To precisely state the pullback formula, we first fix some embeddings and isomorphisms. Then we make a 'good' choice of sections for the Seigel Eisenstein series for our pullback formula. For purposes of interpolation, we explicitly compute the Fourier coefficients of the Seigel Eisenstein series associated to these sections. Here we consider an Eisenstein series of level N, though we will later restrict ourselves to p-power level. Having done that we state the pullback formula in the adelic language and check that the Klingen Eisenstein series obtained by the pullback with the above choices, is up to some normalization, of the type one defined in an earlier chapter. Finally, we interpret the pullback formula in the classical set up, as an inner product. The classical interpretation is done in such a way that the Klingen Eisenstein series can be easily seen to be interpolated via an application of a theorem of Hida.

8.1 Doubling method

As we have already seen in an earlier chapter, the cuspidal automorphic representations for GL_n are generic, i.e. they have global Whittaker models (cf. 5.1). These Whittaker models have played a crucial role in the Euler product decomposition of the integral representation of L-functions for GL_n. But for some reductive groups the cuspidal representations need not be generic. Yet most integral representations of their L-functions rely on Whittaker models. In the 1980's Piatetski-Shapiro and Rallis discovered a family of Rankin-Selberg integrals for classical groups that did not rely on Whittaker models using the *doubling method*, which we discuss below. We follow the exposition of Cogdell, [Cog] and Rallis and Piatetski-Shapiro, [GPSR87].

Let V be a vector space of dimension n over $\mathcal{K}$ (a quadratic extension of $\mathbf{Q}$), equipped with a non-degenerate Hermitian pairing $\langle \cdot, \cdot \rangle$. Let $G = U(V) \subset GL_n(\mathcal{K})$ be the associated unitary group. Let $W = V \oplus (-V)$ be the doubled space with the Hermitian pairing on W defined by

$$\langle\langle (v_1, v_2), (u_1, u_2) \rangle\rangle = \langle v_1, u_1 \rangle - \langle v_2, u_2 \rangle .$$

Let $G' = U(W) \subset GL_{2n}(\mathcal{K})$ be the associated unitary group. Then G' is a quasi-split group with $G' \simeq U(n,n)$ and we have a natural embedding $G \times G \to G'$, so we can identify $G \times G$ as a subgroup in G'. Now define subspaces X and X' of W by

$$X = \{(v, v) \mid v \in V\}$$

$$X' = \{(v, -v) \mid v \in V\}.$$

Then X and X' are totally $\langle\langle \cdot, \cdot \rangle\rangle$ - isotropic spaces and $W = X \oplus X'$. Let $Q_X \subset G'$ be the parabolic subgroup preserving X (we call it the Siegel parabolic). Then $Q = Q_X$ has a levi decomposition $Q = MN$ with $M \simeq GL_n(\mathcal{K})$ and $M \cap (G \times G) = \{(g, g) \mid g \in G\} \subset G \times G$. Let τ be an idele class character of $\mathcal{K}$ and $E(Q_X, s, \tau, f, g')$ be an Eisenstein series given by

$$E(Q_X, s, \tau, f, g') = \sum_{Q_X(\mathbf{Q}) \backslash G'(\mathbf{Q})} f_s(\gamma g')$$

where f_s is a section obtained from the induced representation

$$I_n(\tau) = \mathrm{Ind}_{Q_X(\mathbf{A})}^{G'(\mathbf{A})}(\tau(\det)|\det|^{s-1/2}).$$

If the section f_s is $K_{G'}$-finite; then $E(Q_X, s, \tau, f, g')$ converges for $\mathrm{Re}(s) >> 0$, is automorphic in g', has a meromorphic continuation in s and satisfies a functional equation.

We can now consider the global integral. Let (π, V_π) be a cuspidal representation of $G(\mathbf{A})$ and $\phi \in V_\pi$. Let $(\tilde{\pi}, V_{\tilde{\pi}})$ be the contragredient representation of (π, V_π) with $\tilde{\phi} \in V_{\tilde{\pi}}$. Let $E(Q_X, s, \tau, f, g')$ be an Eisenstein series as above, which we restrict ('pullback') to $G(\mathbf{A}) \times G(\mathbf{A}) \subset G'(\mathbf{A})$. We consider the global integral

$$I(\phi, \tilde{\phi}, s, \tau, f) = \int_{(G \times G)(\mathbf{Q}) \backslash (G \times G)(\mathbf{A})} \phi(g_1)\tilde{\phi}(g_2) E(Q_X, s, \tau, f, (g_1, g_2)) \tau^{-1}(\det g_2) dg_1 dg_2.$$

This integral extends to a meromorphic function of s and satisfies a functional equation thanks to the analytic properties of $E(Q_X, s, \tau, f, g')$.

To see that it has an Euler product decomposition, one inserts the definition of the Eisenstein series into the integral and unfolds it. Then an analysis of the orbits of $G \times G$ on $Q_X \backslash G'$ needs to be carried out. One can check that all but one orbit is negligible (the stabilizer in $G \times G$ contains a unipotent radical of a proper parabolic subgroup of one of the factors of G as a normal subgroup, hence the contribution of the integral over that orbit is zero since ϕ and $\tilde{\phi}$ are cusp forms). The non-negligible orbit is stabilized by $G^d = \{(g,g) \mid g \in G\}$ and for $\mathrm{Re}(s) >> 0$ we get

$$\begin{aligned} I(\phi, \tilde{\phi}, s, \tau, f) &= \int\limits_{G^d(\mathbf{Q})\backslash(G\times G)(\mathbf{A})} \left\langle \pi(g_1)\phi, \tilde{\pi}(g_2)\tilde{\phi} \right\rangle f_s((g_1, g_2))\tau^{-1}(\det g_2) dg_1 dg_2 \\ &= \int\limits_{G(\mathbf{A})} f_s((g,1)) \left\langle \pi(g)\phi, \tilde{\phi} \right\rangle dg \end{aligned}$$

where

$$\left\langle \phi, \tilde{\phi} \right\rangle = \int\limits_{G(\mathbf{Q})\backslash G(\mathbf{A})} \phi(g)\tilde{\phi}(g) dg.$$

Now if we assume that all the functions above are pure tensors then this integral has an Euler decomposition

$$I(\phi, \tilde{\phi}, s, \tau, f) = \prod_v I_v(\phi_v, \tilde{\phi}_v, s, \tau_v, f_v)$$

where

$$I_v(\phi_v, \tilde{\phi}_v, s, \tau_v, f_v) = \int\limits_{G(\mathbf{Q}_v)} f_s((g,1)) \left\langle \pi_v(g)\phi_v, \tilde{\phi}_v \right\rangle dg \qquad \mathrm{Re}(s) >> 0$$

for $\phi = \otimes_v \phi_v$, $\tilde{\phi} = \otimes_v \tilde{\phi}_v$, $\tau = \otimes_v \tau_v$, $f_s = \otimes_v f_{s,v}$; for K_v- fixed vectors ϕ_v^0 and $\tilde{\phi}_v^0$, we assume $\left\langle \phi_v^0, \tilde{\phi}_v^0 \right\rangle = 1$. The identification with L-functions comes from analyzing these local integrals, which are shown to be up to normalization, the Euler factors of the standard L-function for $\pi \otimes \tau$.

8.2 Isomorphisms and embeddings

In this section we choose some isomorphisms and embeddings that we need for the pullback formula. The set up can be viewed as a generalization of the doubling method though with the intention of just constructing an Eisenstein series. The maps discussed below are very similar to those of the doubling formula discussed above. Here we are using the formulation from a preprint of Skinner-Urban, [SU].

Let $V_n = \mathcal{K}^{2n}$ then w_n defines a skew Hermitian pairing $\langle \cdot, \cdot \rangle_n$ on $V_n : \langle x, y \rangle_n = x w_n y^*$ where $y^* = {}^t\bar{y}$. Then $G_n/\mathbf{Q}$ is the unitary similitude group $GU(V_n)$ for the Hermitian space $(V_n, \langle \cdot, \cdot \rangle_n)$.

We denote $v \in V_n$ as $v = (v_1, v_2)$ where $v_i \in \mathcal{K}^n$ and write an element $v \in V_{n+1}$ as $v = (v_1, x, v_2, y)$ where $(v_1, v_2) \in V_n$ and $x, y \in \mathcal{K}$. Now let $W_n = V_{n+1} \oplus V_n$. Then we have a Hermitian pairing on W_n defined by $w_{n+1} \oplus w_n$. Let $GU(W_n)$ denote the associated unitary similitude group. Now consider the maximal isotropic subspace $X_n = \{(v_1, 0, v_2, y) \oplus (v_1, v_2)\}$ and let $Q_{X_n} \subset GU(W_n)$ be its stabilizer.

The map

$$(v_1, x, v_2, y) \oplus (u_1, u_2) \mapsto (v_1, x, u_2, v_2, y, u_1)$$

gives an isomorphism between W_n and V_{2n+1}. This map is given by the matrix

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and the map $g \mapsto R^{-1} g R$ determines a $\mathbf{Q}$-isomorphism $\alpha_n : GU(W_n) \simeq G_{2n+1}$. We define another map

$$(v_1, x, u_2, v_2, y, u_1) \mapsto (v_1 - u_1, x, u_2 - v_2, v_2, y, u_1).$$

This map can be represented by the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G_n$$

and the map $g \mapsto S^{-1} g S$ determines an isomorphism $\beta_n : G_{2n+1} \simeq G_{2n+1}$. Together α_n and β_n determine an isomorphism

$$\gamma_n := \beta_n \alpha_n : GU(W_n) \simeq G_{2n+1}.$$

Then we can observe that $\gamma_n(X_n) = \{(0,0,0,*,*,*)\}$. Since Q_{2n+1} is the stabilizer of the space $\{(0,0,0,*,*,*)\}$, we immediately get that $\gamma_n(Q_{X_n}) = Q_{2n+1}$, the Siegel Parabolic subgroup.

Let $G_{n,n+1} = \{(g,g') \in G_{n+1} \times G_n : \mu_{n+1}(g) = \mu_n(g)\}$ be a subgroup of $GU(W_n)$. Then we have

$$\gamma_n^{-1}(Q_{2n+1}) = \{(m(g,x)n, g) : g \in G_n, x \in \mathrm{Res}_{\mathcal{K}/\mathbf{Q}}\mathbf{G}_m, n \in N_{Q_{n+1}}\}$$

For future use we note that if $g = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix}$ and $g_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then

$$\gamma_n(g, g_1) = \begin{pmatrix} a_1 & a_2 & -b_1 & b_1 & b_2 & 0 \\ a_3 & a_4 & -b_3 & b_3 & b_4 & 0 \\ -\gamma & 0 & \delta & 0 & 0 & \gamma \\ c_1 - \gamma & c_2 & \delta - d_1 & d_1 & d_2 & \gamma \\ c_3 & c_4 & -d_3 & d_3 & d_4 & 0 \\ a_1 - \alpha & a_2 & \beta - b_1 & b_1 & b_2 & \alpha \end{pmatrix}$$

and

$$\alpha_n(g, g_1) = \begin{pmatrix} a_1 & a_2 & & b_1 & b_2 & \\ a_3 & a_4 & & b_3 & b_4 & \\ & & \delta & & & \gamma \\ c_1 & c_2 & & d_1 & d_2 & \\ c_3 & c_4 & & d_3 & d_4 & \\ & & \beta & & & \alpha \end{pmatrix}$$

We also note that $\alpha_n : G_{n+1}(\mathbf{R}) \times G_n(\mathbf{R}) \to G_{2n+1}(\mathbf{R})$ induces a map

$$\mathcal{H}_{n+1} \times \mathcal{H}_n \hookrightarrow \mathcal{H}_{2n+1}$$

given by

$$(z, w) \mapsto \begin{pmatrix} z & \\ & \frac{1}{\bar{w}} \end{pmatrix}.$$

8.3 A particular choice of section

Let κ, N be positive integers and let p be an odd prime. Let τ be a Hecke character of $\mathbf{A}_{\mathcal{K}}^{\times}$ such that

$$\tau_\infty(x_\infty) = \left(\frac{x_\infty}{|x_\infty|}\right)^{-\kappa} \tag{8.1}$$

and

$$\tau_\ell(x_\ell) = 1 \qquad \text{if} \qquad \ell \nmid \infty, \qquad x_\ell \in \mathcal{O}_{\mathcal{K}_\ell}^{\times} \qquad \text{and} \qquad x_\ell - 1 \in N\mathcal{O}_{\mathcal{K}_\ell}. \tag{8.2}$$

Archimedean sections

Let $f_{\kappa,\infty} \in I_n(\tau_\infty)$ be given by $f_{\kappa,\infty}(k) = j_n(k, \mathbf{i})^{-\kappa}$. So we have

$$f_{\kappa,\infty,s}(qk) = \tau_\infty(\det D_q)|\det A_q D_q^{-1}|^s j_n(k, \mathbf{i})^{-\kappa} \qquad q \in Q_n(\mathbf{R}), k \in K_{n,\infty}^h$$

ℓ-adic sections

1. $\ell \mid N$

   Let $f_\ell \in I_n(\tau_\ell)$ be a function given by

$$f_\ell(g) = \begin{cases} \tau_\ell(\det(D_q A_k)) & \text{if } g = q w_n k \in Q_n(\mathbf{Z}_\ell) w_n K_{Q_n,\ell}^h(N) \\ \\ 0 & \text{otherwise.} \end{cases}$$

   Hence f_ℓ is supported on $Q_n(\mathbf{Z}_\ell) w_n N_{Q_n}(\mathbf{Z}_\ell)$.

2. $\ell \nmid N$

   Let $f_\ell \in I_n(\tau_\ell)$ be given by

$$f_\ell(g) = 1 \qquad \text{if } g \in K_{Q_n,\ell}^h(1).$$

Notation VIII.1. We let

$$f_\kappa = f_{\kappa,\infty} \otimes (\otimes_\ell f_\ell) \tag{8.3}$$

and observe that f_ℓ is spherical for all $\ell \nmid N$.

8.4 Fourier coefficients of Siegel Eisenstein series

8.4.1 Haar Measure

To assign a measure on $\mathbf{A}$, for each place of $\mathbf{Q}$ we fix an additive Haar measure on $\mathbf{Q}_v$ such that for each finite place $v = \ell$, $\mathbf{Z}_\ell$ has measure one and at the infinite place $v = \infty$ we use the usual Lebesgue measure. We define a Haar measure on $\mathbf{A}_\mathcal{K}$ by defining an additive Haar measure on $\mathcal{K}_\ell$ such that $\mathcal{O}_{\mathcal{K},\ell}$ has volume $|D_\ell|_\mathcal{K}^{1/2}$ where D_ℓ is the local different and such that on $\mathcal{K}_\infty$ the measure is $2dxdy$ where $z = x + iy \in \mathbf{C}$. With this measure the volume of $\mathbf{A}_\mathcal{K}/\mathcal{K}$ with respect to the induced measure is one. Finally, we define the multiplicative measure on $\mathbf{A}_\mathbf{Q}^\times$ and $\mathbf{A}_\mathcal{K}^\times$ as the ratio of the additive measures with $|\cdot|_\mathbf{Q}$ and $|\cdot|_\mathcal{K}$, respectively. Since we shall refer to Shimura [Shi97] in this section, we note here that the Haar measure used by Shimura is slightly different for $\mathbf{A}_\mathcal{K}$.

8.4.2 Fourier coefficients

The Siegel Eisenstein series

$$E(s,g) := E(Q_n, s, \tau, f, g) = \sum_{\alpha \in Q_n(\mathbf{Q})\backslash G_n(\mathbf{Q})} f_s(\alpha g)$$

has a Fourier expansion given by

$$E(s,g) = \sum_{h \in S_n(\mathbf{Q})} E_h(s,g)$$

where

$$E_h(s,g) = \int_{S_n(\mathbf{Q})\backslash S_n(\mathbf{A})} E(s, n(\sigma)g)\mathbf{e}_\mathbf{A}(\mathrm{tr}(-h\sigma))d\sigma,$$

$S_n = \{\sigma \in M_n(\mathcal{K}) | \sigma^* = \sigma\}$ and $n(\sigma)$ refers to the unipotent element with σ as the upper right hand entry.

Lemma VIII.2. *Let $f = \otimes_v f_v \in I_n(\tau)$ be such that for some prime ℓ the local function $f_\ell(g)$ is supported on $Q_n(\mathbf{Q}_\ell) w_n Q_n(\mathbf{Q}_\ell)$. Then for $h \in S_n(\mathbf{Q})$ and $q \in Q_n(\mathbf{A})$*

$$E_h(s,q) = \prod_v \int\limits_{S_n(\mathbf{Q}_v)} f_{v,s}(w_n n(\sigma) q_v) e_v(-tr(h\sigma)) d\sigma$$

if $Re(s) > 0$.

Proof. If $\alpha \in G_n(\mathbf{Q})$ and $f_\ell((\alpha q)_\ell) \neq 0$, then $\alpha_\ell q_\ell \in Q_n(\mathbf{Q}_\ell) w_n Q(\mathbf{Q}_\ell)$. Hence $\alpha_\ell \in Q_n(\mathbf{Q}_\ell) w_n Q(\mathbf{Q}_\ell)$. This implies that $\det(C_\alpha)_\ell \neq 0$ which in turn implies that $\det(C_\alpha) \neq 0$. We also have that $\alpha \in Q_n w_n Q_n$ if and only if $\det(C_\alpha) \neq 0$. But $Q_n w_n Q_n = Q_n w_n M_{Q_n} N_{Q_n} = Q_n w_n N_{Q_n}$. Hence only the subset $w_n n(S_n(\mathbf{Q}))$ of the coset representatives of $Q_n(\mathbf{Q}) \backslash G_n(\mathbf{Q})$ contributes towards the Eisenstein series. Since f and $e_{\mathbf{A}}(\cdot)$ decompose into local components $E_h(s,q)$ decomposes into its local components as desired. $\square$

Proposition VIII.3. *Let $D_n(N) = G_n(\mathbf{R}) \prod_{\ell \nmid \infty} K^h_{Q_n,\ell}(N) \subset G_n(\mathbf{A})$ be an open subgroup of $G_n(\mathbf{A})$. Then we can find a set of representatives for $G_n(\mathbf{Q}) \backslash G_n(\mathbf{A}) / D_n(N)$ consisting of elements $b = diag(u, \hat{u})$ with $u \in GL_n(\mathbf{A}_\mathcal{K})_f$ such that $u_\ell = 1$ for $\ell \mid N$.*

Proof. See Shimura, [Shi97], lemma 9.8. $\square$

Local Archimedean Fourier coefficients

For $s \in \mathbf{C}$ and $z \in \mathcal{H}_n$ we choose the branches of $\det(z)^s$ and $\overline{\det(z)}^s$ so that the values at $z = i1_n$ are i^{ns} and i^{-ns}, respectively, where i^α is defined by $i^\alpha = \exp(\pi i \alpha / 2)$. Now define a function

$$\xi(y,h;s,s') = \int\limits_{S_n(\mathbf{R})} \det(x+iy)^{-s} \det(x-iy)^{-s'} \mathbf{e}(\mathrm{tr}(-hx)) dx \tag{8.4}$$

for $s, s' \in \mathbf{C}, 0 < y \in S_n(\mathbf{R}), h \in S_n(\mathbf{R})$.

Lemma VIII.4. *Given $h \in S_n(\mathbf{R})$ and $u \in GL_n(\mathbf{C})$ consider the local Fourier coefficient*

$$c_\infty(h, u, s) = \int_{S_n(\mathbf{R})} f_{\kappa,\infty,s}\left(w_n n(\sigma)\begin{pmatrix} u & \\ & \hat{u}\end{pmatrix}\right)\mathbf{e}(-tr(h\sigma))d\sigma.$$

Then

$$c_\infty(h, u, s) = (-1)^{-\kappa}\tau_\infty(\det(u))|\det(u)|^{2s}\xi(uu^*, h, s+\kappa/2, s-\kappa/2).$$

Proof. Let $g = w_n n(\sigma)\begin{pmatrix} u & \\ & \hat{u}\end{pmatrix} = \begin{pmatrix} 0 & \hat{u} \\ -u & -\hat{u}\sigma\end{pmatrix}$. Suppose $g = qk$ for $q \in Q_n(\mathbf{R})$ and $k \in K^h_{n,\infty}$. Observe that $j_n(qk, \mathbf{i}) = j_n(q, k\mathbf{i})j_n(k, \mathbf{i}) = j_n(q, \mathbf{i})j_n(k, \mathbf{i})$ and $|j_n(k, \mathbf{i})| = 1$. So $|j_n(qk, \mathbf{i})| = |j_n(q, \mathbf{i})| = |\det D_q|$. Since

$$\tau_\infty(D_q) = (\det(D_q)^{-1}|\det(D_q)|)^\kappa = |j_n(g, \mathbf{i})|^{-\kappa}j_n(g, \mathbf{i})^\kappa$$

and

$$|\det A_q \det D_q^{-1}|^s = |j_n(g, \mathbf{i})|^{-2s}$$

we get $f_{\kappa,\infty,s}(g) = j_n(g, \mathbf{i})^{-\kappa}|j_n(g, \mathbf{i})|^{\kappa-2s}$. Now we note that $j_n(g, \mathbf{i}) = \det(-u\mathbf{i} - \hat{u}\sigma)$. So we get

$$f_{\kappa,\infty,s}(g) = \det(-u)^{-\kappa}|\det(-u)|^{\kappa+2s}\det(\mathbf{i}uu^* + \sigma)^{-\kappa}|\det(\mathbf{i}uu^* + \sigma)|^{\kappa-2s}. \tag{8.5}$$

Hence

$$c_\infty(h, u, s) = c_u \int_{S_n(\mathbf{R})} \det(\mathbf{i}uu^* + \sigma)^{-\kappa}|\det(\mathbf{i}uu^* + \sigma)|^{\kappa-2s}\mathbf{e}(-\mathrm{tr}(h\sigma))d\sigma$$

where $c_u = \det(-u)^{-\kappa}|\det(-u)|^{\kappa+2s}$. Using 8.4 we get

$$c_\infty(h, u, s) = (-1)^{-\kappa}\tau_\infty(\det(u))|\det(u)|^{2s}\xi(uu^*, h, s+\kappa/2, s-\kappa/2).$$

$\square$

Local non-archimedean Fourier coefficients

We break up the study at the finite places into two cases

1. $\ell \mid N$

Lemma VIII.5. *Let*

$$c_\ell(h,u,s) = \int\limits_{S_{n,\ell}} f_{\ell,s}\left(w_n n(\sigma)\right) \mathbf{e}_\ell(-tr(h\sigma))d\sigma.$$

Then

$$c_\ell(h,u,s) = \begin{cases} |D_\ell|^{\frac{n(n-1)}{4}} & \text{if } h \in S_n(\mathbf{Z}_\ell)^* \\ \\ 0 & \text{otherwise} \end{cases}$$

where $S_n(\mathbf{Z}_\ell)^* = \{h \in S_n(\mathbf{Q}_\ell) : tr(hS) \in \mathbf{Z}, S \in S_n(\mathbf{Z}_\ell)\}$.

Proof. By 8.3 we know that $f_{\ell,s}$ is supported on

$$Q_n(\mathbf{Q}_\ell)w_n K^h_{Q_n,\ell}(\ell^r) = Q_n(\mathbf{Q}_\ell)w_n N_{Q_n}(\mathbf{Z}_\ell).$$

So $w_n n(\sigma) \in Q_n(\mathbf{Q}_\ell)w_n N_{Q_n}(\mathbf{Z}_\ell)$ if and only if $\sigma \in S_n(\mathbf{Z}_\ell)$. Using the definition of the section we get

$$c_\ell(h,u,s) = \int\limits_{S_n(\mathbf{Z}_\ell)} f_{\ell,s}\left(w_n\right) \mathbf{e}_\ell(-\mathrm{tr}(h\sigma))d\sigma \tag{8.6}$$

$$= |D_\ell|^{\frac{n(n-1)}{4}} \tag{8.7}$$

if $h \in S_n(\mathbf{Z}_\ell)^*$ and 0 otherwise. $\square$

2. $\ell \nmid N$

Lemma VIII.6. *Let f_ℓ be the section as in 8.3, $u_\ell \in GL_n(\mathcal{K}_\ell)$. Consider the local Fourier coefficient*

$$c_\ell(h,u,s) := \int\limits_{S_{n,\ell}} f_{\ell,s}\left(w_n n(\sigma)\begin{pmatrix} u & \\ & \hat{u}\end{pmatrix}\right)\mathbf{e}_\ell(-tr(h\sigma))d\sigma$$

Then

$$\begin{aligned} c_\ell(h,u,s) = N^{-n^2}|D_\ell|^{\frac{n(n-1)}{4}}\tau_\ell(\det(u))|\det(uu^*)|^{-s+n}\prod_{\ell\in\mathbf{c}} f_{h,u,\ell}(\bar{\tau}'(\ell)\ell^{-2s}) \\ \cdot\frac{\prod_{i=0}^{n-r-1} L_N(2s-n-i,\bar{\tau}'\epsilon_{\mathcal{K}}^{n+i-1})}{\prod_{i=0}^{n-1} L_N(2s-i,\bar{\tau}'\epsilon_{\mathcal{K}}^{i})}\end{aligned} \tag{8.8}$$

where $r = rank(h)$, $\epsilon_{\mathcal{K}}$ the Hecke character of $\mathbf{Q}$ corresponding to $\mathcal{K}/\mathbf{Q}$, $\tau' = \tau\mid_{\mathbf{Q}_\ell^\times}$, $\mathbf{c}$ a certain finite set of primes and $f_{h,u,\ell}$ is a polynomial with constant term 1 and coefficients in $\mathbf{Z}$ independent of τ.

Proof. This is a well-known result due to Shimura, [Shi97], lemma 18.13. Though one must note that the character inducing our Eisenstein is inverse of that he uses and take into account the difference in Haar measures. □

Proposition VIII.7. *Let $u \in GL_n(\mathbf{A}_{\mathcal{K}})$ such that $u_\ell = 1$ if $\ell \mid N$. Let $c(h,u,s) = \prod c_\ell(h,u,s)$ then $c(h,u,s) \neq 0$ only if $h_\ell \in S_n(\mathbf{Z}_\ell)^*$, in which case*

$$\begin{aligned} c(h,u,s) &= (-1)^{-\kappa}N^{-n^2}|D_\ell|^{\frac{n(n-1)}{4}}\tau(\det(u))|\det(u^*u)_{\mathbf{f}}|^{n-s}|\det(uu^*)_\infty)|^s \\ &\quad\cdot\alpha_N(u^*hu,2s,\bar{\tau}')\xi(u^*u,h,s+\kappa/2,s-\kappa/2)\end{aligned} \tag{8.9}$$

where

$$\alpha_N(u^*hu,2s,\bar{\tau}') = \frac{\prod_{i=0}^{n-r-1} L_N(2s-n-i,\bar{\tau}'\epsilon_{\mathcal{K}}^{n+i-1})}{\prod_{i=0}^{n-1} L_N(2s-i,\bar{\tau}'\epsilon_{\mathcal{K}}^{i})}\prod_{\ell\in\mathbf{c}} f_{h,u,\ell}(\bar{\tau}'(\ell)\ell^{-2s}).$$

Proof. Follows immediately from lemmas VIII.4, VIII.5 and VIII.6. □

Definition VIII.8. We denote by $V(p,q,r)$ the subset of S_n consisting of the elements with p positive, q negative, and r zero eigenvalues.

Definition VIII.9. $\Gamma_n(\alpha) = \pi^{n(n-1)/4} \prod_{k=0}^{n-1} \Gamma(\alpha - k)$.

Lemma VIII.10. *(Shimura) The function $\xi(g, h, \alpha, \beta)$ has an expansion given by*

$$\begin{aligned} \xi(g, h, \alpha, \beta) &= i^{n\beta - n\alpha} 2^{\varphi} 2^{-n(n-1)/2} \pi^{\psi} \Gamma_r(\alpha + \beta - n) \Gamma_{n-q}(\alpha)^{-1} \Gamma_{n-p}(\beta)^{-1} \\ &\cdot \delta(g)^{n-\alpha-\beta} \delta_+(hg)^{\alpha-n+q/2} \delta_-(hg)^{\beta-n+p/2} \omega(2\pi g, h, \alpha, \beta) \end{aligned} \tag{8.10}$$

if $h \in V(p, q, r)$, where

$$\varphi = (2p - n)\alpha + (2q - n)\beta + n(n + r) + pq \tag{8.11}$$

$$\psi = p\alpha + q\beta + r^2 - pq \tag{8.12}$$

where $\delta_+(g)$ is the product of all positive eigenvalues of g and $\delta_-(g) = \delta_+(-g)$ and $\omega(2\pi g, h, \alpha, \beta)$ is as defined in (4.6.K),Shimura [Shi82].

Proof. See Shimura, [Shi82]. The volume form we use is $2^{n(n-1)/2}$ times the volume form used in *loc cit.* □

Fact VIII.11. *(Shimura [Shi82] 4.35.K)*

$$\omega(g, h, n, \beta) = 2^{-pn} \pi^{pr} e(-itr(gh))$$

if $h \in V(p, 0, r)$.

In particular if $\alpha = n$ and $\beta = n - \kappa$ and $h \in V(p, 0, r)$ we have for $\lambda = n(n-1)/2$

(8.13)

$$\begin{aligned} \xi(uu^*, h, n, n - \kappa) &= i^{-n\kappa} 2^{n\kappa - \lambda} \pi^{pn + r^2 + pr} \Gamma_n(n)^{-1} \delta(uu^*)^{\kappa - n} \delta_-(u^* hu)^{p/2 - \kappa} e(-i\mathrm{tr}(uhu^*)) \\ &= i^{-n\kappa} 2^{n\kappa - \lambda} \pi^{n^2} \Gamma_n(n)^{-1} \delta(uu^*)^{\kappa - n} \delta_-(u^* hu)^{p/2 - \kappa} e(-i\mathrm{tr}(uhu^*)) \end{aligned} \tag{8.14}$$

By the correspondence between classical Hermitian modular forms and associated automorphic forms for each $t \in G_n(\mathbf{A}_f)$ (for example we can take $t = b$ as in Prop.

VIII.3) we can define a function $E_t(s,z)$ of $\mathbf{C} \times \mathcal{H}_n$ by

$$E_t(s, g_\infty(\mathbf{i})) = j(g_\infty, \mathbf{i})^\kappa \mu(g_\infty)^{-n\kappa/2} E(s, tg_\infty)$$

where $z = g_\infty(\mathbf{i})$.

Lemma VIII.12. *Suppose* $t = \begin{pmatrix} u_t & \\ & \hat{u}_t \end{pmatrix}$ *with* $u_t \in GL_n(\mathbf{A}_{\mathcal{K},f})$ *and* $z = x + iy \in \mathcal{H}_n$ *put* $c_t(h,y,s) = \det(y)^{-\kappa/2} c(h,u,s)$ *with* $u \in GL_n(\mathbf{A}_\mathcal{K})$ *such that* $u_\mathbf{f} = u_t$ *and* $u_\infty = y^{1/2}$. *Then*

$$E_t(s,z) = \sum_{h \in S_n} c_t(h,y,s) e(tr(hx)). \tag{8.15}$$

Proof. Shimura [Shi97], lemma 18.7. □

Theorem VIII.13. $E_t(s,z)$ *is holomorphic at* $s = n - \kappa/2$

Proof. Follows from theorem 17.12 (iii),[Shi00] □

Lemma VIII.14. *For* $\kappa > n$, *the fourier coefficient* $c_t(h,y,n-\kappa/2) = 0$ *for all* $h < 0$.

Proof. By lemma (18.12), Shimura ([Shi97]),

$$\xi(y,h,s,s')\Gamma_{n-a-b}(n+n-\kappa-n)^{-1}\Gamma_{n-b}(n)\Gamma_{n-a}(n-\kappa)$$

is holomorphic, where a is the number of positive eigenvalues of h, b is the number of negative eigenvalues of h. Since $h < 0$, $a < n$, $b > 0$ and $a + b \le n$. So we have $\Gamma_{n-a-b}(n-\kappa)^{-1}$ has a zero of order $n-a-b$, $\Gamma_{n-b}(n)$ has no poles and $\Gamma_{n-a}(n-\kappa)$ has a pole of order $n-a$. So overall,

$$\Gamma_{n-a-b}(n+n-\kappa-n)^{-1}\Gamma_{n-b}(n)\Gamma_{n-a}(n-\kappa)$$

has a pole of order $(n-a)-(n-a-b) = b > 0$. Hence $\xi(h,y,n,n-\kappa)$ must be 0 for all $h < 0$. So we get $c_t(h,y,n-\kappa/2) = 0$ for all $h < 0$. □

By (8.13) we have for h positive semi-definite

$$\begin{aligned} c_t(h,y,n-\kappa/2) &= (-1)^{-\kappa}N^{-n^2}\det(y)^{-\kappa/2}\tau(\det(u))|D_{\mathcal{K}}|^{\frac{n(n-1)}{4}}|\det(u_tu_t^*)|^{\kappa/2} \\ &\quad \cdot|\det(y)|^{n-\kappa/2}\xi(y,h,n,n-\kappa)\alpha_N(u^*hu,2n-\kappa,\bar\tau') \qquad (8.16)\\ &= (-1)^{-\kappa}i^{-n\kappa}2^{n\kappa}\pi^{n^2}N^{-n^2}\Gamma_n(n)^{-1}\tau(\det(u)) \\ &\quad \cdot|\det(u\bar u)|^{\kappa/2}\alpha_N(u^*hu,2n-\kappa,\bar\tau')e(i\mathrm{tr}(hy)) \qquad (8.17)\end{aligned}$$

Definition VIII.15. With notation as in lemma (VIII.12) let

$$D_t(n-\kappa/2,z)=\frac{(-1)^{\kappa}i^{n\kappa}2^{-n\kappa}\prod_{i=0}^{i=n-1}L_N(2n-\kappa-i,\bar\tau'\epsilon_{\mathcal{K}}^i)}{\pi^{n^2}N^{-n^2}\Gamma_n(n)^{-1}}E_t(n-\kappa/2,z)$$

and

$$D_t(n-\kappa/2,z)=\sum_{h\in S_n,h\geq 0}c_t'(h,n-\kappa/2)e(\mathrm{tr}(hz))$$

where

$$c_t'(h,n-\kappa/2)=\tau(\det(u))|\det(u\bar u)|^{\kappa/2}\alpha_N'(u^*hu,2n-\kappa,\bar\tau'). \qquad (8.18)$$

and $\alpha_N'(u^*hu,2n-\kappa,\bar\tau')=\prod_{i=0}^{n-r-1}L_N(n-\kappa-i,\bar\tau'\epsilon_{\mathcal{K}}^{n+i-1})\prod_{\ell\in\mathbf{c}}f_{h,u,\ell}(\bar\tau'(\ell)\ell^{\kappa-2n})$

Remark VIII.16. We will be interested in the p-adic interpolation of $D_t(n-\kappa/2,z)$.

8.5 The pull-back formula

Let τ be a unitary character of the idele class group $\mathcal{K}^\times\backslash\mathbf{A}_{\mathcal{K}}^\times$, κ a positive integer, and ϕ be a cusp form on G_n. Associated to $f_\kappa\in I_{2n+1}(\tau)$ we define

$$F_{\phi,s}(f_\kappa,g)=\int_{U_n(\mathbf{A})}f_{\kappa,s}(\gamma(g,g_1h))\bar\tau(\det(g_1h))\phi(g_1h)dg_1 \qquad (8.19)$$

where $g\in G_{n+1}(\mathbf{A}),\mu_{n+1}(g)=\mu_n(h)$. We note that $F_{\phi,s}(f_\kappa,g)$ is independent of h.

Proposition VIII.17. *If $f_\kappa\in I_{2n+1}(\tau)$ and $Re(s)>(3n+1)/2$, then $F_{\phi,s}(f_\kappa,g)$ converges absolutely and uniformly for (s,g) in compact sets of $\{Re(s)>n\}\times G_{n+1}(\mathbf{A})$.*

If $h \in G_n(\mathbf{A})$ such that $\mu_n(h) = \mu_{n+1}(g)$ then

(8.20)
$$\int_{U_n(\mathbf{Q})/U_n(\mathbf{A})} E(Q_{2n+1}, s, \tau, f_\kappa, \gamma_n(g, g_1 h))\bar{\tau}(\det g_1 h)\phi(g_1 h)dg_1 = \sum_{\gamma \in A} F_{\phi,s}(f_\kappa, \gamma g)$$

where $A = P_{n+1}(\mathbf{Q})\backslash G_{n+1}(\mathbf{Q})$. The series on the right hand side converges absolutely and uniformly for (s, g) in compact subsets of $\{Re(s) > (3n+1)/2\} \times G_{n+1}(\mathbf{A})$.

Proof. This result is a generalization of the doubling method of Piatetski-Shapiro and Rallis. The proof can be extracted out of Shimura [Shi97]. The general outline of the proof is as follows: One first writes down a coset decompostion of $Q_{2n+1}\backslash G_{2n+1}/\gamma(G_{n,n+1})$ as in Prop. 2.4 [Shi97]. Then one can expresses $Q_{2n+1}\backslash G_{2n+1}$ as a set of coset representatives as in formula [Shi97] (2.7.1). Then an analysis of the orbits of $G_{n,n+1}$ on $Q_{2n+1}\backslash G_{2n+1}$ needs to done. One can check that all but one orbit is "negligible". Hence using the cuspidality of ϕ one notes that only one coset representative contributes to the integral as in [Shi97] (22.9). The integral then defines the section which is summed over the "nonnegligible" orbit giving the above result. □

8.5.1 Klingen Eisenstein series

Of particular interest to us will be the case $n = 1$. Suppose (σ, V) is a cuspidal automorphic representation on $GL_2(\mathbf{A})$ and ψ is a Hecke character of $\mathcal{K}$ such that $\psi \mid_{\mathbf{A}^\times} = w_\sigma$. Using the canonical inclusion of $GL_2(\mathbf{Q})$ in $G_1(\mathbf{Q})$ the pair (σ, ψ) determines a representation of $G_1(\mathbf{A})$ on V by $\sigma_\psi(g)v = \sigma_\psi(xa)v = \psi(a)\sigma(x)$ where $a \in \mathbf{A}_{\mathcal{K}}^\times$ and $x \in GL_2(\mathbf{A})$. Now suppose $f_\kappa \in I_3(\tau)$ then $F_{\phi,s}(f_\kappa, g)$ is convergent if $\mathrm{Re}(s) > 0$ and for such s, $s \mapsto F_{\phi,s}(f_\kappa, -)$ defines a holomorphic map.

Lemma VIII.18. *For $F_{\phi,s}(f_\kappa, g)$ as above, $F_{\phi,s}(f_\kappa, pg) = \rho(p)\delta_P(p)^{\frac{s}{3}} F_{\phi,s}(f_\kappa, g)$ for all $p \in P(\mathbf{Q}_\ell)$ where $\delta_P(p)$ is the modulus character of the Klingen parabolic $P = P_2$.*

Proof. By the choice of embeddings made we have

$$\gamma_1^{-1}(Q_3) = \{(m(g,x)n', g) : g \in G_1, x \in \mathrm{Res}_{\mathcal{K}/\mathbf{Q}}\mathbf{G}_m, n' \in N_P\}.$$

Using this let us write $p = \gamma(m(g',x)n',g')$. Then

$$\begin{aligned}
F_{\phi,s}(f_\kappa, pg) &= \int\limits_{U_1(\mathbf{Q}_\ell)} f_{\kappa,s}(\gamma(pg, g_1h))\bar{\tau}(\det(g_1h))\sigma_\psi(g_1h)\phi dg_1 \\
&= \int\limits_{U_1(\mathbf{Q}_\ell)} f_{\kappa,s}(\gamma(m(g',x)n',g')\gamma(g, g'^{-1}g_1h))\bar{\tau}(\det(g_1h))\sigma_\psi(g_1h)\phi dg_1 \\
&= \tau(\det(g'))x)|\mu x/\bar{x}|^s \int\limits_{U_1(\mathbf{Q}_\ell)} f_{\kappa,s}(\gamma(g, g'^{-1}g_1h))\bar{\tau}(\det(g_1h))\sigma_\psi(g_1h)\phi dg_1 \\
&= \tau(x)|\mu/x\bar{x}|^s\sigma_\psi(g') \int\limits_{U_1(\mathbf{Q}_\ell)} f_{\kappa,s}(\gamma(g, g'^{-1}g_1h))\bar{\tau}(\det(g'^{-1}g_1h))\sigma_\psi(g_1g^{-1}h)\phi dg_1 \\
&= |\mu/x\bar{x}|^s\rho(p)F_{\phi,s}(f_\kappa, g) \\
&= \delta_P(p)^{\frac{s}{3}}\rho(p)F_{\phi,s}(f_\kappa, g) \qquad (8.21)
\end{aligned}$$

□

Remark VIII.19. From this lemma it follows that the right hand side of (8.20) is a Klingen Eisenstein series when $n = 1$.

Definition VIII.20. We will denote the Klingen Eisenstein series obtained above from the pullback as $E(s, F_\phi(f_\kappa), g)$ i.e.

$$E(s, F_\phi(f_\kappa), g) = \int\limits_{U_1(\mathbf{Q})/U_1(\mathbf{A})} E(Q_3, s, \tau, f_\kappa, \gamma_n(g, g_1h))\bar{\tau}(\det g_1h)\sigma_\psi(g_1h)\phi dg_1. \qquad (8.22)$$

In the notation of the section on Klingen Eisenstein series

$$E(s, F_\phi(f_\kappa), g) = E(P, s, F_\phi(f_\kappa)).$$

8.6 Local decomposition of $F_{\phi,s}$

The section defining the Klingen Eisenstein series above can be decomposed as

$$F_\phi(f_\kappa) = F_{\infty,\kappa} \otimes (\otimes_\ell F_\phi(f_\ell)).$$

In this section we recall the results of Shimura, Lapid-Rallis and Skinner-Urban relating the local components in the above decomposition to the local components in the decomposition of ϕ.

8.6.1 Archimedian sections

Let (σ, V) be an irreducible cuspidal automorphic representation on $GL(2)(\mathbf{R})$. We can extend σ to a representation of $G_1(\mathbf{R})$ by setting $\sigma_\psi(g)v = \sigma_\psi(xa) = \psi(a)\sigma(x)$ for $g = (a,x), a \in \mathcal{K}^\times, x \in GL(2)(\mathbf{R})$. Let $\phi \in V$ be the unique (up to scaler) non-zero vector such that $\sigma(k)\phi = j(k,i)^{-\kappa}\phi$ for all $k \in K_{\infty,+}$, the maximal compact in $GL(2)$. Let

$$F_{\infty,\kappa}(s,g) = \int_{U_1(\mathbf{R})} f_{\infty,\kappa,s}(S^{-1}\alpha(g,g_1h))\bar{\tau}(\det(g_1h))\sigma_\psi(g_1h)\phi dg_1 \tag{8.23}$$

where $g \in G(\mathbf{R}), \mu_2(g) = \mu_1(h)$. Then just as in the previous lemma one can see that $F_{\infty,\kappa}(s,pg) = \rho(p)\delta(p)^{s/3}F_{\infty,\kappa}(s,g)$. Also, by Skinner-Urban, [SU] and Shimura, [Shi97], the integral (8.23) converges for $\mathrm{Re}(s) > \max(\kappa/2, 3/2)$ and

$$F_{\infty,\kappa}(s,g) = 2^{-2s-2}\frac{\Gamma(s-\kappa/2-1)}{\Gamma(s-\kappa/2)}F_{\kappa,s}(g)$$

where $F_\kappa(g)$ is the unique vector in $I(\rho_\infty)^{\xi_\infty}$ such that $F_\kappa(1) = \phi$ where ρ_∞ is as in (4.5) and $\xi_\infty = \psi_\infty/\tau_\infty$.

8.6.2 ℓ-adic sections

Let (σ_ℓ, V_ℓ) be an irreducible cuspidal automorphic representation on $GL(2)(\mathbf{Q}_\ell)$ and τ_ℓ, ψ_ℓ as in section (4.2.2). We can extend σ_ℓ to a representation of $G_1(\mathbf{Q}_\ell)$ by setting $\sigma_{\ell,\psi_\ell}(g) = \sigma_{\ell,\psi_\ell}(xa) = \psi_\ell(a)\sigma_\ell(x)$ for $g = (a, x), a \in \mathcal{K}_\ell^\times, x \in GL(2)(\mathbf{Q}_\ell)$. If $\phi_\ell \in V_\ell$ be a new vector of σ_ℓ and σ_ℓ, τ_ℓ and ψ_ℓ are unramified we define F_{ρ_ℓ} as

$$F_{\rho_\ell}(pk) = \rho_\ell(p)\phi_\ell \text{ for } pk \in P(\mathbf{Z}_\ell)K_\ell^h$$

where ρ_ℓ is as in (4.6) and $I(\rho)^{K_\ell^h}$ is spanned by F_{ρ_ℓ}. Let

$$F_\phi(f_\ell, s, g) = \int_{U_1(\mathbf{Q}_\ell)} f_{\ell,s}(\gamma(g, g_1h))\bar{\tau}_\ell(\det(g_1h))\sigma_{\ell,\psi_\ell}(g_1h)\phi_\ell dg_1 \tag{8.24}$$

where $g \in G(\mathbf{R}), \mu_2(g) = \mu_1(h)$. Then just as in the previous lemma one can see that $F_\phi(f_\ell, s, pg) = \rho_\ell(p)\delta_\ell(p)^{s/3}F_\phi(f_\ell, s, g)$. Also, by Skinner-Urban, [SU] and Rallis-Lapid [LR05], the integral (8.24) converges for $\mathrm{Re}(s) > 0$ and

$$F_\phi(f_\ell, s, g) = \frac{L(\tilde{\sigma}, \xi_\ell, s - 1/2)}{\prod_{i=0}^{1} L(2s - i, \bar{\tau}'\epsilon_{\mathcal{K}}^i)} F_{\rho_\ell, s}(g)$$

where $\xi_\ell = \psi_\ell/\tau_\ell$.

Let $\tilde{F} = F_\kappa \otimes (\otimes F_{\rho_\ell})$ and

$$E(s, \tilde{F}, g) = \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \tilde{F}(\gamma g)$$

be the Klingen Eisenstein series associated to it. Let $W_\phi(1) = 2^{-2s-2}\frac{\Gamma(s-\kappa/2-1)}{\Gamma(s-\kappa/2)}\frac{L(\tilde{\sigma},\xi_\ell,s-1/2)}{\prod_{i=0}^{1} L(2s-i,\bar{\tau}')}$. Then

$$E(s, F, g) = W_\phi(1)E(s, \tilde{F}, g).$$

8.7 Summary

Now we put together the results in this section and relate them to the global integral of Furusawa, [Fur93]. Let f_κ be the section as defined in (8.3) and $E(Q_3, s, \tau, f_\kappa, g)$ be the Siegel Eisenstein series associated to it. Let $\tilde{E}(Q_3, s, \tau, f_\kappa, \gamma_n(g, g_1h))$ be the normalized Siegel Eisenstein series

$$\tilde{E}(Q_3, s, \tau, f_\kappa, \gamma_n(g, g_1h)) = E(Q_3, s, \tau, f_\kappa, \gamma_n(g, g_1h))\prod_{i=0}^{2} L_N(2s - i, \bar{\tau}'\epsilon_{\mathcal{K}}^i).$$

Let

$$\tilde{E}(s,F_\phi,g)=\int\limits_{U_1(\mathbf{Q})/U_1(\mathbf{A})}\tilde{E}(Q_3,s,\tau,f_\kappa,\gamma_n(g,g_1h))\bar{\tau}(\det g_1h)\sigma_\psi(g_1h)\phi dg_1 \tag{8.25}$$

be the Klingen Eisenstein series associated to normalized Siegel Eisenstein series. Then by the pull back formula (8.22) we have

$$\tilde{E}(s,F_\phi,g)=W_\phi(1)\prod_i^2 L_N(2s-i,\bar{\tau}')E(s,\tilde{F},g) \tag{8.26}$$

$$=2^{-2s-2}\frac{\Gamma(s-\kappa/2-1)}{\Gamma(s-\kappa/2)}\frac{L(\tilde{\sigma},\xi_\ell,s-1/2)}{L(2s-2,\bar{\tau}')}E(s,\tilde{F},g) \tag{8.27}$$

Using the notation in section (7.1) and applying the integral representation of Furusawa [Fur93] to the Klingen Eisenstein series $\tilde{E}(s,F_\phi,g)$ we get for π a cuspidal automorphic representation on $GSp(4)(\mathbf{A})$ as in section (7.1) and $\ell\notin\Omega$

(8.28)

$$Z_\ell(s)=\left(\prod_{i=0}^{i=1}L(2s-2,\bar{\tau}'_\ell\epsilon^i_{\mathcal{K}_\ell})\right)\frac{L(s-1,\tilde{\pi}_\ell\times\tilde{\sigma}_\ell)}{L(2s-2,\bar{\tau}'_\ell)L(s-1/2,\omega_\ell,\tilde{\sigma}_\ell)}\frac{L(s-1/2,\omega_\ell,\tilde{\sigma}_\ell)}{\prod_{i=0}^{i=1}L(2s-2,\bar{\tau}'_\ell\epsilon^i_{\mathcal{K}_\ell})}B_{\varphi_\ell}(1)$$

$$=L(s-1,\tilde{\pi}_\ell\times\tilde{\sigma}_\ell)$$

8.8 Classical interpretation of Pullback formula

In this section we interpret the pullback formula in the classical set up as a Petersson inner product of the Siegel Eisenstein series and a cusp form (with slight modification). This interpretation will relate the Klingen Eisenstein series to a Petersson inner product which is easily seen to be interpolated.

Let p be an odd prime that splits in $\mathcal{K}$. Let $f\in S_\kappa(\Gamma_0(p^r),\chi)$ be an eigenform and (σ,V) be the irreducible automorphic cuspidal representation associated to it on $GL_2(\mathbf{A})$. Let $V=\otimes V_v$, $\sigma=\otimes\sigma_v$, $\phi=\otimes\phi_v\in V$ be the completely reducible automorphic cusp form associated to f. Let χ be the central character of σ and $\psi=\otimes\psi_v$ and $\tau=\otimes\tau_v$ unitary Hecke characters of $\mathbf{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ such that

1. σ_ℓ is ramified only at p

2. ϕ_ℓ is a spherical vector for all finite places $\ell \neq p$

3. $\psi = \tau \cdot \xi$ where ξ is an unramified character.

4. $\sigma_\infty(k)\phi_\infty = j(k,i)^{-\kappa}\phi_\infty$ for $k \in K'_{\infty,+}$

5. $\psi|_{\mathbf{A}_{\mathbf{Q}}^\times} = \chi$

6. $\tau_\infty(x) = (x/|x|)^{-\kappa} = \psi_\infty(x)$

Assumptions (1) and (2) have been made since we will finally be interested only in elliptic and Siegel modular forms of level a power of p. Let

$$\phi'(g) = \sigma\left(\left(\begin{smallmatrix} 1 & \\ & p^{-r} \end{smallmatrix}\right)_p\right)\phi(g)$$

Using the notation as in section (8.2) let

$$E(s, f_\kappa, \alpha(g, g_1)) = E(Q_3, s, \tau, f_\kappa, \alpha(g, g_1)).$$

Then

$$\begin{aligned} E(s, f_\kappa, \alpha(g, g_1)) &= E(s, f_\kappa, S^{-1}\alpha(g, g_1)) \\ &= E(s, \tilde{f}_\kappa, \gamma(g, g_1)) \end{aligned}$$

where $\tilde{f}_\kappa(s, g) = f_\kappa(s, gS^{-1})$ and $S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. Then by the pullback formula we know that

$$\begin{aligned} E(s, F_{\phi'}(\tilde{f}_\kappa), g) &= \int\limits_{U_1(\mathbf{Q})/U_1(\mathbf{A})} E(s, \tilde{f}_\kappa, \gamma(g, g_1))\bar{\tau}(\det g_1)\phi'_\psi(g_1)dg_1 \\ &= \int\limits_{U_1(\mathbf{Q})/U_1(\mathbf{A})} E(s, f_\kappa, \alpha(g, g_1))\bar{\tau}(\det g_1)\phi'_\psi(g_1)dg_1 \end{aligned}$$

Now we study the integral

$$I(g) = \int\limits_{U_1(\mathbf{Q})/U_1(\mathbf{A})} E(s, \alpha_1(g, g_1))\bar{\tau}(\det g_1)\phi'_\psi(g_1)dg_1 \tag{8.29}$$

where $E(s, \alpha(g, g_1)) = E(Q_3, s, \tau, f_\kappa, \alpha(g, g_1))$ and ϕ'_ψ is a cusp form on $G_1(\mathbf{A})$ such that $\phi'((a, g)) = \psi(a)\phi'(g), a \in \mathbf{A}_{\mathcal{K}^\times}, g \in GL_2(\mathbf{A})$.

We want to interpret this integral in a classical setup. For the rest of this section we will suppress s and the section f_κ unless needed. Let

$$J(g, g_1) = E(\alpha_1(g, g_1))\bar{\tau}(\det g_1)\phi'_\psi(g_1)$$

and

$$K^0(p^r) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_1(\mathbf{Z}) \mid b \equiv 0 \mod p^r\}.$$

Lemma VIII.21. *$J(g, g_1k) = J(g, g_1)$ for $k \in K^0(p^r)$.*

Proof.

$$J(g, g_1 \left(\begin{smallmatrix} a & p^r b \\ c & d \end{smallmatrix}\right)) = E(\alpha_1(g, g_1 \left(\begin{smallmatrix} a & p^r b \\ c & d \end{smallmatrix}\right)))\bar{\tau}(\det(g_1 ad))\phi'_\psi(g_1 \left(\begin{smallmatrix} a & p^r b \\ c & d \end{smallmatrix}\right)) \tag{8.30}$$

since $\alpha(1, \left(\begin{smallmatrix} a & p^r b \\ c & d \end{smallmatrix}\right)) = \begin{pmatrix} 1_2 & & & \\ & d & & c \\ & & 1_2 & \\ & p^r b & & a \end{pmatrix}$ we have

$$= E(\alpha_1(g, g_1))\tau(d)\bar{\tau}(\det g_1)\bar{\tau}(ad)\psi(a)\phi'_\psi(g_1 \begin{pmatrix} 1 & p^r b/a \\ c/a & d/a \end{pmatrix}) \tag{8.31}$$

since $\tau = \psi \cdot \xi$ where ξ is unramified, we get

$$= J(g, g') \tag{8.32}$$

□

Remark VIII.22. Note here that we work with ϕ' instead of ϕ since we want invariance under $K^0(p^r)$ when we classically interpret the pullback formula.

Let $h_\mathcal{K}$ denote the class number of $\mathcal{K}$ and let $a_1, \cdots, a_{h_\mathcal{K}}$ be representatives for the class group of $\mathcal{K}$. Then by Cebotarev density theorem, we can assume that each representative can be represented by a uniformizer at a degree one prime. So we assume $a_i\bar{a}_i = q_i^{-1}$ where q_i splits in $\mathcal{K}$.

Now we note that $U_1(\mathbf{A}) = \bigsqcup_{a_i} U_1(\mathbf{Q})U_1(\mathbf{R}) \left(\begin{smallmatrix} a_i & \\ & \bar{a_i}^{-1} \end{smallmatrix}\right) K^0(p^r)$ (cf. Hida, [Hid93], chapter (9)) so we have

$$U_1(\mathbf{Q})\backslash U_1(\mathbf{A})/K_\infty K^0(p^r) = \Gamma\backslash SL_2(\mathbf{R})/SO_2(\mathbf{R}) \times \bigsqcup_{a_i} \left(\begin{smallmatrix} a_i & \\ & \bar{a_i}^{-1} \end{smallmatrix}\right)$$

where $\Gamma = U_1(\mathbf{Q}) \cap K^0(p^r)$. Let

$$\Gamma_i = U_1(\mathbf{Q}) \cap \begin{pmatrix} a_i & \\ & \bar{a}_i^{-1} \end{pmatrix} K^0(p^r) \left(\begin{smallmatrix} {a_i}^{-1} & \\ & \bar{a}_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & \\ & q_i \end{smallmatrix}\right) \Gamma \begin{pmatrix} 1 & \\ & q_i^{-1} \end{pmatrix}$$

and

$$R_i = \Gamma_i\backslash SL_2(\mathbf{R})/SO_2(\mathbf{R}) = \Gamma_i\backslash \mathfrak{H}^1.$$

For $I(g)$ as in 8.29 we have

$$\begin{aligned} I(g) &= \mathrm{vol} \sum_{a_i} \int_{R_i} E\left(\alpha(g, g_1 \begin{pmatrix} a_i & \\ & \bar{a}_i^{-1} \end{pmatrix})\right) \bar{\tau}(\det(\begin{pmatrix} a_i & \\ & \bar{a}_i^{-1} \end{pmatrix}))\phi'_\psi(g_1 \begin{pmatrix} a_i & \\ & \bar{a}_i^{-1} \end{pmatrix})dg_1 \\ &= \mathrm{vol} \sum_{a_i} \bar{\tau}(a_i/\bar{a}_i)\psi(a_i) \int_{R_i} E\left(\alpha(g, g_1 \begin{pmatrix} a_i & \\ & \bar{a}_i^{-1} \end{pmatrix})\right) \phi'_\psi(g_1 \left(\begin{smallmatrix} 1 & \\ & q_i \end{smallmatrix}\right)_{q_i})dg_1 \end{aligned}$$

where $\mathrm{vol} = \frac{1}{[U_1(\hat{\mathbf{Z}}):K^0(p^r)]}$.

We now want to rewrite this integral in a form more suitable for a classical interpretation. For this we note that

$$\begin{aligned} \phi'_\psi(g_1 \left(\begin{smallmatrix} 1 & \\ & q_i \end{smallmatrix}\right)) &= \phi'_\psi\left(\begin{pmatrix} 1 & \\ & q_i^{-1} \end{pmatrix}_\infty \begin{pmatrix} 1 & \\ & q_i^{-1} \end{pmatrix}_p g_1 \left(\begin{smallmatrix} 1 & \\ & q_i \end{smallmatrix}\right)_p\right) \\ &= \phi'_\psi\left(\begin{pmatrix} 1 & \\ & q_i^{-1} \end{pmatrix}_\infty g_1\right) \\ &= \phi_\psi\left(\left(\begin{smallmatrix} 1 & \\ & p^r \end{smallmatrix}\right)_\infty \begin{pmatrix} 1 & \\ & q_i^{-1} \end{pmatrix}_\infty g_1\right). \end{aligned} \tag{8.33}$$

Also,

$$E\left(\alpha(g, g_1 \begin{pmatrix} a_i & \\ & \bar{a}_i^{-1} \end{pmatrix})\right) = E\left(\alpha(g, g_1)\alpha\left(1, \begin{pmatrix} a_i & \\ & \bar{a}_i^{-1} \end{pmatrix}\right)\right)$$

$$= E\left(\alpha(g, g_1)\begin{pmatrix} 1_2 & & & \\ & \bar{a}_i^{-1} & & \\ & & 1_2 & \\ & & & a_i \end{pmatrix}_{q_i}\right)$$

$$= \tau({a_i}^2)_{q_i} E\left(\alpha(g, g_1)\begin{pmatrix} {a_i}^{-1} & & & \\ & (a\bar{a}_i)^{-1} & & \\ & & {a_i}^{-1} & \\ & & & 1 \end{pmatrix}_{q_i}\right)$$

$$= \tau({a_i}^2)_{q_i} E\left(\alpha(g, g_1)\begin{pmatrix} {a_i}^{-1} & & & \\ & q_i & & \\ & & {a_i}^{-1} & \\ & & & 1 \end{pmatrix}_{q_i}\right) \tag{8.34}$$

Now observe that

$$\begin{pmatrix} 1 & & & \\ & q_i^{-1} & & \\ & & q_i^{-1} & \\ & & & 1 \end{pmatrix}\begin{pmatrix} {a_i}^{-1} & & & \\ & q_i & & \\ & & {a_i}^{-1} & \\ & & & 1 \end{pmatrix}_{q_i}$$

$$= \{\begin{pmatrix} 1 & & & \\ & q_i^{-1} & & \\ & & q_i^{-1} & \\ & & & 1 \end{pmatrix}_{\infty}, \cdots, \begin{pmatrix} {a_i}^{-1} & & & \\ & q_i^{-1} & & \\ & & \bar{a}_i & \\ & & & 1 \end{pmatrix}_{q_i}, \cdots, \begin{pmatrix} 1 & & & \\ & q_i^{-1} & & \\ & & q_i^{-1} & \\ & & & 1 \end{pmatrix}_{p}, \cdots, \}$$

hence we get

$$E\left(\alpha(g, g_1 \begin{pmatrix} a_i & \\ & \bar{a}_i^{-1} \end{pmatrix})\right) = \tau_p(q_i^{-1})\tau_{q_i}({a_i}^2)E\left(\alpha\left(\begin{pmatrix} 1_2 & \\ & q_i^{-1}1_2 \end{pmatrix} g \begin{pmatrix} {a_i}^{-1} & \\ & \bar{a}_i \end{pmatrix}, \begin{pmatrix} 1 & \\ & q_i^{-1} \end{pmatrix} g_1\right)\right)$$

$$= \tau_{q_i}(a_i/\bar{a}_i)E\left(\alpha\left(\begin{pmatrix} 1_2 & \\ & q_i^{-1}1_2 \end{pmatrix} g \begin{pmatrix} {a_i}^{-1} & \\ & \bar{a}_i \end{pmatrix}, \begin{pmatrix} 1 & \\ & q_i^{-1} \end{pmatrix} g_1\right)\right) \tag{8.35}$$

since

$$\tau_p(q_i^{-1}) = \tau(1, \cdots, \underset{p}{q_i^{-1}}, \cdots)$$

$$= \tau(q_i, \cdots, \underset{p}{1}, \cdots, \underset{q_i}{q_i}, \cdots)$$

$$= \tau_\infty(q_i)\tau_{q_i}(a_i\bar{a}_i)^{-1}$$

$$= \tau_{q_i}(a_i\bar{a}_i)^{-1}. \tag{8.36}$$

Observe that since τ is unitary

$$\bar{\tau}(a_i/\bar{a}_i)\psi(a_i)\tau_{q_i}(a_i/\bar{a}_i) = \psi(a_i) = \tau(a_i).$$

So far we have

(8.37)

$$I(g) = \mathrm{vol} \sum_i \tau(a_i) \int_{R_i} E\left(\alpha\left(\left(\begin{smallmatrix} 1_2 & \\ & q_i^{-1} 1_2 \end{smallmatrix}\right) g \left(\begin{smallmatrix} a_i{}^{-1} & \\ & \bar{a}_i \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \\ & q_i^{-1} \end{smallmatrix}\right) g_1\right)\right) \phi'_\psi(g_1 \left(\begin{smallmatrix} 1 & \\ & q_i \end{smallmatrix}\right)_{q_i}) dg_1.$$

We now transform the functions above into their classical analogues.Suppose $g \in U_2(\mathbf{R})$ and $g(i) = Z$ and $g_1(i) = w$.

Since ϕ is the automorphic form associated to f we get

$$\begin{aligned} \phi_\psi\left(\left(\begin{smallmatrix} 1 & \\ & p^r \end{smallmatrix}\right)_\infty \left(\begin{smallmatrix} 1 & \\ & q_i^{-1} \end{smallmatrix}\right)_\infty g_1\right) &= j(\left(\begin{smallmatrix} 1 & \\ & p^r \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & \\ & q_i^{-1} \end{smallmatrix}\right) g_1, i)^{-\kappa} p^{-r\kappa/2} q_i^{\kappa/2} f(wq/p^r) \\ &= j(g_1, i)^{-\kappa} f(w) \mid_\kappa \left(\begin{smallmatrix} q_i & \\ & p^r \end{smallmatrix}\right). \end{aligned} \tag{8.38}$$

Observe that if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{R})$ then $w_1 g w_1^{-1} = \left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right)$,

$$j(g', i) = \overline{j(w_1 g_1 w_1^{-1}, i)} \qquad \textit{and} \qquad j(g_1, i)\overline{j(w_1 g_1 w_1^{-1}, i)} = i\bar{w}/\mathrm{Im}(w).$$

For $\kappa \geq 6$ we can define classical Hermitian modular forms on $\mathcal{H}_3$: $E_{t_i}(Z') = j(g, i)^\kappa \mu(g)^{3\kappa/2} E(t_i g), g \in K_{3,\infty}^{h,+}$, $g(i) = Z'$. So for $t_i = \begin{pmatrix} a_i^{-1} & & \\ & 1 & \\ & & \bar{a}_i & \\ & & & 1 \end{pmatrix}$ we get

$$\begin{aligned} & E\left(\alpha\left(\left(\begin{smallmatrix} 1_2 & \\ & q_i^{-1} 1_2 \end{smallmatrix}\right) g \left(\begin{smallmatrix} a_i{}^{-1} & \\ & \bar{a}_i \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \\ & q_i^{-1} \end{smallmatrix}\right) g_1\right)\right) \\ &= \overline{j(w_1 \left(\begin{smallmatrix} 1 & \\ & q_i^{-1} \end{smallmatrix}\right) g_1 w_1^{-1}, i)^{-\kappa}} q_i^{-\kappa/2} j(\left(\begin{smallmatrix} 1_2 & \\ & q_i^{-1} 1_2 \end{smallmatrix}\right) g, i)^{-\kappa} \mu(g)^\kappa q_i^{-\kappa} E_{t_i} \left(\begin{smallmatrix} q_i Z & \\ & \frac{1}{q_i \bar{w}} \end{smallmatrix}\right) \\ &= \overline{j(w_1 g_1 w_1^{-1}, i)^{-\kappa}} q_i^{\kappa/2} j(g, i)^{-\kappa} \mu(g)^\kappa E_{t_i} \left(\begin{smallmatrix} q_i Z & \\ & \frac{1}{q_i \bar{w}} \end{smallmatrix}\right) \\ &= \overline{j(w_1 g_1 w_1^{-1}, i)^{-\kappa}} q_i^\kappa (-\bar{w})^\kappa j(g, i)^{-\kappa} \mu(g)^\kappa E_{t_i} \left(\begin{smallmatrix} q_i Z & \\ & -\bar{w} \end{smallmatrix}\right) \mid_\kappa \left(\begin{smallmatrix} & -1 \\ q_i & \end{smallmatrix}\right) \end{aligned}$$

Let

$$E_{\tilde{f}}(Z) = j(g, i)^\kappa \mu(g)^{-\kappa} I(g)$$

then putting together the above facts, we get

$$\begin{aligned} E_{\tilde{f}}(Z) = \mathrm{vol} \sum_i \tau(a_i) q_i^\kappa \int_{R_i} & E_{t_i} \left(\begin{smallmatrix} q_i Z & \\ & -\bar{w} \end{smallmatrix}\right) \mid_\kappa \left(\begin{smallmatrix} & -1 \\ q_i & \end{smallmatrix}\right) \\ & \cdot f(w) \mid_\kappa \left(\begin{smallmatrix} q_i & \\ & p^r \end{smallmatrix}\right) (-\bar{w})^\kappa (i\bar{w}/\mathrm{Im}(w))^{-\kappa} d\mathrm{vol}(w) \end{aligned}$$

On making a change of variable $u = -\bar{w}$ this becomes

$$= \mathrm{vol} \sum_i \tau(a_i) q_i^{\kappa} i^{-\kappa} (-1)^{\kappa-1} \int_{R_i} E_{t_i} \left(\begin{smallmatrix} q_i Z & \\ & u \end{smallmatrix} \right) |_{\kappa} \begin{pmatrix} & -1 \\ q_i & \end{pmatrix}$$

$$\cdot\, f(-\bar{u}) |_{\kappa} \begin{pmatrix} q_i & \\ & p^r \end{pmatrix} (\mathrm{Im}(-\bar{u}))^{\kappa} d\mathrm{vol}(u)$$

$$= \mathrm{vol} \sum_i \tau(a_i) q_i^{\kappa} i^{-\kappa} (-1)^{\kappa-1} \left\langle E_{t_i} \left(\begin{smallmatrix} q_i Z & \\ & w \end{smallmatrix} \right) |_{\kappa} \begin{pmatrix} & -1 \\ q_i & \end{pmatrix}, f^{\rho}(w) |_{\kappa} \begin{pmatrix} q_i & \\ & p^r \end{pmatrix} \right\rangle_{\Gamma_i}$$

$$= \mathrm{vol} \sum_i \tau(a_i) q_i^{\kappa} i^{-\kappa} (-1)^{\kappa-1} \left\langle E_{t_i} \left(\left(\begin{smallmatrix} q_i Z & \\ & w \end{smallmatrix} \right) \right), f^{\rho}(w) |_{\kappa} \begin{pmatrix} & -1 \\ p^r & \end{pmatrix} \right\rangle_{\Gamma_i}$$

this follows from the definition of Γ_i

$$= \mathrm{vol} \sum_i \tau(a_i) q_i^{\kappa} i^{-\kappa} (-1)^{\kappa-1} \left\langle E_{t_i} \left(\begin{smallmatrix} q_i Z & \\ & w \end{smallmatrix} \right), f^{\rho}(w) |_{\kappa} \begin{pmatrix} & -1 \\ p^r & \end{pmatrix} \right\rangle_{\Gamma_i}$$

Hence we have

Theorem VIII.23. *Let $E_{\tilde{f}}(Z)$ be the Hermitian Eisenstein series associated to $E(s, F(\tilde{f}), g)$ where $g_{\infty}(i) = Z$, then for Z in $\mathcal{H}_2$*

$$E_{\tilde{f}}(Z) = C \sum_i \tau(a_i) q_i^{\kappa} i^{-\kappa} (-1)^{\kappa-1} \left\langle E_{t_i} \left(s, \begin{smallmatrix} q_i Z & \\ & w \end{smallmatrix} \right), f^{\rho}(w) |_{\kappa} \begin{pmatrix} & -1 \\ p^r & \end{pmatrix} \right\rangle_{\Gamma_i}$$

where $C = \frac{1}{[U_1(\hat{\mathbf{Z}}):K^0(p^r)]}$ and a_i's are the representatives of the class group of $\mathcal{K}$.

The above classical interpretation of the pull back formula plays an important role in the interpolation of the Klingen Eisenstein series.

CHAPTER IX

p-adic interpolation

In this chapter we first recall the Leopoldt-Kubota-Iwasawa p-adic Dirichlet L-function. We then discuss p-adic families of modular forms for GL_2, $GSp(2n)$ and $GU(n,n)$, especially Hida families. As important examples we construct p-adic families interpolating Siegel Eisenstein and then combine this will the pullback formula to construct a p-adic family of Klingen Eisenstein series. The latter is then used to construct a p-adic L-function on $GSp(4) \times GL(2)$.

9.1 p-adic Dirichlet L-functions

In this section we give a brief outline of p-adic Dirichlet L-functions which are p-adic analogues of Dirichlet L-functions. The usual series of the Dirichlet L-functions do not converge p-adically. But the values of $L(s,\chi)$ at negative integers are algebraic, so we look for a p-adic function which agrees with $L(s,\chi)$ at the negative integers. Here we do not explicitly construct the measures or the corresponding power series but simply recall the results available. We refer the reader to [Lan90] and [Hid93] for more details.

Let p be a fixed odd prime. Let χ be any Dirichlet character of $(\mathbf{Z}/p^n\mathbf{Z})^\times$ having values in a finite extension of $\mathbf{Q}_p$. Let ω be the Teichmuller character. The p-adic L-function $\mathcal{L}_p(s,\chi)$ is a continuous function on $\mathbf{Z}_p$ except when $\chi = \mathrm{id}$, and in this case

$\zeta_p(s) = \mathcal{L}_p(s, \mathrm{id})$ is a continuous function defined on $\mathbf{Z}_p - \{1\}$, having the following interpolation property

$$\mathcal{L}_p(-m, \chi) = L(-m, (\chi\omega^{-m-1})_0)(1 - \chi\omega^{-m-1}{}_0(p)p^m) \qquad \text{for all } m \in \mathbf{N}$$

where $\chi_0 = \chi$ if χ is non-trivial and χ_0 is the constant function 1 on $\mathbf{Z}_p$ if χ is trivial.

Let

$$H_\chi(T) := \begin{cases} 1 & \text{if } \mathrm{cond}(\chi) \neq \text{ a power of } p, \mathbf{1} \\ \chi(1+p)(1+T) - 1 & \text{otherwise} \end{cases}$$

Theorem IX.1. *[Was97] Given $H_\chi(T)$ as above there exists $G_\chi(T) \in \mathcal{O}[[T]]$ ($\mathcal{O} = \mathbf{Z}_p[\chi]$) such that*

$$\mathcal{L}_p(1-s, \chi) = \frac{G_\chi((1+p)^s - 1)}{H_\chi((1+p)^s - 1)} \qquad s \in \mathbf{Z}_p$$

and $s \neq 1$ if $\chi = 1$.

9.2 Λ-adic forms: GL_2

In this section we recall the general theory of Λ-adic modular forms. These have come to be known as Hida families. We shall give definitions and examples and state some well-known results about Λ-adic forms. For more details one can consult [Hid93]. Finally, we introduce a linear map ℓ_f which plays an important role in the interpolation of the Klingen Eisenstein series.

Let p be a fixed odd prime. Let N be a fixed integer $(N, p) = 1$. Let

$$\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \bar{\mathbf{Q}}_p^\times$$

be a Dirichlet character. Here let $\mathcal{O}$ be the ring of integers of some finite extension of $\mathbf{Q}_p$ containing $\mathbf{Z}[\chi]$. Let $\Lambda = \Lambda_{\mathcal{O}} = \mathcal{O}[[T]]$. For κ a positive integer and ξ a p-power root of unity let $\vartheta_{\kappa,\xi} : \Lambda \to \mathcal{O}[\xi]$ be given by $1 + T \mapsto \xi(1+p)^\kappa$.

Definition IX.2. A Λ-adic modular form of level N and character χ is a collection

$$\mathcal{F} = \{c_n(T) \in \Lambda \mid n = 0, 1, 2, \cdots\}$$

such that

$$\vartheta_{\kappa,\xi}(\mathcal{F}) := \sum_{n=0}^{\infty} \vartheta_{\kappa,\xi}(c_n) q^n \in M_\kappa(Np^r, \chi\omega^{-\kappa}\psi_\xi, \mathcal{O}[\xi])$$

for all but finitely many pairs (κ, ξ), $(\kappa \geq 2)$. Here by q^n we mean $e^{2\pi i z}$ and ψ_ξ is the p-power order character of conductor a power of p such that $\psi_\xi(1+p) = \xi$ and where ξ is a p^{r-1} root of unity. We denote the space of such forms by $\mathcal{M}(N, \chi)$. We say $\mathcal{F}$ is a Λ-*adic cusp form* if $\vartheta_{\kappa,\xi}(\mathcal{F}) \in \mathcal{S}_\kappa(Np^r, \chi\omega^{-\kappa}\psi_\xi, \mathcal{O}[\xi])$ for all but finitely many pairs (κ, ξ) as above. We write $\mathcal{S}(N, \chi)$ to denote the space of such forms.

We give an example of a Λ-adic modular form below. Besides serving as an example we will use it to introduce some notation that we will need for interpolation arguments.

Example IX.3. Let $\chi : (\mathbf{Z}/N'\mathbf{Z})^\times \to \mathcal{O}^\times$ be a primitive Dirichlet character with $N' = N$ or Np and $\gcd(N, p) = 1$. Recall that

$$(\mathbf{Z}/p\mathbf{Z})^\times \times \mathbf{Z}_p \simeq \mathbf{Z}_p^\times$$

via the map $(\delta, a) \mapsto \delta(1+p)^a$. For $\ell \neq p$ define $a_\ell \in \mathbf{Z}_p$ by the equation

$$\ell = \omega(\ell)(1+p)^{a_\ell}.$$

We now define $c_n(T)'$s as follows

$$c_1(T) = 1$$

$$c_\ell(T) = 1 + \chi(\ell)\ell^{-1}(1+T)^{a_\ell}$$

$$c_{\ell^r}(T) = c_\ell(T)c_{\ell^{r-1}}(T) - \chi(\ell)\ell^{-1}(1+T)^{a_\ell}c_{\ell^{r-2}}(T) \qquad (r \geq 2)$$

$$c_n(T) = c_{\ell_1}^{r_1}(T) \cdots c_{\ell_s}^{r_s}(T) \qquad (\gcd(n,p)=1, n = \Pi_{\ell_i}^{r_i})$$

$$c_{p^r n}(T) = c_n(T) \qquad (\gcd(n,p)=1)$$

$$c_0(T) = \frac{1}{2}\frac{\hat{G}_\chi(T)}{\hat{H}_\chi(T)}$$

where $\hat{G}_\chi(T) = G_\chi((1+T)-1)$ and $\hat{H}_\chi(T) = H_\chi((1+T)-1)$. Suppose that $\chi \neq 1$ so that $H_\chi(T) = 1$. Consider $\mathcal{E}_\chi = \{c_n(T)\}$. Then $\vartheta_{\kappa,\xi}(\mathcal{E}_\chi) = \sum_{n=0}^{\infty} \vartheta_{\kappa,\xi}(c_n) q^n$. Below we check that $\mathcal{E}_\chi$ is a Λ-adic form.

$$\begin{aligned} \vartheta_{\kappa,\xi}(c_0) &= \frac{1}{2} G_\chi(\xi(1+p)^\kappa - 1) \\ &= \frac{1}{2}\mathcal{L}_p(1-\kappa, \chi\psi_\xi) \\ &= \frac{1}{2} L(1-\kappa, \chi\omega^{-k}\psi_\xi)(1 - \chi\omega^{-\kappa}\psi_\xi(p)p^{\kappa-1}) \end{aligned} \tag{9.1}$$

$$\begin{aligned} \vartheta_{\kappa,\xi}(c_\ell) &= 1 + \chi(\ell)\ell^{-1}\xi^{a_\ell}(1+p)^{a_\ell(\kappa)} \\ &= 1 + \chi(\ell)\ell^{-1}\psi_\xi(\ell)(1+p)^{a_\ell(\kappa)} \\ &= 1 + \chi\omega^{-k}\psi_\xi(\ell)\ell^{\kappa-1}. \end{aligned} \tag{9.2}$$

The above calculations show that $\vartheta_{\kappa,\xi}(\mathcal{E}_\chi)$ gives the Eisenstein series

$$E(z) = E^\kappa_{1,\chi\psi_\xi}(z) - \chi\omega^{-\kappa}\psi_\xi(p)p^{\kappa-1}E^\kappa_{1,\chi\psi_\xi}(pz).$$

Suppose L is a finite extension of F_Λ, the fractional field of Λ. Let $\mathcal{O}_L$ be the integral closure of Λ in L. We can extend the notion of a Λ-adic modular form to that of an $\mathcal{O}_L$-modular form. Set

$$\mathcal{X}_L = \{\phi : \mathcal{O}_L \to \bar{\mathbf{Q}}_p \text{ extending some } \vartheta_{\kappa,\xi}\ (\kappa \geq 2)\} \tag{9.3}$$

where $\vartheta_{\kappa,\xi} : \Lambda \to \mathcal{O}[\xi]$ is the specialization map $1+T \mapsto \xi(1+p)^k$. Then we define a $\mathcal{O}_L$-modular form of character χ to be

$$\mathcal{F} = \{c_n \in \mathcal{O}_L \mid n = 0, 1, 2, \cdots\}$$

such that

$$\phi(\mathcal{F}) = \sum_{n=0}^{\infty} \phi(c_n) q^n \in M_\kappa(Np^r, \chi\omega^{-\kappa}\psi_\xi, \phi(\mathcal{O}_L)) \tag{9.4}$$

for almost all $\phi \in \mathcal{X}_L$. Denote by $\mathcal{M}(N, \chi, \mathcal{O}_L)$ the $\mathcal{O}_L$-modular forms of character χ and by $\mathcal{S}(N, \chi, \mathcal{O}_L)$ the $\mathcal{O}_L$-cusp forms of character χ.

It is desirable to have a notion of Hecke operators in this setting just as in the setting of classical modular forms. One can extend our Hecke operators to act on these spaces as follows. Let $\mathcal{F} = \{c_n\} \in \mathcal{M}(N, \chi, \mathcal{O}_L)$ then define

$$T_\ell \mathcal{F} = \{c'_n\}$$

where

$$c'_0 = c_0 + \chi(\ell)\ell^{-1}(1+T)^{a_\ell} c_0$$

$$c'_n = c_{\ell n} + \begin{cases} 0 & \text{if } \ell \nmid n, \ell \nmid Np \\ \chi(\ell)\ell^{-1}(1+T)^{a_\ell} c_{n/\ell} & \ell | n, \ell \nmid Np \end{cases}$$

$$c'_n = e_{\ell n} \qquad \ell | Np.$$

Fact IX.4. *$\phi(T_\ell \mathcal{F}) = T_\ell \phi(\mathcal{F})$ whenever $\phi(\mathcal{F})$ is a modular form.*

By multiplicativity we can define the action of T_n on $\mathcal{M}(N, \chi, \mathcal{O}_L)$ and $\mathcal{S}(N, \chi, \mathcal{O}_L)$. It can be checked that $\mathcal{S}(N, \chi, \mathcal{O}_L))$ is stable under T_n. The modules $\mathcal{M}(N, \chi, \mathcal{O}_L)$ and $\mathcal{S}(N, \chi, \mathcal{O}_L)$ are generally not finitely generated. To remedy this problem we "cut down the space" via the *ordinary projector*. For any finite extension K of $\mathbf{Q}_p(\xi)$ and A its p-adic ring of integers recall the ordinary projector:

$$e = \lim_{m\to\infty} T_p^{m!} \in \mathrm{End}_A M_\kappa(Np^r, \chi\omega^{-\kappa}\psi_\xi, A).$$

Definition IX.5. We shall denote the space of ordinary modular forms by

$$M^0_\kappa(Np^r, \chi\omega^{-\kappa}\psi_\xi, A) := e M_\kappa(Np^r, \chi\omega^{-\kappa}\psi_\xi, A)$$

and ordinary cusp forms by

$$S_\kappa(Np^r, \chi\omega^{-\kappa}\psi_\xi, A) := eS^0_\kappa(Np^r, \chi\omega^{-\kappa}\psi_\xi, A).$$

Definition IX.6. We define $\mathcal{M}^0(N, \chi, \mathcal{O}_L)$ as

$$\{\mathcal{F} \in \mathcal{M}(N, \chi, \mathcal{O}_L) : \phi(\mathcal{F}) \in M^0_\kappa(Np^r, \chi\omega^{-\kappa}\psi_\xi, \phi(\mathcal{O}_L)) \text{ for a.e. } \phi \in \mathcal{X}_L\}$$

Similarly we define $\mathcal{S}^0(N, \chi, \mathcal{O}_L) \subset \mathcal{M}^0(N, \chi, \mathcal{O}_L)$.

Fact IX.7. *$rk_{\mathcal{O}[\xi]} eS_\kappa(Np^r, \chi\psi_\xi\omega^{-\kappa}, \mathcal{O}[\xi])$ and $rk_{\mathcal{O}[\xi]} eM_\kappa(Np^r, \chi\psi_\xi\omega^{-\kappa}, \mathcal{O}[\xi])$ are independent of κ and ξ.*

Fact IX.8. *The spaces $\mathcal{M}^0(N, \chi, \mathcal{O}_L)$ and $\mathcal{S}^0(N, \chi, \mathcal{O}_L)$ are finitely generated torsion-free $\mathcal{O}_L$-modules and $\mathcal{M}^0(N, \chi, \mathcal{O}_L)$ and $\mathcal{S}^0(N, \chi, \mathcal{O}_L)$ are free Λ-modules.*

Remark IX.9. One can also define an ordinary projector

$$e : \mathcal{M}(N, \chi, \mathcal{O}_L) \to \mathcal{M}^0(N, \chi, \mathcal{O}_L)$$

and

$$e : \mathcal{S}(N, \chi, \mathcal{O}_L) \to \mathcal{S}^0(N, \chi, \mathcal{O}_L)$$

such that $\phi(e\mathcal{F}) = e\phi(\mathcal{F})$ for a.e. $\phi \in \mathcal{X}$.

Remark IX.10. T_ℓ acts on $\mathcal{M}^0(N, \chi, \mathcal{O}_L)$ and $\mathcal{S}^0(N, \chi, \mathcal{O}_L)$ as well.

Fact IX.11. *The eigenvalues of T_ℓ acting on $\mathcal{M}^0(N, \chi, \mathcal{O}_L) \otimes_{\mathcal{O}_L} L$ are integral over $\mathcal{O}_L$.*

Fact IX.12. *If $\ell = p$ or if $ord_\ell(N) = ord_\ell(cond\ (\chi))$, then T_ℓ can be diagonalized on $\mathcal{M}^0(N, \chi, \mathcal{O}_L) \otimes_{\mathcal{O}_L} \bar{L}$.*

Definition IX.13. The ordinary Hecke algebra $\mathcal{H}^0(N, \chi)$ (resp. $\mathcal{H}^0_{\text{cusp}}(N, \chi)$) is the subalgebra of $\text{End}_\Lambda(\mathcal{M}^0(N, \chi))$(resp. $\text{End}_\Lambda(\mathcal{S}^0(N, \chi))$ generated by all the T_n's over

Λ. For any Λ-algebra A, we define $\mathcal{H}^0(N,\chi,A)$ (resp. $\mathcal{H}^0_{\text{cusp}}(N,\chi,A)$) by $\mathcal{H}^0(N,\chi)\otimes_\Lambda A$ (resp. $\mathcal{H}^0_{\text{cusp}}(N,\chi)\otimes_\Lambda A$).

Proposition IX.14. *(semi-simplicity) Let χ be a primitive Dirichlet character modulo $N, (N,p)=1$ then $\mathcal{H}^0(N,\chi)$ (resp. $\mathcal{H}^0_{cusp}(N,\chi)$) is reduced; i.e. $\mathcal{H}^0(N,\chi,F_\Lambda)$ (resp. $\mathcal{H}^0_{cusp}(N,\chi,F_\Lambda)$), for F_Λ the quotient field of Λ, is semisimple.*

Proof. Follows from fact IX.12. □

9.2.1 An inner product relation

Let f be a primitive ordinary elliptic eigen cusp form of level N and character χ with coefficients in a number field L. Then we have a map from $H^0_\kappa(N,\chi,L)$ to L given by $T(n)\mapsto\lambda_n$ where $T(n)f=\lambda_n f$. Since $H^0_\kappa(N,\chi,L)$ is semi-simple we can write

$$H^0_\kappa(N,\chi,L)\simeq L\oplus A.$$

Let $1_f\in H^0_\kappa(N,\chi,L)$ denote the idempotent projecting onto L. Now we associate to f a linear form ℓ_f on $M_\kappa(N,\chi,L)$ given by

$$\ell_f(g)=a(1,eg\mid 1_f)$$

where e is the ordinary projector. Then its a theorem of Hida that

Theorem IX.15. *Let f be a p-stabilized new form of level p^r, weight $\kappa\geq 2$, and character χ. Then, for g in $M_\kappa(p^r,\chi)$,*

$$\ell_f(g)=\frac{\langle g,h\rangle}{\langle f,h\rangle}$$

where $h=f^\rho\mid_\kappa\left(\begin{smallmatrix}0&-1\\p^r&0\end{smallmatrix}\right)$.

Proof. Page 175, [Hid85] □

Similarly, in the context of $\mathcal{O}_L$-adic forms,given $\mathcal{F}$ an ordinary normalized $\mathcal{O}_L$-adic cusp eigenform, the homomorphism

$$\mathcal{H}^0(N, \chi, L) \to L \text{ given by } H \mapsto \lambda_H, H\mathcal{F} = \lambda_H \mathcal{F}$$

is split. We denote by $1_{\mathcal{F}}$ the projection as above. Let

$$\mathbf{H}_{\mathcal{F}} := \text{Denominator}(1_{\mathcal{F}}) := \{a \in \mathcal{O}_L \mid a\mathbf{1}_{\mathcal{F}} \in \mathcal{H}^0(N, \chi, \mathcal{O}_L)\}.$$

We pick an element $H_{\mathcal{F}} \in \mathbf{H}_{\mathcal{F}}$. All our future interpolations associated to p-adic modular forms will depend on this choice. We now define

$$T_{\mathcal{F}} = H_{\mathcal{F}} \cdot 1_{\mathcal{F}} \in \mathcal{H}^0(N, \chi, L).$$

Let

$$\ell_{\mathcal{F}}(\mathcal{G}) = a(1, e\mathcal{G} \mid T_{\mathcal{F}})$$

for $\mathcal{G} \in \mathcal{M}(N, \chi, \mathcal{O}_L)$. For any ϕ extending $\vartheta_{\kappa,\xi}$, $\phi(\ell_{\mathcal{F}}(\mathcal{G})) \in \phi(\mathcal{O}_L)$ and for almost all ϕ, $\phi(\ell_{\mathcal{F}}(\mathcal{G})) = \phi(H_{\mathcal{F}})\ell_{\phi(\mathcal{F})}(\phi(\mathcal{G}))$.

9.3 Λ-adic forms: $GSp(4)$

In this section we recall the theory of Λ-adic Siegel modular forms and ordinary Λ-adic Siegel modular forms. Most of the material in this section is a generalization of the results in the case of p-adic families of elliptic modular forms. The results referred to in this section can be found in the work of Hida [Hid02], Tilouine-Urban[TU99] and Urban [Urb05]. Throughout this section we shall work with Siegel modular forms on $\mathbf{H}_n$.

Let N be a fixed integer $(N, p) = 1$. Let $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \bar{\mathbf{Q}}_p^\times$ be a Dirichlet character. Here let $\mathcal{O}$ be the ring of integers of some finite extension of $\mathbf{Q}_p$ containing

$\mathbf{Z}[\chi]$. Let $\Lambda = \mathcal{O}[[T]]$. For κ a positive integer and ξ a p^{th}- power root of unity let $\vartheta_{\kappa,\xi} : \Lambda \to \mathcal{O}[\xi]$ be given by $1+T \mapsto \xi(1+p)^\kappa$. Put

$$B_m = \{\theta \in M_m(\mathbf{Q}) | \theta = {}^t\theta, \theta_{ii}, 2\theta_{ij} \in \mathbf{Z}, \theta \geq 0\}$$

Definition IX.16. A Λ-adic Siegel modular form of level N and character χ is a collection

$$\mathbf{F} = \{c_B(T) \in \Lambda \mid B \in B_m\}$$

such that

$$\vartheta_{\kappa,\xi}(\mathbf{F}) = \sum_{B \in B_m} \vartheta_{\kappa,\xi}(c_B) q^B \in M_{s,\kappa}(Np^r, \chi\omega^{-\kappa}\psi_\xi, \mathcal{O}[\xi])$$

for all but finitely many pairs (κ, ξ), $(\kappa \geq 3)$ where ξ is a p^{r-1} root of unity and ψ_ξ is the character of p-power order and conductor such that $\psi_\xi(1+p) = \xi$. Here, by q^B we mean $e^{2\pi i \mathrm{tr}(BZ)}$. We denote the space of such forms by $\mathbf{M}_s(N, \chi)$. We say $\mathbf{F}$ is a Λ-*adic cusp form* if $\vartheta_{\kappa,\xi}(\mathbf{F}) \in S_{s,\kappa}(Np^r, \chi\omega^{-\kappa}\psi_\xi, \mathcal{O}[\xi])$ for all but finitely many pairs (κ, ξ) as above. We write $\mathbf{S}_s(N, \chi)$ to denote the space of such forms.

Remark IX.17. One can explicitly construct examples of such Λ-adic Siegel modular forms by interpolating the Fourier coefficients of Siegel Eisenstein series obtained from some good sections. One can refer to Courtieu-Panchishkin [CP04] for details.

Let $A \subset \mathbf{C}$ be any p-adic ring containing $\mathcal{O}[\xi]$ and $U_{s,p}$ be the Hecke operator defined in (3.2) then as in the case of the elliptic modular forms we can define the *ordinary projector* :

$$e = \lim_{m\to\infty} U_{s,p}^{m!} \in \mathrm{End}_A M_{s,\kappa}(Np^r, \chi\omega^{-\kappa}\psi_\xi, A).$$

Definition IX.18. We denote the space of ordinary modular forms by

$$M^0_{s,\kappa}(Np^r, \chi\omega^{-\kappa}\psi_\xi, A) := eM_{s,\kappa}(Np^r, \chi\omega^{-\kappa}\psi_\xi, A)$$

and cusp forms by

$$S^0_{s,\kappa}(Np^r, \chi\omega^{-\kappa}\psi_\xi, A) := eS_{s,\kappa}(Np^r, \chi\omega^{-\kappa}\psi_\xi, A).$$

Fact IX.19. *$rk_{\mathcal{O}[\xi]} eS_{s,\kappa}(Np^r, \chi\psi_\xi\omega^{-\kappa}, \mathcal{O}[\xi])$ and $rk_{\mathcal{O}[\xi]} eM_{s,\kappa}(Np^r, \chi\psi_\xi\omega^{-\kappa}, \mathcal{O}[\xi])$ are bounded independent of κ and ξ.*

Suppose L is a finite extension of F_Λ, the fractional field of Λ. Let $\mathcal{O}_L$ be the integral closure of Λ in L. We can extend the notion of a Λ-adic Siegel modular form to that of an $\mathcal{O}_L$-adic Siegel modular form. Set

$$\mathcal{X}_L = \{\phi : \mathcal{O}_L \to \bar{\mathbf{Q}}_p \text{ extending some } \vartheta_{\kappa,\xi}\ (\kappa \geq 2)\} \tag{9.5}$$

where $\vartheta_{\kappa,\xi} : \Lambda \to \mathcal{O}[\xi]$ is the specialization map $1 + T \mapsto \xi(1+p)^k$. Then we define a $\mathcal{O}_L$-adic Siegel modular form of degree n and character χ to be

$$\mathbf{F} = \{c_B \in \mathcal{O}_L \mid B \in \text{ symmetric } n \times n \text{ semi positive definite matrices}\}$$

such that

$$\phi(\mathbf{F}) = \sum_B^\infty \phi(c_B) q^B \in M_{s,\kappa}(Np^r, \chi\omega^{-\kappa}\psi_\xi, \phi(\mathcal{O}_L)) \tag{9.6}$$

for almost all $\phi \in \mathcal{X}_L$. Denote by $\mathbf{M}_s(N, \chi, \mathcal{O}_L)$ the $\mathcal{O}_L$-adic Siegel modular forms of character χ and by $\mathbf{S}_s(N, \chi, \mathcal{O}_L)$ the $\mathcal{O}_L$-adic Siegel cusp forms of character χ.

Just as in IX.6 we can define the space of *ordinary $\mathcal{O}_L$-adic Siegel modular forms* and *ordinary $\mathcal{O}_L$-adic Siegel cusp forms* and we can denote them by $\mathbf{M}^0_s(N, \chi, \mathcal{O}_L)$ and $\mathbf{S}^0_s(N, \chi, \mathcal{O}_L)$ respectively.

Fact IX.20. *The spaces $\mathbf{M}^0_s(N, \chi, \mathcal{O}_L)$ and $\mathbf{S}^0_s(N, \chi, \mathcal{O}_L)$ are finitely generated torsion-free $\mathcal{O}_L$-modules and $\mathbf{M}^0_s(N, \chi, \mathcal{O}_L)$ and $\mathbf{S}^0_s(N, \chi, \mathcal{O}_L)$ are free Λ-modules.*

Definition IX.21. Let the ordinary Hecke algebra $\mathbf{H}^0_{s,N}(N, \chi)$ (resp. $\mathbf{H}^0_{s,N,\mathrm{cusp}}(N, \chi)$) be the subalgebra of $\mathrm{End}_\Lambda(\mathbf{M}^0_s(N, \chi))$(resp. $\mathrm{End}_\Lambda(\mathbf{S}^0_s(N, \chi))$ generated by all the

$T_{s,n}$'s , $(n,N)=1$ over Λ. For any Λ-algebra A, we define $\mathbf{H}^0_{s,N}(N,\chi,A)$ (resp. $\mathbf{H}^0_{s,N,\mathrm{cusp}}(N,\chi,A)$) by $\mathbf{H}^0_{s,N}(N,\chi)\otimes_\Lambda A$ (resp. $\mathbf{H}^0_{s,N,\mathrm{cusp}}(N,\chi)\otimes_\Lambda A$).

Remark IX.22. We note that unlike in the case of elliptic modular forms, there is no non-degenerate pairing between the space of Siegel modular forms and Hecke operators. This can be observed from the fact that Hecke operators move the Fourier coefficients of a theta series in square classes.

Proposition IX.23. *(semi-simplicity) The Hecke algebras* $\mathbf{H}^0_{s,N}(N,\chi)$ *and* $\mathbf{H}^0_{s,N,cusp}(N,\chi)$ *are reduced.*

9.3.1 Another inner product relation

We need an analogue of the inner product relation of Hida in the setting of Siegel modular forms.

Let F be an ordinary Siegel eigen cusp form of weight $\kappa\geq 2n$ and level p^r and character χ with Fourier coefficients in a finite extension L of $\mathbf{Q}_p$. Then there exists a natural map from $H_s(\Gamma^s_{Q_n}(p^r),\chi,L)\twoheadrightarrow L$, $T_{s,n}\mapsto\lambda_n$, where $T_{s,n}F=\lambda_nF$. By a theorem of Hida (cf. page 46, [Hid98]) we know that $H^0_{\kappa,s}(p^r,\chi,L)$ is semisimple. Hence we have $H^0_{\kappa,s}(p^r,\chi,L)=L\oplus A$. We let $1_F\in H^0_\kappa(N,\chi,L)$ denote the idempotent projecting onto L.

Lemma IX.24. *Let F and G be Siegel cusp forms of degree n, weight $\kappa\geq 2n$, level N and character χ.Then*

$$\left\langle\left\langle T_{s,\ell}G, F^\rho|_{\left(\begin{smallmatrix}0&-1\\N&0\end{smallmatrix}\right)}\right\rangle\right\rangle=\left\langle\left\langle G,((T_{s,\ell}F^\rho)|_{\left(\begin{smallmatrix}0&-1\\N&0\end{smallmatrix}\right)}\right\rangle\right\rangle.$$

where

$$T_{s,\ell}=K_{Q_n}(N)\,diag(1,1,\ell,\ell)K_{Q_n}(N).$$

Proof. Let ϕ be the automorphic form associated to F^ρ and ϕ' be the automorphic form associated to G.

$$\left\langle\left\langle T_{s,\ell}\phi', \phi|_{\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)}\right\rangle\right\rangle$$

$$\begin{aligned}
&= \int\limits_{GSp_{2n}(\mathbf{Q})\backslash GSp_{2n}(\mathbf{A})} T_{s,\ell}\phi'(g)\overline{\phi(g\left(\begin{smallmatrix} & N^{-1} \\ -1 & \end{smallmatrix}\right))\chi^{-1}(\det(g))dg} \\
&= \sum_{u\in S_n(\mathbf{Z}) \mod \ell} \int\limits_{GSp_{2n}(\mathbf{Q})\backslash GSp_{2n}(\mathbf{A})} \phi'(g\left(\begin{smallmatrix} 1 & u \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right))\overline{\phi(g\left(\begin{smallmatrix} & N^{-1} \\ -1 & \end{smallmatrix}\right))\chi^{-1}(\det(g))dg} \\
&= \sum_{u\in S_n(\mathbf{Z}) \mod \ell} \int\limits_{GSp_{2n}(\mathbf{Q})\backslash GSp_{2n}(\mathbf{A})} \chi^{-1}(\ell^n)\phi'(g)\overline{\phi(g\left(\begin{smallmatrix} \ell^{-1} & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -u \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} & N^{-1} \\ -1 & \end{smallmatrix}\right))\chi^{-1}(\det(g))dg} \\
&= \sum_{u\in S_n(\mathbf{Z}) \mod \ell} \int\limits_{GSp_{2n}(\mathbf{Q})\backslash GSp_{2n}(\mathbf{A})} \chi^{-1}(\ell^n)\phi'(g)\overline{\phi(g\left(\begin{smallmatrix} \ell^{-1} & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} & N^{-1} \\ -1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ uN & 1 \end{smallmatrix}\right))\chi^{-1}(\det(g))dg} \\
&= \ell^{n(n+1)/2} \int\limits_{GSp_{2n}(\mathbf{Q})\backslash GSp_{2n}(\mathbf{A})} \chi^{-1}(\ell^n)\phi'(g)\overline{\phi(g\left(\begin{smallmatrix} \ell^{-1} & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} & N^{-1} \\ -1 & \end{smallmatrix}\right))\chi^{-1}(\det(g))dg}
\end{aligned}$$

while

$$\left\langle\left\langle \phi', (T_{s,\ell}\phi)|_{\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)}\right\rangle\right\rangle$$

$$\begin{aligned}
&= \int\limits_{GSp_{2n}(\mathbf{Q})\backslash GSp_{2n}(\mathbf{A})} \phi'(g)\overline{T_{s,\ell}\phi(g\left(\begin{smallmatrix} & N^{-1} \\ -1 & \end{smallmatrix}\right))\chi^{-1}(\det(g))dg} \\
&= \sum_{u\in S_n(\mathbf{Z}) \mod \ell} \int\limits_{GSp_{2n}(\mathbf{Q})\backslash GSp_{2n}(\mathbf{A})} \phi'(g)\overline{\phi(g\left(\begin{smallmatrix} & N^{-1} \\ -1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & u \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right))\chi^{-1}(\det(g))dg} \\
&= \sum_{u\in S_n(\mathbf{Z}) \mod \ell} \int\limits_{GSp_{2n}(\mathbf{Q})\backslash GSp_{2n}(\mathbf{A})} \phi'(g)\overline{\phi(g\left(\begin{smallmatrix} 1 & \\ -Nu & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ & \ell \end{smallmatrix}\right)\left(\begin{smallmatrix} & N^{-1} \\ -1 & \end{smallmatrix}\right))\chi^{-1}(\det(g))dg}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{u\in S_n(\mathbf{Z}) \mod \ell} \int\limits_{GSp_{2n}(\mathbf{Q})\backslash GSp_{2n}(\mathbf{A})} \phi'(g\left(\begin{smallmatrix} 1 & \\ uN & 1 \end{smallmatrix}\right))\overline{\phi(g\left(\begin{smallmatrix} 1 & \\ & \ell \end{smallmatrix}\right)\left(\begin{smallmatrix} & N^{-1} \\ -1 & \end{smallmatrix}\right))\chi^{-1}(\det(g)\ell^{-1})dg} \\
&= \ell^{n(n+1)/2} \int\limits_{GSp_{2n}(\mathbf{Q})\backslash GSp_{2n}(\mathbf{A})} \chi^{-1}(\ell^n)\phi'(g)\overline{\phi(g\left(\begin{smallmatrix} \ell^{-1} & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} & N^{-1} \\ -1 & \end{smallmatrix}\right))\chi^{-1}(\det(g)\ell^{-1})dg}
\end{aligned}$$

Hence

$$\left\langle\left\langle \phi', (T_{s,\ell}\phi)|_{\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)} \right\rangle\right\rangle = \left\langle\left\langle T_{s,\ell}\phi', \phi|_{\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)} \right\rangle\right\rangle.$$

The lemma follows for $\ell \mid N$ from the relationship between adelic Hecke operators and classical Hecke operators for Siegel modular forms.. Similarly, one handles the case $\ell \nmid N$ using the coset decomposition for $T_{s,\ell}$ given in (III.6). $\square$

Corollary IX.25. *If F and G are Siegel cusp eigenforms of degree n, weight $\kappa \geq 2n$, level N and character χ then*

$$\left\langle\left\langle G, F^{\rho} \mid_{\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)} \right\rangle\right\rangle = 0$$

unless F and G have the same eigenvalues.

Lemma IX.26. *For G a Siegel modular form of degree n, weight $\kappa \geq 3$, level p^r and character χ and F an ordinary Siegel cusp eigenform of degree 2, weight $\kappa \geq 3$, level p^r and character χ where $p^r = cond(\chi)$ if $\chi \neq 1$ and $p^r = p$ if $\chi = 1$. Then*

$$\langle\langle G, F^{\rho} \mid_{\kappa} \left(\begin{smallmatrix} & -1 \\ p & \end{smallmatrix}\right)\rangle\rangle = \langle\langle eG, F^{\rho} \mid_{\kappa} \left(\begin{smallmatrix} & -1 \\ p & \end{smallmatrix}\right)\rangle\rangle$$

Proof. Let $U_{s,p}F = \alpha_0 F$. We can decompose $M_{\kappa}(p^r, \chi)$ into generalized eigenspaces obtained from the action of $U_{s,p}$ operator as

$$M_{\kappa}(p^r, \chi) = \oplus_{\alpha} M_{\alpha}.$$

So we have

$$G = \sum_{\alpha} G_{\alpha}$$

where G_{α} is a generalized eigenvector with eigenvalue α and

$$\langle\langle G, F^{\rho} \mid_{\kappa} \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle = \sum_{\alpha} \langle\langle G_{\alpha}, F^{\rho} \mid_{\kappa} \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle \tag{9.7}$$

Now $(U_{s,p}-\alpha)^t G_\alpha = 0$ for some t. So we have

$$0 = \left\langle\left\langle (U_{s,p}-\alpha)^t G_\alpha, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle$$

by lemma (IX.24)

$$= \left\langle\left\langle G_\alpha, ((U_{s,p}-\alpha)^t F)^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle$$

On expanding $(U_{s,p}-\alpha)^t$ and using corollary (IX.25) we get

$$\left\langle\left\langle G_\alpha, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle = 0 \tag{9.8}$$

unless $\alpha = \alpha_0$. So from (9.7) we have

$$\left\langle\left\langle G, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle = \left\langle\left\langle G_{\alpha_0}, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle \tag{9.9}$$

hence

$$\left\langle\left\langle eG, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle = \left\langle\left\langle eG_{\alpha_0}, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle. \tag{9.10}$$

Recall $e = \lim_{m\to\infty} U_{s,p}^{m!}$ and α_0 is a p-adic unit hence $\lim_{m\to\infty}\alpha_0^{m!} = 1$. So

$$\begin{aligned}
\left\langle\left\langle eG_{\alpha_0}, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle &= \lim_{m\to\infty} \left\langle\left\langle U_{s,p}^{m!} G_{\alpha_0}, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle \\
&= \lim_{m\to\infty} \left\langle\left\langle G_{\alpha_0}, (U_{s,p}^{m!} F)^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle \\
&= \left\langle\left\langle G_{\alpha_0}, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle
\end{aligned}$$

Hence we have the result

$$\left\langle\left\langle eG, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle = \left\langle\left\langle G, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle.$$

$\square$

Definition IX.27. We say a Siegel modular form F of degree n, weight κ and level N satisfies the multiplicity one hypothesis if π_F occurs with multiplicity one in the $GSp(2n, \mathbf{A}_f)$ space generated by $\{\varphi_F \mid F$ holomorphic of degree n, weight κ level$N\}$.

Just as in the case of GL_2 we associate to an ordinary $\mathcal{O}_L$-adic Siegel cusp eigen form an idempotent $1_{\mathbf{F}}$. Given $\mathbf{F}$ an ordinary normalized $\mathcal{O}_L$-adic Siegel eigen cusp form, the homomorphism

$$\mathbf{H}^0_{s,N}(N, \chi, L) \to L \qquad \text{given by} \qquad H \mapsto \lambda_H$$

where $H\mathbf{F} = \lambda_H \mathbf{F}$ is split. We denote by $1_{\mathbf{F}}$ the projection as above. Let

$$\mathbf{H}_{\mathbf{F}} := \text{Denominator}(1_{\mathbf{F}}) = \{a \in \mathcal{O}_L \mid a1_{\mathbf{F}} \in \mathbf{H}^0_{s,N}(N, \chi, \mathcal{O}_L)\}.$$

We pick an element $H_{\mathbf{F}} \in \mathbf{H}_{\mathbf{F}}$. Many of our future interpolations associated to p-adic Siegel modular forms will depend on this choice. Let

$$\ell_{\mathbf{F}} = H_{\mathbf{F}} 1_{\mathbf{F}} \tag{9.11}$$

then by the theory of Λ-adic Siegel modular forms we have

$$\phi(\ell_{\mathbf{F}}\mathbf{G}) = \phi(H_{\mathbf{F}})\phi(1_{\mathbf{F}})\phi(\mathbf{G}) \tag{9.12}$$

Lemma IX.28. *Let F and G be ordinary Siegel eigenforms of degree 2, level p and trivial character such that π_F(resp π_G) is irreducible and F(resp G) satisfies the multiplicity one hypothesis. Then G must be a multiple of F.*

Proof. By the definition of 1_F, $1_F G = G$ implies that F and G have the same $T_{s,q}$ eigenvalues for all $q \neq p$. The trace of galois representation for F and G on a Frobenius at a prime q (for almost all q) is the eigenvalue of $T_{s,q}$ since the L-function of the representation is the spin-Lfunction for the Siegel modular form, Eric

([Urb05]). Hence by Chebotarev density theorem, the ℓ-adic Galois representations associated to F and G must be the same for every prime ℓ. Now observe that π_F and π_G are unramified away from p by the assumptions on F and G. At an unramifed place the Galois representation gives back the Satake parameters by the identification of the L-functions. Hence the Satake parameters of $\pi_q(F)$ and $\pi_q(G)$ are determined by the ℓ-adic representations of F and G respectively for $\ell \neq q$. But since the ℓ-adic representations of F and G are the same, we conclude that $\pi_q(F) = \pi_q(G)$ for $q \neq p$. By prop 3.2 in Tilouine-Urban, [TU99], both F and G are p-stabilizations of a form of full level. Hence $\pi_p(F)$ and $\pi_p(G)$ are unramified. But then by a similar argument as above this implies that $\pi_p(F) = \pi_p(G)$. So $\pi_q(F) = \pi_q(G)$ for all primes q hence $\pi(F) \simeq \pi(G)$. But by the multiplicity one hypothesis on F and G, this implies $\pi(F) = \pi(G)$. But we know that the spherical vector in $\pi_q(F)(q \neq p)$ is unique and by Hida, [Hid98], the ordinary vector in $\pi_p(F)$ is unique. Hence $F = cG$ $\square$

Lemma IX.29. *Let G be a Siegel modular forms of degree 2, weight $\kappa \geq 3$, level p^r and character χ and F an ordinary Siegel cusp eigenform of degree n, weight $\kappa \geq 3$, level p^r and character χ where $p^r = cond(\chi)$ if $\chi \neq 1$ and $p^r = p$ if $\chi = 1$. If $1_F eG = cF$ then*

$$c = \frac{\left\langle\left\langle G, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p & \end{smallmatrix}\right)\right\rangle\right\rangle}{\left\langle\left\langle F, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p & \end{smallmatrix}\right)\right\rangle\right\rangle}$$

Proof. Since $1_F eG = cF$

$$\left\langle\left\langle 1_F eG, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle = c\left\langle\left\langle F, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle.$$

So

$$c = \frac{\left\langle\left\langle 1_F eG, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle}{\left\langle\left\langle F, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\right\rangle\right\rangle}.$$

But now $1_F eG = e1_F G$ so by lemma (IX.26),

$$\langle\langle 1_F eG, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle = \langle\langle 1_F G, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle$$
$$= \langle\langle G, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle \tag{9.13}$$

by lemma (IX.25). Hence

$$c = \frac{\langle\langle G, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle}{\langle\langle F, F^\rho \mid_\kappa \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle}.$$

□

Just as in the case of GL_2 we associate to an ordinary $\mathcal{O}_L$-adic Siegel cusp eigen form an idempotent $1_{\mathbf{F}}$. Given $\mathbf{F}$ an ordinary normalized $\mathcal{O}_L$-adic Siegel eigen cusp form, the homomorphism

$$\mathbf{H}^0_{s,N}(N,\chi,L) \to L \qquad \text{given by} \qquad H \mapsto \lambda_H$$

where $H\mathbf{F} = \lambda_H \mathbf{F}$ is split. We denote by $1_{\mathbf{F}}$ the projection as above. Let

$$\mathbf{H}_{\mathbf{F}} := \text{Denominator}(1_{\mathbf{F}}) = \{a \in \mathcal{O}_L \mid a1_{\mathbf{F}} \in \mathbf{H}^0_{s,N}(N,\chi,\mathcal{O}_L)\}.$$

We pick an element $H_{\mathbf{F}} \in \mathbf{H}_{\mathbf{F}}$. Many of our future interpolations associated to p-adic Siegel modular forms will depend on this choice. Let

$$\ell_{\mathbf{F}} = H_{\mathbf{F}} 1_{\mathbf{F}} \tag{9.14}$$

then by the theory of Λ-adic Siegel modular forms we have

$$\phi(\ell_{\mathbf{F}}\mathbf{G}) = \phi(H_{\mathbf{F}})\phi(1_{\mathbf{F}})\phi(\mathbf{G}) \tag{9.15}$$

Theorem IX.30. *Let $\mathbf{F} \in \mathbf{S}^0_s(1,\chi,\mathcal{O}_L)$ be an $\mathcal{O}_L$-adic Siegel eigen cusp form such that for a Zariski dense subset $\phi \in \mathcal{X}_L$, $\phi(\mathbf{F})$ is such that $\phi(\mathbf{F})$ generates an irreducible cuspidal representation for which multiplicity one hypothesis holds. Let $\mathbf{G} \in \mathbf{M}_s(1,\chi,\mathcal{O}_L)$ then $\ell_{\mathbf{F}}(e\mathbf{G}) = \mathbf{cF}$ and*

$$\phi(\mathbf{c}) = \phi(H_{\mathbf{F}})\frac{\langle\langle \phi(\mathbf{G}), \phi(\mathbf{F})^\rho \mid \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle}{\langle\langle \phi(\mathbf{F}), \phi(\mathbf{F})^\rho \mid \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle} \tag{9.16}$$

Proof. Write $\mathbf{G} = \sum_i \mathbf{c}_i \mathbf{F}_i$ as where $\mathbf{F}_1 = \mathbf{F}$ and $\mathbf{F}_i$'s are distinct eigenforms. Then

$$1_{\mathbf{F}}(e\mathbf{G}) = \sum_i \mathbf{c}_i e(1_{\mathbf{F}}\mathbf{F}_i).$$

But then $1_{\mathbf{F}}\mathbf{F}_i = 0$ if $i \geq 2$. Suppose not, without loss of generality assume $i = 2$ and $1_{\mathbf{F}}\mathbf{F}_2 \neq 0$. Since $\mathbf{F}$ and $\mathbf{F}_2$ are linearly independent over L, there are two symmetric martices T_1 and T_2 such that $\det(C_{T_i}(\mathbf{F}_j)) \neq 0$. The determinant of this matrix is in $\mathcal{O}_L$. Since the determinant is non-zero, on specialization, $\det(C_{T_i}(\mathbf{F}_j))$ being an Iwasawa function it can have only finitely many zeros. So the specialization is non-zero for infinitely many specializations with character unramified at p. But then by lemma (IX.28), the specialization $\phi(\mathbf{F}_1)$ and $\phi(\mathbf{F}_2)$ are linearly dependent. So the matrix of the specialized Fourier coefficients must be zero, a contradiction. Hence $1_{\mathbf{F}}\mathbf{F}_2 = 0$. So $1_{\mathbf{F}}(e\mathbf{G}) = \mathbf{c}1_F$ and by lemma (IX.29) the result follows. □

9.4 Λ-adic forms: $GU(n, n)$

We want to interpolate the normalized Siegel Eisenstein series $D_t(n-\kappa/2, z)$ from VIII.15. To achieve this we interpolate the Fourier coefficients $c'_t(h, n-\kappa/2)$ for h a symmetric semi-definite $n \times n$ matrix. Here $t = \mathrm{diag}(u, \hat{u})$ with $u_\ell = 1$ for $\ell \mid N$. Recall that

$$c'_t(h, n-\kappa/2) = \tau(\det(u))|\det(u)|^{\kappa}\alpha'_N(u^*hu, 2n-\kappa, \bar{\tau}')$$

where $\alpha'_N(u^*hu, 2n-\kappa, \bar{\tau}') = \prod_{i=0}^{n-r-1} L_N(n-\kappa-i, \bar{\tau}'\epsilon_{\mathcal{K}}^{n+i-1}) \prod_{\ell\in\mathbf{c}} f_{h,u,\ell}(\bar{\tau}'(\ell)\ell^{\kappa-2n})$. In order to carry out this interpolation we first construct a p-adic character and discuss Λ-adic Hermitian modular forms.

9.4.1 Construction of a p-adic character

We first discuss the strategy for interpolation of the values of the character τ. Let L be a number field and L_∞ its maximal free $\mathbf{Z}_p$ extension. Let

$$\Gamma_L := \mathrm{Gal}(L_\infty/L) \simeq \mathbf{Z}_p^d,$$

d some positive integer. Then by class field theory we have a surjection

$$\phi_L : L^\times \backslash \mathbf{A}_L^\times / L_\infty^\times \times \prod_{v \nmid p} \mathcal{O}_{L,v}^\times \twoheadrightarrow \Gamma_L.$$

This map then extends to a map

$$\Phi_L : L^\times \backslash \mathbf{A}_L^\times \xrightarrow{\phi_L} \Gamma_L \hookrightarrow \mathbf{Z}_p[[\Gamma_L]]^\times$$

where the second inclusion is the tautological map.

For the current work we restrict ourselves to $L = \mathcal{K}$. We also assume that p splits in $\mathcal{K}$. In this case the action of the nontrivial automorphism of $\mathrm{Gal}(\mathcal{K}/\mathbf{Q})$ decomposes $\Gamma_\mathcal{K}$ as

$$\Gamma_\mathcal{K} = \Gamma_\mathcal{K}^+ \oplus \Gamma_\mathcal{K}^-$$

where $\Gamma_\mathcal{K}^+ \simeq \mathbf{Z}_p$ and $\Gamma_\mathcal{K}^- \simeq \mathbf{Z}_p$ with topological generators say γ_+ and γ_- respectively. So we have

$$\Phi_\mathcal{K} : \mathcal{K}^\times \backslash \mathbf{A}_\mathcal{K}^\times \xrightarrow{\phi_\mathcal{K}} \Gamma_\mathcal{K} \hookrightarrow \mathbf{Z}_p[[\Gamma_\mathcal{K}^+ \oplus \Gamma_\mathcal{K}^-]] \simeq \mathbf{Z}_p[[X_+, X_-]]$$

where $\gamma_+ \mapsto 1 + X_+$ and $\gamma_- \mapsto 1 + X_-$. Since p splits, $\mathcal{O}_{\mathcal{K},p} \simeq \mathbf{Z}_p \oplus \mathbf{Z}_p$ and $\mathcal{O}_{\mathcal{K},p}^\times \simeq \mathbf{Z}_p^\times \times \mathbf{Z}_p^\times$. Now we note that $\mathcal{O}_{\mathcal{K},p}^\times \subset \mathcal{K}^\times \backslash \mathbf{A}_\mathcal{K}^\times$. By the action of complex conjugation we can decompose $\mathcal{O}_{\mathcal{K},p}^\times$ as

$$\mathcal{O}_{\mathcal{K},p}^\times = \mathcal{O}_{\mathcal{K},p}^{\times,+} \times \mathcal{O}_{\mathcal{K},p}^{\times,-}$$

where

$$\mathcal{O}_{\mathcal{K},p}^{\times,+} = \{(a,a) \mid a \in \mathbf{Z}_p^\times\} = \mathbf{Z}_p^\times \simeq (\mathbf{Z}/p\mathbf{Z})^\times \times 1 + p\mathbf{Z}_p$$

$$\mathcal{O}_{\mathcal{K},p}^{\times,-} = \{(a,a^{-1}) \mid a \in \mathbf{Z}_p^\times\} \simeq \mathbf{Z}_p^\times \simeq (\mathbf{Z}/p\mathbf{Z})^\times \times 1 + p\mathbf{Z}_p$$

By class field theory we have

$$\begin{array}{ccccc} \mathcal{O}_{\mathcal{K},p}^{\times} & = & \mathcal{O}_{\mathcal{K},p}^{\times,+} & \times & \mathcal{O}_{\mathcal{K},p}^{\times,-} \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_{\mathcal{K}} & = & \Gamma_{\mathcal{K}}^{+} & \times & \Gamma_{\mathcal{K}}^{-} \end{array}$$

where the central map is a surjection and the map to the right has image with finite index say m (m reflects the possibility that p may divide the class number of $\mathcal{K}$). Now suppose u_p^+, u_p^- are the topological generators $1+p$ of $1+p\mathbf{Z}_p \subset \mathcal{O}_{\mathcal{K},p}^{\times,+}, \mathcal{O}_{\mathcal{K},p}^{\times,-}$ respectively. We may further assume that under the maps above

$$\begin{array}{ccl} u_p^+ & \mapsto & \gamma_+ \\ u_p^- & \mapsto & \gamma_-^{m'}, \quad m' \in \mathbf{Z}_p. \end{array}$$

Let $\underline{\kappa} = (\kappa_+, \kappa_-)$ and $\underline{\xi} = (\xi_+, \xi_-)$. Now consider the specialization

$$\psi_{\underline{\kappa},\underline{\xi}} = \Phi_{\mathcal{K}} \mod (X_+ - (\xi_+ (u_p^+)^{k_+} - 1), X_- - (\xi_- (u_p^-)^{k_-/m'} - 1))$$

where $\kappa_+, \kappa_- \in \mathbf{Z}/2\mathbf{Z}$, $\kappa_+ + \kappa_- \in \mathbf{Z}$ and ξ_+ and ξ_- are roots of unity of p^{th} power order. Then $\psi_{\underline{\kappa},\underline{\xi}}$ is a $\mathbf{Z}_p[[\xi_+, \xi_-(u_p^-)^{1/m'}]]^\times$ valued character of $\mathcal{K}^\times \backslash \mathbf{A}_{\mathcal{K}}^\times$. To this Galois character we can associate a Hecke character (i.e. a character of $\mathcal{K}^\times \backslash \mathbf{A}_{\mathcal{K}}^\times$ with finite conductor).

$$\psi'_{\underline{\kappa},\underline{\xi}}(x) = (x_\infty \bar{x}_\infty)^{\kappa_+} (x_\infty/\bar{x}_\infty)^{\kappa_-} (x_{p_1} x_{p_2})^{-\kappa_+} (x_{p_1}/x_{p_2})^{-\kappa_-} \psi_{\underline{\kappa},\underline{\xi}}(x) \tag{9.17}$$

$$= x_\infty^{\kappa_+ + \kappa_-} \bar{x}_\infty^{\kappa_+ - \kappa_-} x_{p_1}^{-(\kappa_+ + \kappa_-)} x_{p_2}^{-\kappa_+ + \kappa_-} \psi_{\underline{\kappa},\underline{\xi}}(x) \tag{9.18}$$

Then the values of $\psi'_{\underline{\kappa},\underline{\xi}}(x)$ are interpolated by $\psi_{\underline{\kappa},\underline{\xi}}(x)$ whenever $x_\infty = 1 = x_p$ (the case of interest).

9.4.2 One variable character

As a first step towards interpolating the Siegel Eisenstein series we interpolate the contribution of $\tau(\det(u))|\det(u)|^{\kappa}$. The p-adic character constructed above has two variables namely X_+, X_-. Since we are interested in a one variable p-adic family we would like to turn it into a one variable character $\Phi'_{\mathcal{K}}$ (say in T) such that for $1+T \to \xi(1+p)^{\kappa}$ the resulting Hecke character has the infinity type of the character $\tau|\cdot|^{\kappa}$ i.e. $(x_\infty/|x_\infty|)^{-\kappa}|x_\infty|^{\kappa} = \bar{x}_\infty^{\kappa}$. Now note that when $\kappa+ = \kappa/2$ and $\kappa_- = -\kappa/2$, the infinity type of $\Phi_{\mathcal{K}}$ is given by $\bar{x}_\infty^{\kappa}$.

Let $\Lambda = \mathbf{Z}_p[[T]]$ and $\Lambda' = \mathbf{Z}_p[[S]]$ which we transform into a Λ-algebra via $1+T \mapsto (1+S)^{m'}$. Consider,

$$\Phi'_{\mathcal{K}} : \mathbf{A}_{\mathcal{K}}^{\times}/\mathcal{K}^{\times} \to \mathbf{Z}_p[[X_+, X_-]]^{\times} \to \mathbf{Z}_p[[S]]^{\times}$$

where the map on the right is given by

$$1 + X_+ \mapsto (1+S)^{m'/2}, \qquad 1 + X_- \mapsto (1+S)^{-1/2}.$$

Let ξ be a p power root of unity and κ be an integer. Let $\vartheta_{\kappa,\xi} : \Lambda \to \bar{\mathbf{Q}}_p$ be a character such that $\vartheta_{\kappa,\xi}(1+T) = \xi(1+p)^{\kappa}$ and $\phi : \Lambda' \to \bar{\mathbf{Q}}_p$ such that $1+S \mapsto \xi'(1+p)^{-1/m'}$ where $\xi'^{m'} = \xi$ where is a p^{th} power root of unity. Or to be consistent with the notation earlier ϕ extends some $\vartheta_{\kappa,\xi}$. Let

$$\psi_\phi := \phi \circ \Phi'_{\mathcal{K}}$$

and ψ'_ϕ denote the associated Hecke Character. Then this Hecke character has infinity type $\bar{x}_\infty^{\kappa}$.

9.4.3 Λ-adic forms: $GU(n,n)$

Let τ_0 be a Hecke character of conductor Np, $(N,p) = 1$ We let $\mathcal{O}$ be the integer ring of some finite extension of $\mathbf{Q}_p$ containing $\mathbf{Z}_p[\tau_0]$. Let $\Lambda = \Lambda_{\mathcal{O}} = \mathcal{O}[[T]]$ and let

$\Lambda' = \mathcal{O}[[S]]$ be as in 9.4.2. Let L be a finite extension of $F_{\Lambda'}$ and $\mathcal{O}_L$ be the integral closure of Λ' in L. We will choose this extension to be as large as needed for our applications. Also in this section by q^T for any $n \times n$ Hermitian matrix T we mean $e(\mathrm{tr}(TZ))$ for $Z \in \mathcal{H}_n$.

Let $\vartheta_{\kappa,\xi} : \Lambda' \to \mathcal{O}[\xi]$ be given by $1 + S \mapsto \xi(1+p)^\kappa$ for ξ a p^{r-1} root of unity and $\kappa \geq 2n$.

Definition IX.31. A Λ'-adic Hermitian modular form of level N character τ_0 is a collection

$$\mathbf{E}_{h,\tau_0} = \{c_B(S) \in \Lambda' \mid B \in \mathcal{N}\}$$

such that

$$\vartheta_{\kappa,\xi}(\mathbf{E}_{h,\tau_0}) = \sum_{B \in \mathcal{N}} \vartheta_{\kappa,\xi}(c_B) q^B \in M_{h,\kappa}(Np^r, \tau_0 \psi'_{\kappa,\xi} \mathcal{O}[\xi])$$

for all but finitely many pairs (κ, ξ), $(\kappa \geq 2n)$ where $\psi'_{\kappa,\xi}$ is as in (9.17) and $\mathcal{N}$ is the space of semi positive definite Hermitian matrices. We denote the space of such forms by $\mathbf{M}_h(N, \tau_0)$. We say $\mathbf{E}_{h,\tau_0}$ is a Λ'-*adic cusp form* if $\vartheta_{\kappa,\xi}(\mathbf{E}_{h,\tau_0}) \in S_{h,\kappa}(Np^r, \tau_0\psi'_{\kappa,\xi}, \mathcal{O}[\xi])$ for all but finitely many pairs (κ, ξ) as above. We write $\mathbf{S}_h(N, \tau_0)$ to denote the space of such forms.

We can extend the notion of Λ'-adic Hermitian modular forms to that of $\mathcal{O}_L$-adic Hermitian modular forms. Set

$$\mathcal{X}_L = \{\phi : \mathcal{O}_L \to \bar{\mathbf{Q}}_p \text{ extending some } \vartheta_{\kappa,\xi}\ (\kappa \geq 2)\}. \tag{9.19}$$

Then we define a $\mathcal{O}_L$-adic Hermitian modular form of character τ_0 to be

$$\mathbf{E}_{h,\tau_0} = \{c_B \in \mathcal{O}_L \mid B \in \mathcal{N}\}$$

such that

$$\phi(\mathbf{E}_{h,\tau_0}) = \sum_B \phi(c_B) q^B \in M_\kappa(Np^r, \tau_0\phi(\psi'_{\kappa,\xi}), \phi(\mathcal{O}_L)) \tag{9.20}$$

for almost all $\phi \in \mathcal{X}_L$. Denote by $\mathbf{M}_h(N, \tau_0, \mathcal{O}_L)$ the $\mathcal{O}_L-$modular forms of character τ_0 and by $\mathbf{S}_h(N, \tau_0, \mathcal{O}_L)$ the $\mathcal{O}_L$-cusp forms of character τ_0.

9.4.4 Example: Siegel Eisenstein series

As an example of a Λ-adic Hermitian modular form we discuss the interpolation of the normalized Siegel Eisenstein $D_t(n-\kappa/2, z)$ from section (VIII.15). We do this by interpolating the Fourier coefficients $c'_t(h, n-\kappa/2)$ for h a symmetric semi-definite $n \times n$ matrix. Here $t = \mathrm{diag}(u, \hat{u})$ with $u_\ell = 1$ for $\ell \mid N$. Recall that

$$c'_t(h, n-\kappa/2) = \tau(\det(u))|\det(u)|^{\kappa}\alpha'_N(u^*hu, 2n-\kappa, \bar{\tau}')$$

where $\alpha'_N(u^*hu, 2n-\kappa, \bar{\tau}') = \prod_{i=0}^{n-r-1} L_N(n-\kappa-i, \bar{\tau}'\epsilon_{\mathcal{K}}^{n+i-1}) \prod_{\ell \in \mathbf{c}} f_{h,u,\ell}(\bar{\tau}'(\ell)\ell^{\kappa-2n})$.

Theorem IX.32. *Let τ_0 be a finite order Hecke character and N_0 a positive integer such that cond$(\tau_0) \mid N_0$. Let $t = $ diag$(u, \hat{u})$, $u \in GL_n(\mathbf{A}_{\mathcal{K}})$ such that $u_\ell = 1$ for all $\ell \mid N_0p$. Let $\Lambda = \mathcal{O}[[T]]$ and $\Lambda' = \mathcal{O}[[S]]$ as before. Let $\vartheta_{\kappa,\xi} : \Lambda = \mathcal{O}[[T]] \to \bar{\mathbf{Q}}_p$ such that $1+T \mapsto \xi(1+p)^{\kappa}$ and $\phi : \Lambda' \to \bar{\mathbf{Q}}_p$ extends $\vartheta_{\kappa,\xi}$. Let $\tau_{\kappa,\xi}$ be the Hecke character associated to $\tau_0\Phi'_{\mathcal{K}}$ composed with ϕ. Then there exists a Λ'-adic form*

$$\mathbf{D}_{t,\tau_0} = \{C_h, \in \Lambda' \mid h \in S_n(\mathbf{Q}), h \geq 0, uhu^* \in S_n(\hat{\mathbf{Z}})^*\}$$

such that

$$\phi(\mathbf{D}_{t,\tau_0}) = \sum_h \phi(C_n)q^h = D_t(n-\kappa/2, z; \tau_{\kappa,\xi})$$

for almost all κ and ξ.

Proof. By the construction of the one variable character above we know that values of the Hecke character associated to $\phi \circ (\tau_0\Phi'_{\mathcal{K}})$ are p-adically interpolated. We also observe that the infinity type of this character is $\bar{x}_\infty^{\kappa}$. Hence the values

$\tau(\det(u))|\det(u)|^\kappa$ for a Hecke character τ of infinity type $(x_\infty/|x_\infty|)^{-\kappa}$ are also interpolated. It remains to interpolate

$$\prod_{i=0}^{n-r-1} L_N(n-\kappa-i, \bar{\tau}'\epsilon_{\mathcal{K}}^{n+i-1}) \prod_{\ell\in\mathbf{c}} f_{h,u,\ell}(\tau^{-1}(\ell)\ell^{\kappa-2n}).$$

But the values of the incomplete Dirichlet L-function are see to be interpolated by the existence of the Kubota-Leopoldt p-adic Dirichlet L-function (discussed in section(IX.1)). Finally, $f_{h,u,\ell}(\bar{\tau}'(\ell)\ell^{\kappa-2n})$ is seen to be p-adically interpolated by considering the restriction of the character $\phi\circ(\tau_0\Phi'_{\mathcal{K}})$ to $\mathbf{Q}$ and then explicitly writing out the Λ-adic form as in example (IX.3). □

9.5 p-adic interpolation of Klingen Eisenstein series

In this section we first discuss some generalities about the Fourier-Jacobi series of a Hermitian modular form. We then apply this theory to the interpolation of the Klingen Eisenstein series obtained from the pullback of the Siegel Eisenstein series on $GU(3,3)$. The notation chosen is suggestive of the setup to which we will apply this theory.

9.5.1 Some generalities

Let $z \in \mathcal{H}_2$ and $w \in \mathcal{H}_1$ and let $\mathcal{H}_2 \times \mathcal{H}_1 \hookrightarrow \mathcal{H}_3$ given by $z \times w \to (z, w)$. Let E be an Hermitian modular form on $\mathcal{H}_3$ then we can write $E\left(\begin{smallmatrix} z & \\ & w\end{smallmatrix}\right)$ as

$$E\left(\begin{smallmatrix} z & \\ & w\end{smallmatrix}\right) = \sum_{B=\left(\begin{smallmatrix} B' & b \\ \bar{b} & n\end{smallmatrix}\right)\in\mathcal{N}_3} a(B)e(nw)e(\mathrm{tr}(B'z)) \tag{9.21}$$

where $\mathcal{N}_3$ is some lattice in the space of 3×3 positive semi-definite Hermitian modular forms. We can also write this series in terms of its Fourier-Jacobi coefficients $C(B', w)$ as

$$E\left(\begin{smallmatrix} z & \\ & w\end{smallmatrix}\right) = \sum_{B'} C(B', w)e(\mathrm{tr}(B'z)) \tag{9.22}$$

where

$$C(B',w) = \sum_{B=\left(\begin{smallmatrix} B' & b \\ \bar{b} & n \end{smallmatrix}\right)} a(B)e(nw).$$

Lemma IX.33. *The Fourier-Jacobi coefficient*

$$C(B',w) = \sum_{B=\left(\begin{smallmatrix} B' & x \\ \bar{x} & n \end{smallmatrix}\right)} a(B)e(nw)$$

is a Hermitian modular form on $\mathcal{H}_1$ of weight κ.

Proof. Let $\gamma = \begin{pmatrix} \mu^{-1}\cdot 1_2 & & & \\ & a & & b \\ & & \mu\cdot 1_2 & \\ & c & & d \end{pmatrix} \in \Gamma^h_{Q_3}(N)$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^h_{Q_1}(N)$ and μ is the automorphy factor of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have

$$\begin{aligned} E\left(\begin{smallmatrix} z & \\ & w \end{smallmatrix}\right) &= E\left(\begin{smallmatrix} z & \\ & w \end{smallmatrix}\right)|_\gamma = \mu^{3\kappa/2} j(\gamma, \left(\begin{smallmatrix} z & \\ & w \end{smallmatrix}\right))^{-\kappa} E(\gamma\left(\begin{smallmatrix} z & \\ & w \end{smallmatrix}\right)) \\ &= \mu^{3\kappa/2}\mu^{-\kappa} \det(cw+d)^{-\kappa} E\left(\begin{smallmatrix} z & \\ & (aw+b)(cw+d)^{-1} \end{smallmatrix}\right) \\ &= \mu(\gamma)^{\kappa/2} \det(cw+d)^{-\kappa} E\left(\begin{smallmatrix} z & \\ & (aw+b)(cw+d)^{-1} \end{smallmatrix}\right). \end{aligned}$$

Now using the definition of the Fourier-Jacobi coefficients we see that $C(B',w)$ is an Hermitian modular form of weight κ for $\Gamma^h_{Q_1}(N)$. $\square$

Let L be a finite extension of the field of fractions of Λ' as in section (9.4.2). Now we restrict ourselves to $\mathcal{O}_L$-adic Hermitian modular forms on $\mathcal{H}_3$. Let

$$\mathbf{E}_{h,\tau_0} = \{c_B \in \mathcal{O}_L \mid B \in \mathcal{N}_3\} \tag{9.23}$$

be such that for each B', a semi positive definite 2×2 Hermitian matrix

$$\mathbf{E}_{h,\tau_0,B'} := \{c_{n,B'} = \sum_b c_B \in \mathcal{O}_L \mid B = \begin{pmatrix} B' & b \\ \bar{b} & n \end{pmatrix} \in \mathcal{N} \text{ for some } b, \text{ nonnegative integer } n\}$$

is an $\mathcal{O}_L$-adic modular form such that

$$\phi(\mathbf{E}_{h,\tau_0,B'}) = \sum_n \phi(c_{n,B'})q^n \in M_\kappa(\Gamma^h_{Q_1}(Np^r), \tau_0\psi'_{\kappa,\xi}, \phi(\mathcal{O}_L)). \tag{9.24}$$

Let $\mathcal{F} \in \mathcal{M}(N, \tau_0, \mathcal{O}_L)$ be an eigen cusp form on $GL(2)$.We put

$$\mathbf{E}_{\mathcal{F}} = \{\ell_{\mathcal{F}}(\mathbf{E}_{h,\tau_0,B'}) \in \mathcal{O}_L \mid B' \text{ semi positive definite } 2 \times 2 \text{ Hermitian matrix}\}.$$

Now we note that

$$\begin{aligned}
\phi(\mathbf{E}_{\mathcal{F}}) &= \sum_{B'} \phi(\ell_{\mathcal{F}}(\mathbf{E}_{h,\tau_0,B'})q^{B'} \\
&= \sum_{B'} \ell_{\phi(\mathcal{F})}\phi(\mathbf{E}_{h,\tau_0,B'})q^{B'} \\
&= \phi(H_{\mathcal{F}}) \sum_{B'} \frac{\langle \phi(\mathbf{E}_{h,\tau_0,B'}), \phi(\mathcal{F})^{\rho} \mid_{Np^r} \rangle}{\langle \phi(\mathcal{F}), \phi(\mathcal{F})^{\rho} \mid_{Np^r} \rangle} q^{B'} \\
&= \phi(H_{\mathcal{F}}) \sum_{B'} \frac{\left\langle \sum_n \phi(c_{n,B'})q^n, \phi(\mathcal{F})^{\rho} \mid_{Np^r} \right\rangle}{\langle \phi(\mathcal{F}), \phi(\mathcal{F})^{\rho} \mid_{Np^r} \rangle} q^{B'} \qquad (9.25) \\
&= \phi(H_{\mathcal{F}}) \sum_{B'} \frac{\left\langle \sum_n \phi(c_{n,B'})q^n q^{B'}, \phi(\mathcal{F})^{\rho} \mid_{Np^r} \right\rangle}{\langle \phi(\mathcal{F}), \phi(\mathcal{F})^{\rho} \mid_{Np^r} \rangle} \\
&= \phi(H_{\mathcal{F}}) \sum_{B'} \frac{\left\langle \sum_n \sum_b \phi(c_B)q^n q^{B'}, \phi(\mathcal{F})^{\rho} \mid_{Np^r} \right\rangle}{\langle \phi(\mathcal{F}), \phi(\mathcal{F})^{\rho} \mid_{Np^r} \rangle}
\end{aligned}$$

where $B = \left(\begin{smallmatrix} B' & b \\ \bar{b} & n \end{smallmatrix}\right)$

$$= \phi(H_{\mathcal{F}}) \frac{\left\langle \sum_B \phi(c_B)q^n q^{B'}, \phi(\mathcal{F})^{\rho} \mid_{Np^r} \right\rangle}{\langle \phi(\mathcal{F}), \phi(\mathcal{F})^{\rho} \mid_{Np^r} \rangle} \in \phi(\mathcal{O}_L) \qquad (9.26)$$

where $B = \left(\begin{smallmatrix} B' & b \\ \bar{b} & n \end{smallmatrix}\right)$.

For each Z, $\phi(\mathbf{E}_{h,\tau_0})\left(\begin{smallmatrix} Z & \\ & w \end{smallmatrix}\right)$ is a modular form of level N and weight κ and we have

$$\phi(\mathbf{E}_{\mathcal{F}}) = \phi(H_{\mathcal{F}}) \frac{\langle \phi(\mathbf{E}_{h,\tau_0})\left(\begin{smallmatrix} Z & \\ & w \end{smallmatrix}\right), \phi(\mathcal{F})^{\rho} \mid_{Np^r} \rangle}{\langle \phi(\mathcal{F}), \phi(\mathcal{F})^{\rho} \mid_{Np^r} \rangle}$$

9.5.2 Application: p-adic interpolation of Klingen Eisenstein series

In this section we use the classical interpretation of the pullback formula and the preceding generalities to interpolate the Klingen Eisenstein series.

Assumption IX.34. *Let*

- χ *be an even Dirichlet character mod* $(\mathbf{Z}/p\mathbf{Z})^{\times}$

- $\mathcal{F} \in S_\kappa^0(1, \chi, \mathcal{O}_L)$ *an ordinary* $\mathcal{O}_L$*-adic eigenform*
- $\mathcal{K}$ *be the imaginary quadratic extension*
- τ_0 *a finite order Hecke character of* $\mathcal{K}$ *such that*
  - $\tau_0 \mid_{\mathbf{A}^\times} = \chi$
  - τ_0 *takes values in* $\mathcal{O}_L$
  - *unramified away from* p.

Recall that in section (9.4.4) we proved that for $t = \mathrm{diag}(u, \hat{u})$, where $u \in GL_2(\mathbf{A}_\mathcal{K})$, $u_p = 1$, there exists

$$\mathbf{D}_{t,\tau_0} = \{C_{B,t} \mid C_{B,t} \in \Lambda' = \mathcal{O}[[S]], B \in \mathcal{N}_3\}$$

such that $\phi(\mathbf{D}_{t,\tau_0}) = \sum \phi(C_{B,t})q^B = D_t(n-\kappa, z; \tau_0\psi')$ where ϕ extends $\vartheta_{\kappa,\xi}$ and $\kappa > 6$.

With notation as in section (8.8) let

$$\mathbf{E}_{h,\tau_0} = \left\{ \sum_{a_i} \tau_0(a_i)\Phi'_\kappa(a_i) C_{B_i,t_i} \in \Lambda' \mid B \in \mathcal{N}_3 \right\} \tag{9.27}$$

where $B = \left(\begin{smallmatrix} B' & b \\ {}^t\bar{b} & n \end{smallmatrix}\right)$, $B_i = \begin{pmatrix} q_i B' & b \\ {}^t\bar{b} & n \end{pmatrix}$ and $t_i = \begin{pmatrix} \bar{a}_i^{-1} & & & \\ & 1 & & \\ & & \bar{a}_i & \\ & & & 1 \end{pmatrix}$.

Lemma IX.35. $\mathbf{E}_{h,\tau_0}$ *satisfies condition (9.24).*

Proof. $\mathbf{E}_{h,\tau_0}$ is a linear combination of Λ'-adic forms C_{B_i,t_i}. From lemma (IX.33) it follows that each of these Λ'-adic forms satisfies condition (9.24). Hence the result follows. $\square$

Theorem IX.36. *Let* $\mathcal{F} \in S^0(1, \chi, \mathcal{O}_L)$ *and* $\mathbf{E}_{h,\tau_0}$ *be a* Λ'*-adic form as in formula (9.27). Then* $\phi(\mathbf{E}_\mathcal{F})$ *is a* Λ'*-adic Klingen Eisenstein series on* $\mathcal{H}_2$ *such that*

$$\phi(\mathbf{E}_\mathcal{F}) = \phi(H_\mathcal{F}) \frac{\langle \phi(\mathbf{E}_{h,\tau_0}) \left(\begin{smallmatrix} Z & \\ & w \end{smallmatrix} \right), \phi(\mathcal{F})^\rho \mid_{Np^r} \rangle}{\langle \phi(\mathcal{F}), \phi(\mathcal{F})^\rho \mid_{Np^r} \rangle}$$

for almost all ϕ *where* ϕ *extends* $\vartheta_{\kappa,\xi}$.

This theorem gives the interpolation of the Klingen Eisenstein series obtained from the pullback of the normalized Siegel Eisenstein series.

9.6 The global integral - a classical interpretation

As discussed in chapter (7.1) the degree eight L-function has a global integral representation as a Rankin Selberg integral obtained by integration of a Klingen Eisenstein series on $GU(2,2)$ restricted to $GSp(4)$ against a Siegel eigen cusp form on $GSp(4)$. We interpret his global integral as an inner product and finally construct the p-adic L-function. In this section we interpret the global integral as an inner product.

Let

- $f \in S_\kappa(p^r, \chi)$ be an eigen cuspform and let $\phi = \phi_f$ be the associated automorphic form.
- $F \in S_{s,\kappa}(p^r, \chi)$ be a nonzero Siegel modular form on $\mathbf{H}_2$ with $a(S, F) \neq 0$ for some $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ with $-D = d(S)$ as in section (5.3)
- $\varphi = \varphi_F$ be the automorphic form associated to F.
- $\mathcal{K}$ be the quadratic extension $\mathbf{Q}(\sqrt{-D})$.
- τ be an Hecke character of $\mathcal{K}$ such that
  - $\tau_\infty = (z/|z|)^{-\kappa}$
  - $\tau \mid_{\mathbf{A}^\times} = \chi$
  - τ is unramified away from p

- ξ be any unramified character of $\mathbf{A}_{\mathcal{K}}^{\times}$ of finite order.
- $\psi = \tau\xi$
- ρ be the representation associated to (σ_f, ψ, τ) as in (4.3)
- $F_{\phi}(f_\kappa) \in I(\rho)$ as in (8.8)
- E_f be the classical Eisenstein series associated to $E(s, F_{\phi}(f), g)$ as in section (8.8)

We recall the pullback formula 8.20 for the Klingen Eisenstein series i.e.

$$\int_{U_n(\mathbf{Q})/U_n(\mathbf{A})} E(Q_{2n+1}, s, \tau, f_\kappa, \gamma_n(g, g_1 h))\bar{\tau}(\det g_1 h)\phi(g_1 h) dg_1$$

$$= \sum_{\gamma \in P_{n+1}(\mathbf{Q})\backslash G_{n+1}(\mathbf{Q})} F_{\phi,s}(f_\kappa, \gamma g).$$

Now note that

$$\begin{aligned} E(Q_{2n+1}, s, \tau, f_\kappa, \alpha_n(g, g_1 h)) &= E(Q_{2n+1}, s, \tau, f_\kappa, S^{-1}\alpha_n(g, g_1 h)) \\ &= E(Q_{2n+1}, s, \tau, \tilde{f}_\kappa, \gamma_n(g, g_1 h)) \end{aligned}$$

where $\tilde{f}_\kappa(s, g) = f_\kappa(s, gS^{-1})$.

Next we observe that for $\left(\begin{smallmatrix} A & B \\ Cp^r & D \end{smallmatrix}\right) \in K_{Q_2}^h(p^r)$,

$$\sum_{\gamma \in P_{n+1}(\mathbf{Q})\backslash G_{n+1}(\mathbf{Q})} F_{\phi,s}(f_\kappa, \gamma g \left(\begin{smallmatrix} A & B \\ Cp^r & D \end{smallmatrix}\right))$$

$$\begin{aligned} &= \int_{U_n(\mathbf{Q})/U_n(\mathbf{A})} E(Q_{2n+1}, s, \tau, f_\kappa, \alpha_1(g \left(\begin{smallmatrix} A & B \\ Cp^r & D \end{smallmatrix}\right), g_1 h))\bar{\tau}(\det g_1 h)\phi(g_1 h) dg_1 \\ &= \tau(\det(A)) \int_{U_n(\mathbf{Q})/U_n(\mathbf{A})} E(Q_{2n+1}, s, \tau, f_\kappa, \alpha_1(g \left(\begin{smallmatrix} A & B \\ Cp^r & D \end{smallmatrix}\right), g_1 h))\bar{\tau}(\det g_1 h)\phi(g_1 h) dg_1 \\ &= \tau(\det(A)) \sum_{\gamma \in P_{n+1}(\mathbf{Q})\backslash G_{n+1}(\mathbf{Q})} F_{\phi,s}(f_\kappa, \gamma g) \end{aligned}$$

Let $F \in S^0_{\kappa,s}(\Gamma^0(p^r), \tau')$ and φ be the automorphic form associated to

$$F^\rho \mid_\kappa \left({}_{p^r} {}^{-1} \right)$$

then we know that

$$\varphi(g \left(\begin{smallmatrix} A & B \\ p^r C & D \end{smallmatrix} \right)) = \tau'(\det(A))\varphi(g)$$

for $\left(\begin{smallmatrix} A & B \\ p^r C & D \end{smallmatrix} \right) \in K^s_{Q_2}(p^r)$

Hence

$$E(s, F_\phi(f), h)\bar{\varphi}(h)$$

is invariant on the right by $K^s_{Q_2}(p^r)$.

Now we consider the global integral of Furusawa.

$$\begin{aligned} Z(s, F_{\phi,s}(f), \varphi) &= \int_{H(\mathbf{Q})Z_H(\mathbb{A})\backslash H(\mathbb{A})} E(F_{\phi,s}(f), h)\bar{\varphi}(h)dh \\ &= \int_{H(\mathbf{Q})Z_H(\mathbb{A})\backslash H(\mathbb{A})/K^s_{Q_2}(p^r)K^s_{2,\infty}} E(F_{\phi,s}(f), h)\bar{\varphi}(h)dh \\ &= \int_{\Gamma^s_{Q_2}(p^r)\backslash \mathbf{H}_2} E_f(s, Z)\overline{F^\rho(Z) \mid_\kappa \left({}_{p^r} {}^{-1} \right)}(\det Y)^\kappa d\mu Z \end{aligned}$$

So by 8.21 at $s = (1 - \kappa/2)/3$ the point of holomorphicity of $E_f(s, Z)$ we have

Lemma IX.37.

$$Z(s, F_{\phi,s}(f), \varphi) = \left\langle\left\langle E_f(1/3 - \kappa/6, Z), F^\rho \mid_\kappa \left({}_{p^r} {}^{-1} \right) \right\rangle\right\rangle$$

9.7 Construction of a p-adic L-function

In this section we construct a p-adic L-function associated to the degree eight L-function on $GSp(4) \times GL(2)$. Let

- χ be an even character modulo p

- $\mathcal{F} \in \mathcal{S}^0(1, \chi, \mathcal{O}_L)$ be an $\mathcal{O}_L$-adic ordinary eigenform

- $\mathbf{F} \in \mathbf{S}_s^0(1, \chi, \mathcal{O}_L)$ be an $\mathcal{O}_L$-adic Siegel eigen cusp form.

Assumption IX.38. *Let*

- $\mathcal{O}_L$ *is an extension of* $\Lambda' = \mathcal{O}_L[[S]]$

- *for a Zariski dense subset of* $\phi \in \mathcal{X}_L$, $\phi(\mathbf{F})$ *is such that* $\varphi_{\phi(\mathbf{F})}$ *generates an irreducible cuspidal representation for which multiplicity one holds*

- $S =$ *semi-integral matrix such that* $d(S) =$ *discriminant of an imaginary quadratic field* $\mathcal{K}$ *in which* p *splits.*

- ξ *is an unramified idele class character of* $\mathcal{K}$ *of finite order with values in* $\mathcal{O}_L$

We want to associate a p-adic L-function to these objects. To do this we also need to assume that there exists τ_0 a finite order Hecke character such that $\tau_0 \mid_{\mathbf{A}_{\mathbf{Q}}^\times} = \chi$ and τ_0 is unramified away from p. But this can be seen to be always satisfied: χ is an even character. Hence $\chi = \omega^{2a}$, where ω is the Teichmuller character. So we can take $\tau_0 = \omega^a \circ \mathrm{Nm}_{\mathcal{K}/\mathbf{Q}}$.

Let $\psi_0 = \tau_0 \xi$. Then associated to the datum $(\mathcal{F}, \tau_0, \psi_0)$ we have a $\mathcal{O}_L$-adic Klingen Eisenstein series

$$\mathbf{E}_{\mathcal{F}} = \{C_B \in \mathcal{O}_L \mid B \in \mathcal{N}_2\}$$

as in theorem (IX.36). Let

$$\mathbf{E}'_{\mathcal{F}} = \{A_B = \sum_{B' \text{such that } \mathrm{tr}(B'Z) = \mathrm{tr}(BZ), B' \in \mathcal{N}_2} C_{B'} \mid B \in \text{ symmetric } 2 \times 2 \text{ matrices } \}.$$

Then $\mathbf{E}'_{\mathcal{F}}$ is an $\mathcal{O}_L$-adic Siegel modular form in $\mathbf{M}_s(1, \chi, \mathcal{O}_L)$ and $\phi(\mathbf{E}'_{\mathcal{F}})$ is the restriction to $\mathbf{H}_2$ of $\phi(\mathbf{E}_{\mathcal{F}})$.

Let $\ell_{\mathbf{F}}$ be the projection associated to $\mathbf{F}$ defined in (9.14). Then by the mutliplicity one assumption on $\mathbf{F}$ in (IX.38) and theorem (IX.30) we have

$$\ell_{\mathbf{F}}(e\mathbf{E}'_{\mathcal{F}}) = \mathbf{c}\mathbf{F} \tag{9.28}$$

where e is the ordinary projection. For κ large enough on specialization we get

$$\phi(\ell_{\mathbf{F}}(e\mathbf{E}'_{\mathcal{F}})) = \phi(\mathbf{c})\phi(\mathbf{F}). \tag{9.29}$$

By (9.15), $\phi(\ell_{\mathbf{F}}(e\mathbf{E}'_{\mathcal{F}})) = \phi(H_{\mathbf{F}})1_{\phi(\mathbf{F})}(\phi(e\mathbf{E}'_{\mathcal{F}}))$ hence

$$\phi(\mathbf{c})\phi(\mathbf{F}) = \phi(H_{\mathbf{F}})1_{\phi(\mathbf{F})}(e\phi(\mathbf{E}'_{\mathcal{F}})) \tag{9.30}$$

By the inner product relation (IX.30) for Siegel modular forms we have

$$\phi(\mathbf{c}) = \phi(H_{\mathbf{F}})\frac{\langle\langle \phi(\mathbf{E}'_{\mathcal{F}}), \phi(\mathbf{F})^{\rho} \mid \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle}{\langle\langle \phi(\mathbf{F}), \phi(\mathbf{F})^{\rho} \mid \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle} \tag{9.31}$$

Now $\phi(\mathbf{E}'_{\mathcal{F}})$ is the Klingen Eisenstein series in Furusawa up to the section at p and a factor of the period $\langle \phi(\mathcal{F}), \phi(\mathcal{F})^{\rho} \mid \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle$. From the classical interpretation of the global integral as in section (9.6) it follows that $\langle\langle \phi(\mathbf{E}'_{\mathcal{F}}), \phi(\mathbf{F})^{\rho} \mid \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle\rangle$ is the zeta integral up to the period $\langle \phi(\mathcal{F}), \phi(\mathcal{F})^{\rho} \mid \left(\begin{smallmatrix} & -1 \\ p^r & \end{smallmatrix}\right)\rangle$. Hence $\mathbf{c}$ gives us a p-adic L-function with the required interpolation property so on putting together the information at the infinite place in (8.7) we have

Theorem IX.39. *Let $\mathcal{F}$(resp.$\mathbf{F}$) be an ordinary family $\mathcal{O}_L$-adic elliptic eigenform of tame level 1 and character χ(resp. an ordinary $\mathcal{O}_L$ adic Siegel modular form of tame level 1 and character χ). Suppose $\mathbf{F}$ satisfies the multiplicity one hypothesis. Let S be a symmetric semi-integral matrix such that $\det(S) > 0$ is a fundamental discriminant. Let ξ be an unramified Hecke character of $\mathcal{K} = \mathbf{Q}(\sqrt{-\det(S)})$ of finite order. There exists $\mathcal{L} \in L$ such that if $\kappa >> 0$ and $\phi : \mathcal{O}_L \to \mathbf{Q}_p$ is a $\mathbf{Z}_p$-homomorphism such that $\phi(1+T) = \zeta(1+p)^{\kappa}$ for ζ a p^{r-1} root of unity, $r \geq 1$*

then

$$\phi(\mathcal{L}) = a_p(\phi)\frac{L^{\{p\}}(\phi(\mathbf{F})^\rho \times \phi(\mathcal{F}), \kappa)}{\left\langle \phi(\mathcal{F}), \phi(\mathcal{F})^\rho \mid_\kappa \left({}_{p^r} {}^{-1} \right) \right\rangle \left\langle \left\langle \phi(\mathbf{F}), \phi(\mathbf{F})^\rho \mid_\kappa \left({}_{p^r} {}^{-1} \right) \right\rangle \right\rangle} \tilde{B}_{S,\xi,\phi(\mathbf{F})}(1_4)$$

where $a_p(\phi)$ *is some normalizing factor depending on* $\phi(\mathbf{F})$ *and* $\phi(\mathcal{F})$ *and* $\tilde{B}_{S,\xi,\phi(\mathbf{F})}(1_4)$ *is the value at* 1_4 *of the Bessel model of* $\phi(F)$ *associated to* S *and* ξ.

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