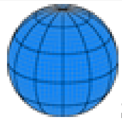

Spherical Measures without Spherical Trigonometry

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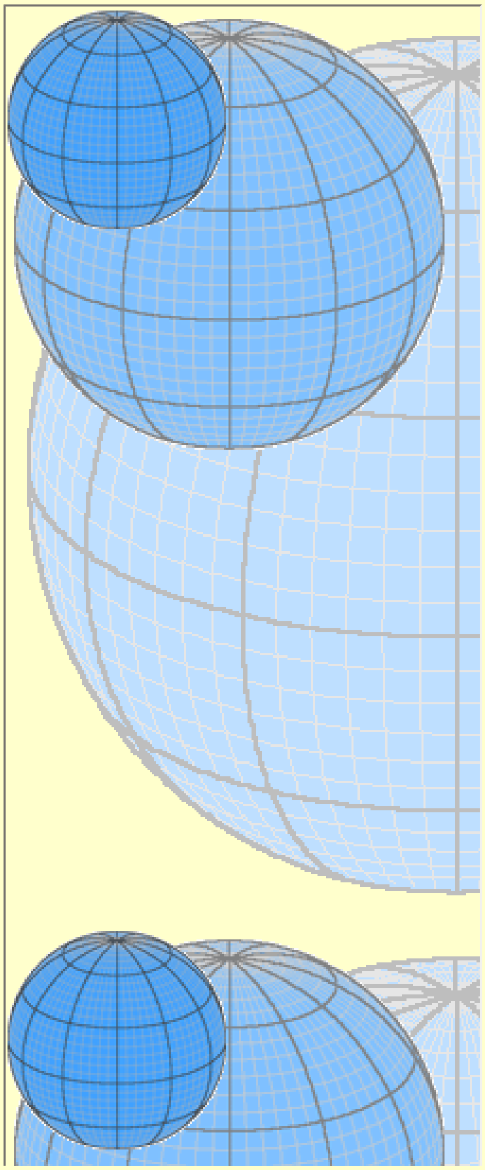

A common misconception suggests that it is necessary to understand spherical trigonometry in order to calculate properties of a spherical surface. That this is false can be demonstrated by deriving the formula for distance and direction on a sphere without any knowledge of spherical trigonometry. However, some knowledge of the properties of a sphere is required.

Suppose one wants to know the great circle distance between two locations given by their names in latitude and longitude. Call these P_1 (ϕ_1, λ_1) and P_2 (ϕ_2, λ_2).

Using plane trigonometry one can compute the location of these points in a three dimensional rectangular (Euclidean) coordinate system, centered at the origin of the sphere. This yields the locations as vectors in space as V_1 (X_1, Y_1, Z_1) and V_2 (X_2, Y_2, Z_2). The straight line distance between these two points is given by the well known formula: $D_{12} = [(X_1-X_2)^2 + (Y_1-Y_2)^2 + (Z_1-Z_2)^2]^{1/2}$. This is the chord distance between the two points, penetrating the sphere. The chord distance between the two points can be considered to be one side of a triangle, with origin at the center of a circle, whose other two sides have a length equal to the radius of the sphere. Take this radius to be one unit, and call it R. Divide the triangle into two equal parts, by constructing the perpendicular to the chord from the center. Now we know two sides of a right triangle, and the length of the third side (from the center to the chord) can be calculated. From this the angle at the center can be calculated, using any of the trigonometric formulae relating points in a right triangle. Twice this angle is the angle that spans the distance between the two points, that is, it is the angular separation of the points on the sphere.

In more detail, use standard spherical coordinates:

$X_1 = R \cos \phi_1 \cos \lambda_1$

$Y_1 = R \cos \phi_1 \sin \lambda_1$

$Z_1 = R \sin \phi_1$

$X_2 = R \cos \phi_2 \cos \lambda_2$

$Y_2 = R \cos \phi_2 \sin \lambda_2$

$Z_2 = R \sin \phi_2$

The length, L, of the line from the center of the circle to the perpendicular to the chord is, using C as the length of the chord, $L = [R^2 - (C/2)^2]^{1/2}$. Now, from elementary trigonometry, $C/2L = \tan \alpha$, $L/R = \cos \alpha$, $C/2R = \sin \alpha$. Doubling α gives the angle subtended at the center of the circle, and at the center of the sphere. Use the simplest (the third) of these relations to calculate the distance on a sphere.

Example:
Near Santa Barbara, CA: $\phi_1 = 34°$, $\lambda_1 = -120°$.

Near Zürich, CH: $\phi_2 = 47.5°$, $\lambda_2 = 8.5°$

$X_1 = R \cos \phi_1 \cos \lambda_1 = -0.414529$

$Y_1 = R \cos \phi_1 \sin \lambda_1= -0.717978$

$Z_1 = R \sin \phi_1=0.559193$

$X_2 = R \cos \phi_2 \cos \lambda_2 = 0.668169$

$Y_2 = R \cos \phi_2 \sin \lambda_2= 0.099859$

$Z_2 = R \sin \phi_2= 0.737277$

$C = [(-.414529 - .668169)^2 + (-.717978 - .099859)^2 + (.559193 - .737277)^2]^{1/2}=$ 1.368491, L = 0.729252 but is not really needed, and continue to use R = 1. Then, from $\alpha = \sin^{-1}(C/2)$, α = 43.176 degrees of arc. Twice this value gives 86.35 degrees for the length of the arc on the circle. Now take R= 6378 kilometers to approximate the radius of the sphere. Then the distance between Santa Barbara and Zürich, on the spherical assumption, is $\pi*\alpha*R/90$ or 9612 km.

No spherical trigonometry has been used. But if one is familiar with vector algebra the result can be obtained more immediately, namely by calculating the angle between the two vectors directly as
$2\alpha = \cos^{-1}(V_1 \cdot V_2/|V_1||V_2|)$.

For the direction of the second point from the first recall that directions are measured between great circles. In this case we need the direction of the great circle from Santa Barbara to Zürich with respect to the reference direction from North. The plane passing through Santa Barbara, the North Pole, and the origin, cuts the sphere along the North-South great circle and has equation -.717978 X + .414529 Y = 0, from the three-point equation of a plane. The plane spanning the region between Santa Barbara, Zürich, and the origin intersects the sphere along the connecting great circle and has equation -.5851891 X + .6792581 Y + .4383362 Z = 0. The angle between these two planes is 31.9 degrees East of North. (see C. Oakley, 1954, “Analytic Geometry”, Barnes & Noble, pages 179, 185). The area of a polygon on a sphere is equal to $R^2\{\pi(n-2)+\Sigma\beta_i\}$ where the sum is taken over the n interior angles. Again, no spherical trigonometry has

been used.

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