

Supersonic Flow onto a Solid Wedge

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Abstract

We consider the problem of two-dimensional supersonic flow onto a solid wedge, or equivalently in a concave corner formed by two solid walls. For mild corners, there are two possible steady state solutions, one with a strong and one with a weak shock emanating from the corner. The weak shock is observed in supersonic flights. A longstanding natural conjecture is that the strong shock is unstable in some sense.

We resolve this issue by showing that a sharp wedge will eventually produce weak shocks at the tip when accelerated to a supersonic speed. More precisely, we prove that for upstream state as initial data in the entire domain, the time-dependent solution is self-similar, with a weak shock at the tip of the wedge. We construct analytic solutions for self-similar potential flow, both isothermal and isentropic with arbitrary $\gamma \geq 1$.

In the process of constructing the self-similar solution, we develop a large number of theoretical tools for these elliptic regions. These tools allow us to establish large-data results rather than a small perturbation. We show that the wave pattern persists as long as the weak shock is supersonic-supersonic; when this is no longer true, numerics show a physical change of behavior. In addition, we obtain rather detailed information about the elliptic region, including analyticity as well as bounds for velocity components and shock tangents.

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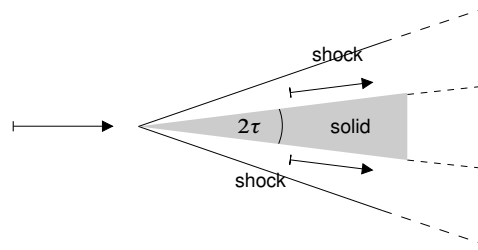


FIGURE 1.1. Reflect either part of Figure 1.2 across the upstream wall. There are four combinations; each is a steady solutions to supersonic flow onto a solid wedge. The one with two weak shocks is the one observed in nature and numerics.

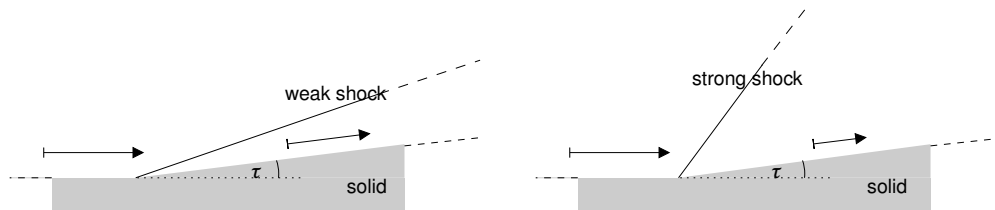


FIGURE 1.2. Two steady solutions of supersonic flow along an infinite wall with a corner.

1 Introduction

1.1 Background

Gas flow onto a solid wedge, like forward edges of airplane wings or engine inlets, is a fundamental problem for aerodynamics (see Figure 1.1). An equivalent problem is flow in a convex corner of an otherwise straight wall (see Figure 1.2). For *supersonic* flow and sufficiently small τ , it is well-known that this problem has steady solutions with a straight shock emanating from the corner, separating two constant-state regions (“upstream” and “downstream”). The shock is the consequence of compression of the gas by the downstream wall.

A longstanding open and puzzling problem is that, for τ close to 0 (corresponding to a sharp wedge or mild corner), there are *two* possible steady solutions of the corner flow, one with a strong and one with a (comparatively) weak shock (see Figure 1.2). Both shocks satisfy the entropy condition.¹ However, only the *weak* shocks are observed in nature. To quote [8]: “The question arises which of the two actually occurs. It has frequently been stated that the strong one is unstable and that, therefore, only the weak one could occur. A convincing proof of this instability has apparently never been given.”

The goal of the present paper is to understand this.

¹ There is a third shock that violates the entropy condition.

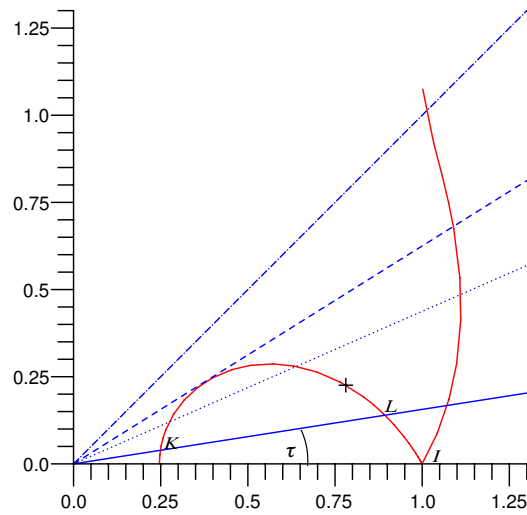


FIGURE 1.3. Curve: downstream velocities for all possible shock normals. The left branch corresponds to compression shocks, the right branch to unphysical expansion shocks. The + is exactly sonic, points left of it are subsonic, right of it supersonic. Solid line: there are three solutions; strong shock K , weak shock L , unphysical. Dotted line: still three solutions, but the weak shock is supersonic-subsonic now. Dashed line: critical angle; weak and strong shock coincide. Dashed-dotted line: τ above critical, no physical solution.

For many purposes, in particular for many questions concerning flow around airplane wings, viscosity, heat conduction, and kinetic effects can be neglected. It is natural to consider inviscid models, such as the full or isentropic compressible Euler equations or compressible potential flow. The appropriate boundary condition at solid surfaces is the *slip condition*: the gas velocity is tangential.

In each model the shock and its upstream and downstream states satisfy the *Rankine-Hugoniot relations*, a system of nonlinear algebraic equations. These relations determine the *shock polar*: the curve of downstream velocities that result when varying the shock normal while holding the shock steady and the upstream velocity and density fixed (see Figure 1.3).

In Figure 1.3 the upstream velocity is $(1, 0)$, labeled I . Possible downstream velocities are intersections of the shock polar with the ray at a counterclockwise angle τ from the positive horizontal axis. Obviously for small τ there are three intersection points. The leftmost intersection, called K , corresponds to the strong shock. The rightmost is an unphysical expansion shock that need not be considered. The middle point, called L , is the weak shock. (The shock normals are parallel

to the difference between downstream velocity (intersection point) and upstream velocity $(1, 0)$.)

For $\tau \downarrow 0$, the strong shock approaches a *normal shock*, whereas the weak shock vanishes (L approaches I).

There is a *critical angle* $\tau = \tau_*$ where L and K coincide; for larger τ no steady entropy-satisfying shock can be attached to the wedge tip or corner.

The $+$ on the shock polar indicates a downstream state that is exactly *sonic* (Mach number $M = 1$). Polar points to the left of it are *subsonic* ($M < 1$); polar points to the right of it are *supersonic*. In numerical experiments, the weak shock detaches from the corner/wedge tip when its downstream changes from supersonic to subsonic (e.g., by increasing τ).

In [13, 14] we have reported on numerical experiments: to our surprise and somewhat contrary to the aforementioned conjecture, both the strong and weak shocks are time-asymptotically stable under large, compactly supported perturbations. Instead, the strong shock is unstable under (generic) perturbations of the downstream state at infinity; depending on the perturbation, either the weak shock appears or the shock detaches from the wedge tip/corner entirely. It may be possible to obtain a strong shock in very special cases, for example by placing a perfectly feedback-controlled nozzle somewhere downstream.

The weak shock is stable under *both* kinds of perturbation.

While various conjectures and empirical observations have been made regarding weak vs. strong shock, previously no *mathematical* arguments for either were known. To obtain one, we devise an “unbiased” test: at time $t = 0$, fill the entire domain with upstream data; check which shock appears for $t > 0$.

In numerics, the *weak* shock appears spontaneously (see Figure 1.4). Motivated by this, we construct an *analytical* solution.

An equivalent experiment is to accelerate a solid wedge in motionless air instantaneously to supersonic speed. More generally, if a finite wedge is accelerated from rest at time 0 to a fixed supersonic speed at time $\epsilon \ll L/c$ (L wedge length), we may expect the solution to be a good approximation for times t in the scale $\epsilon \ll t \ll L/c$.

1.2 Numerical Results

Figure 1.4 shows the flow pattern (density) solving our test problem, for some positive time. Here $\gamma = \frac{7}{5}$, $M_I = 2.94$, and $\tau = 10^\circ$. \vec{v}_I is horizontal from left to right. Darker shades represent higher density ρ ; the black curves are constant- ρ level sets.

A straight shock emanates from the wedge tip/wall corner. Calculation shows that it is the weak shock. There is another straight shock on the right, parallel to the downstream wall. Below each straight shock lies a constant region. Both shocks are connected by a curved shock, with a nontrivial (elliptic) region below.

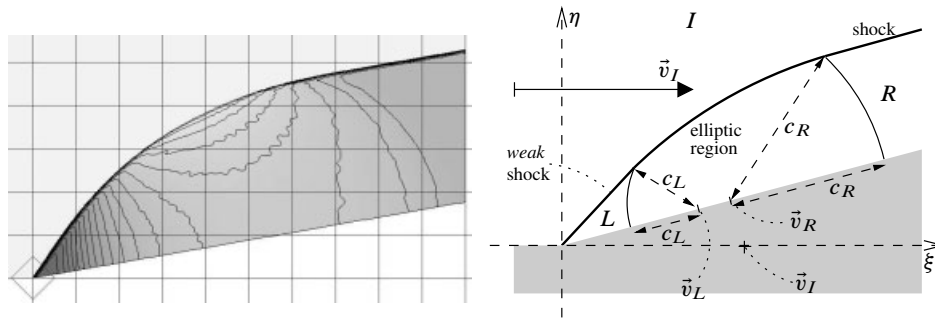


FIGURE 1.4. Left: numerical solution of the wedge flow problem. Right: Structure. Three trivial hyperbolic regions I , L , and R are separated by straight shocks. They enclose a nontrivial elliptic region bounded by a curved shock and two parabolic circle arcs centered in \vec{v}_L and \vec{v}_R with radius c_L and c_R , respectively. Density and velocity are functions of $\xi = x/t$ and $\eta = y/t$ only.

The flow pattern is self-similar: density and velocity are constant along rays $x = \xi t, y = \eta t$ for fixed ξ, η . This can be visualized as t being a “zoom” parameter, with $t = 0$ corresponding to “infinitely far away” and $t \uparrow \infty$ to “infinitely close to the origin” (wedge tip/wall corner). In particular, the flow structure is the same for all times.

Figure 1.5 shows the elliptic region in more detail. In the top picture density is shown (again darker shades are higher density); its minimum in the elliptic region is attained at the shock (Proposition 3.13 will show that the minimum is a “pseudonormal” point). In the middle picture the velocity tangential to the downstream wall is shown (darker shades are smaller, more negative velocity; the right shock is not visible because that velocity component is continuous across it). The bottom picture shows velocity normal to the downstream wall. Again an extremum can be observed at the curved portion of the shock.

The diamond in the center of the bottom domain boundaries indicates the origin in self-similar coordinates, where we use standard coordinates (Figure 4.1). The velocities in Figure 1.5 are relative to it.

It should be emphasized that numerical computations only suggest the structure of the solution. For instance, it is not clear that the constant states L and R extend to the pseudosonic circles P_L and P_R . Although, in one dimension with viscosity, some techniques can convert a numerical solution with sufficiently small residual into an existence proof for an exact solution (see, e.g., [18]), only partial results are available in multiple dimensions without viscosity (see [11, 19, 20]); it is not clear whether a general result is even true (see [10]).

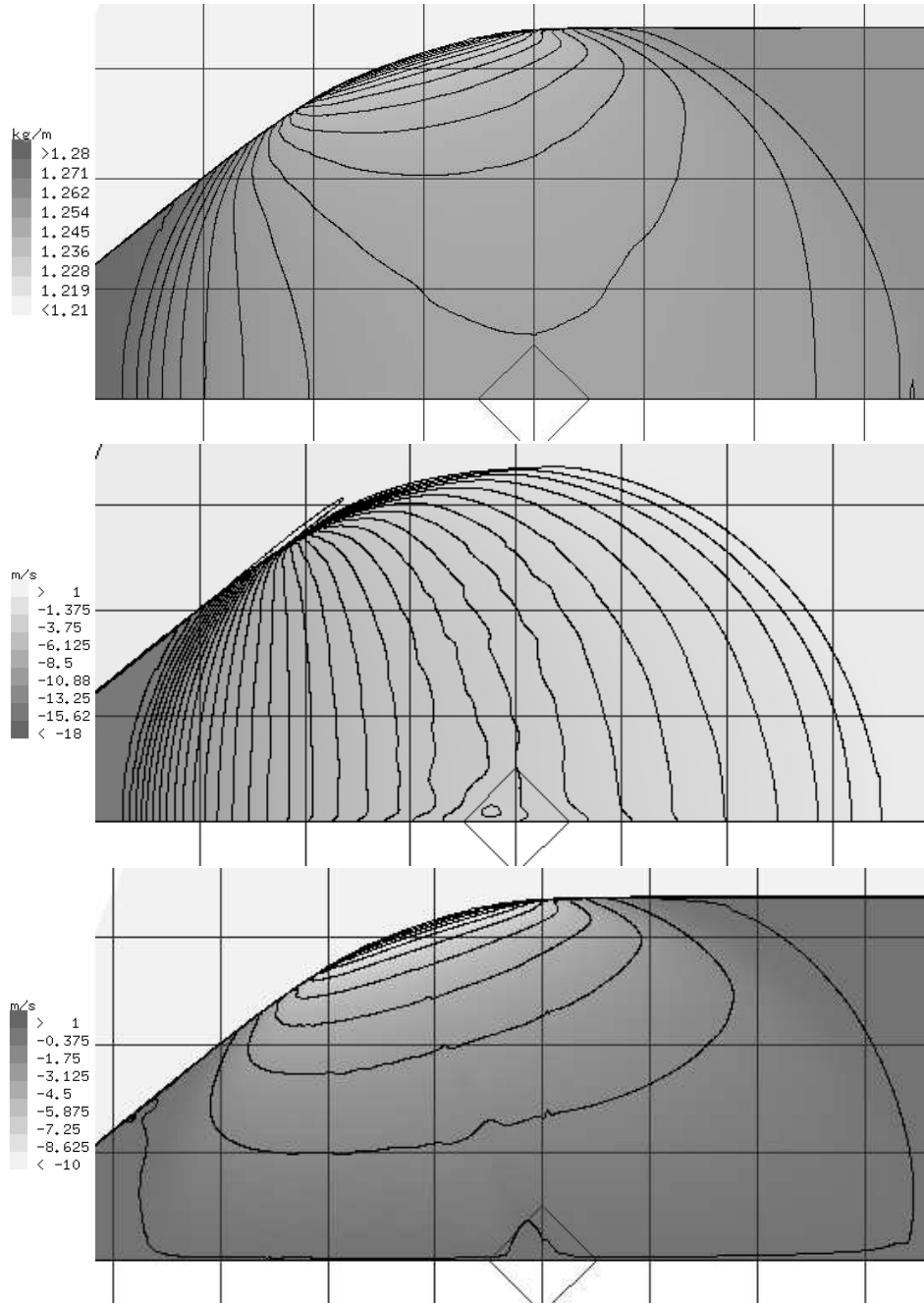


FIGURE 1.5. Elliptic region. From top to bottom: ρ , \vec{v} tangential, and \vec{v} normal to the wedge surface. Corner/wedge tip is the intersection of the left shock and bottom domain boundary (left outside diagrams). The diamond indicates the origin in Figure 4.1 coordinates; velocities are relative to it.

1.3 Main Result

To obtain a mathematical argument, we construct the self-similar solution *exactly* rather than numerically. We use compressible potential flow as a model:

THEOREM 1.1 *Let $\tau \in (0, \frac{\pi}{2})$, $M_I, \rho_I, c_I \in (0, \infty)$, and $\gamma \in [1, \infty)$; set $\vec{v}_I := (M_I c_I, 0)$. Define the wedge*

$$W := \{(x, y) \in \mathbb{R}^2 : y < x \tan \tau\} \quad \text{and} \quad \Omega := (0, \infty) \times \mathbb{C}W.$$

Assume the following conditions are satisfied:

- (1) *Unsteady potential flow with γ -law pressure admits a steady straight shock with upstream data ρ_I and \vec{v}_I and downstream velocity \vec{v}_L and sound speed c_L so that²*

$$\angle(\vec{v}_I, \vec{v}_L) = \tau.$$

- (2) *The shock is supersonic-supersonic:*

$$M_L := \frac{|\vec{v}_L|}{c_L} > 1.$$

- (3) *Of the two intersection points of the shock with the circle $\partial B_{c_L}(\vec{v}_L)$, let $\vec{\xi}_L^*$ be the one closer to the corner (origin). Let the R shock be the unique shock parallel to \vec{v}_L , with upstream data ρ_I and \vec{v}_I , downstream sound speed c_R , and downstream velocity \vec{v}_R parallel to \vec{v}_L as well. Of the two intersection points with $\partial B_{c_R}(\vec{v}_R)$, let $\vec{\xi}_R^*$ be the one farther from the corner (see Figure 1.6). We require that*

$$(1.1) \quad \{\text{Line segment from } \vec{\xi}_L^* \text{ to } \vec{\xi}_R^*\} \cap \overline{B_{c_I}(\vec{v}_I)} = \emptyset.$$

Then there is a weak solution (see Remark 1.2) $\phi \in C^{0,1}(\overline{\Omega})$ of

$$(1.2) \quad \text{unsteady potential flow} \quad \text{in } \Omega,$$

$$(1.3) \quad \nabla \phi \cdot \vec{n} = 0 \quad \text{on } \partial W,$$

$$(1.4) \quad \rho = \rho_I, \quad \nabla \phi = \vec{v}_I \quad \text{for } t = 0.$$

In addition to existence, detailed results about the structure of the weak solution can be obtained (see Remark 4.32). At this point we emphasize only that each solution consists, in some neighborhood of the origin, of the *weak* shock separating two constant-state regions.

Remark 1.2. See Section 2.1 for an introduction to and a precise definition of potential flow. By “weak solution” we mean that

$$(1.5) \quad \nabla \phi(0, \vec{x}) = \vec{v}_I \quad \text{for a.e. } \vec{x} \in \mathbb{C}W$$

² \angle is the counterclockwise angle from first to second vector, ranging from 0 to 2π .

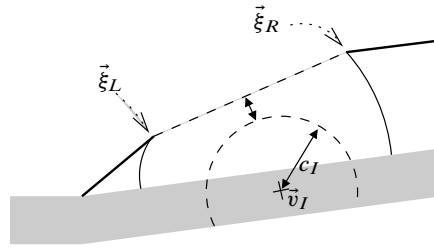


FIGURE 1.6. Solutions are constructed for all cases that satisfy the condition (1.1): the dashed circle and line must be separated. A shock has positive strength if and only if it does not touch the circle.

and

$$\int_{\Omega} \rho \vartheta_t + \rho \nabla \phi \cdot \nabla \vartheta \, d\vec{x} \, dt + \int_{\mathbb{C}W} \vartheta(0, x, y) \rho_I \, d\vec{x} = 0$$

for all test functions $\vartheta \in C_c^\infty(\overline{\Omega})$. (For $\phi \in C^{0,1}(\overline{\Omega})$, the velocity $\nabla \phi$ is a.e. well-defined on $\{0\} \times \mathbb{C}W$, but ϕ_t and hence ρ may not be well-defined.)

Remark 1.3. As Remark 4.31 shows, there is a large set of tip shocks and parameters that satisfy the conditions of Theorem 1.1.

The first and second condition are physically necessary, not technical limitations. If the first is violated (for $M_I < 1$ or large τ), there is no straight steady physical shock attached to the corner at all. If either of them is violated, numerical experiments show a flow pattern with a shock *detached* from the corner, moving upstream (left).

The third condition is technical. It is needed in some cases to prove that the shock does not vanish (which is never observed in numerics); none of the other estimates requires this condition. We expect that the condition will be removed with some additional analysis.

It should be emphasized that the theorem and its proof are global in nature: large parameter changes are possible.

Remark 1.4. Incidentally, we also solve the problem for *asymmetric* wedges, as long as both sides allow a supersonic-supersonic weak shock and as long as (1.1) is satisfied on both sides.

1.4 Related Work

Courant and Friedrichs [8, secs. 117, 122, 123] explain in detail shock polars and the corner flow problem. Despite its age, its discussion of weak vs. strong shock is still a good reflection of the state of prior research. Another useful reference is [16].

Čanić, Keyfitz, and Kim [1] consider the classical problem of regular reflection of a shock off a symmetric wedge; this problem, like ours, has a self-similar solution. They consider the unsteady transonic small disturbance equation as a model.

Zheng [32] studies the same problem for the pressure gradient system. The monographs [21, 31] compute various self-similar flows numerically and present some analysis and simplified models.

Chen and Feldman [4] prove existence of small perturbations of a plane shock in steady potential flow.

Chen [6] constructs steady solutions for three-dimensional cones rather than two-dimensional wedges. Lien and Liu [25] discuss stability of three-dimensional flow past a perturbed cone; Chen and Li [7] show existence and linear stability in the case of the isentropic Euler equations. Chen, Zhang, and Zhu [5] study existence and stability of supersonic flows onto perturbed wedges with attached shocks; the introduction gives a detailed discussion of previous work.

So far the only other paper that proves global existence of some nontrivial time-dependent solution of potential flow is by Chen and Feldman [2]: they construct exact solutions for regular reflection, assuming sufficiently blunt wedges.

1.5 Overview

In Section 2 we give an introduction to unsteady potential flow. We derive self-similar potential flow, discuss its shock conditions, and analyze the properties of shocks in detail.

In Section 3 we discuss a collection of maximum principles for elliptic regions of self-similar potential flow. Some of these identify circumstances in which certain quantities, such as density, can or cannot have maxima or minima in the interior. Other results discuss local extrema at solid (slip condition) walls and finally shocks with a constant-state hyperbolic region on the other side.

Since the hyperbolic regions are trivial (see Figure 1.4), the heart of the problem is the construction of the elliptic region. This is accomplished in Section 4. Readers interested in more overview should go to Section 4.2, where all proof steps are surveyed.

Crucial ingredients are the maximum principles from Section 3, combined with ODE-type arguments at the parabolic arcs in Sections 4.6 to 4.10, and techniques to control shock location and normals (Sections 4.11 and 4.12). Section 4.16 combines the elliptic region with its hyperbolic counterparts to construct the full-flow pattern. The remaining sections are standard but delicate applications of nonlinear elliptic theory. Some literature results, such as regularity in corners and at free boundaries, need to be adapted, which is done in the appendix.

1.6 Notation

For the most part we use standard notation. Subscripts and superscripts may denote tensor indices, partial derivatives, or powers, depending on the context.

$\angle(\vec{x}, \vec{y})$ is the *counterclockwise* angle from \vec{x} to \vec{y} . For $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$,

$$\vec{x}^\perp := (-x_2, x_1)$$

(counterclockwise rotation by 90°),

$$\vec{x} \times \vec{y} := x_1 y_2 - x_2 y_1.$$

\vec{x}^2 is the rank-one matrix $\vec{x}\vec{x}^\top$ whereas $|\vec{x}|^2$ is the norm. Correspondingly, $\nabla^2 = \nabla\nabla^\top$ is the Hessian (*not* the Laplacian).

Normals \vec{n} are outer normals to a domain, except on the shock S (defined later) where they are downstream, so usually inner. Tangents \vec{t} are always defined as $\vec{t} := \vec{n}^\perp$.

2 Potential Flow

2.1 Unsteady Potential Flow

We consider the isentropic Euler equations of compressible gas dynamics in d space dimensions:

$$(2.1) \quad \rho_t + \nabla \cdot (\rho\vec{v}) = 0,$$

$$(2.2) \quad (\rho\vec{v})_t + \sum_{i=1}^d (\rho v^i \vec{v})_{x^i} + \nabla(p(\rho)) = 0.$$

Hereafter, ∇ denotes the gradient with respect to either the space coordinates $\vec{x} = (x^1, x^2, \dots, x^d)$ or the similarity coordinates $t^{-1}\vec{x}$. Vector $\vec{v} = (v^1, v^2, \dots, v^d)$ is the velocity of the gas, ρ the density, and $p(\rho)$ pressure. In this article we consider only polytropic pressure laws (γ -laws) with $\gamma \geq 1$:

$$(2.3) \quad p(\rho) = \frac{c_0^2 \rho_0}{\gamma} \left(\frac{\rho}{\rho_0} \right)^\gamma$$

(here c_0 is the sound speed at density ρ_0). Many subsequent results extend with little or no change to $\gamma < 1$ or to general pressure laws, but in special cases some steps require more work or break down entirely. To keep the presentation simple, we don't strive for generality with respect to pressure laws.

For smooth solutions, substituting (2.1) into (2.2) yields the simpler form

$$(2.4) \quad \vec{v}_t + \vec{v} \cdot \nabla^\top \vec{v} + \nabla(\pi(\rho)) = 0.$$

Here π is defined as

$$\pi(\rho) = c_0^2 \cdot \begin{cases} \frac{(\rho/\rho_0)^{\gamma-1} - 1}{\gamma-1}, & \gamma > 1, \\ \log(\rho/\rho_0), & \gamma = 1. \end{cases}$$

This π is C^∞ in $\rho \in (0, \infty)$ and $\gamma \in [1, \infty)$ and has the property

$$\pi_\rho = \frac{p_\rho}{\rho}.$$

If we assume *irrotationality*

$$v_j^i = v_i^j$$

(where $i, j = 1, 2, \dots, d$), then the Euler equations are reduced to potential flow:

$$\vec{v} = \nabla_{\vec{x}}\phi$$

for some scalar *potential* function ϕ .³ For smooth flows, substituting this into (2.4) yields, for $i = 1, 2, \dots, d$,

$$0 = \phi_{it} + \nabla\phi_i \cdot \nabla\phi + \pi(\rho)_i = \left(\phi_t + \frac{|\nabla\phi|^2}{2} + \pi(\rho) \right)_i.$$

Thus, for some constant A ,

$$(2.5) \quad \rho = \pi^{-1}\left(A - \phi_t - \frac{|\nabla\phi|^2}{2}\right).$$

Substituting this into (2.1) yields a single second-order quasi-linear hyperbolic equation, the *potential flow* equation, for a scalar field ϕ :

$$(2.6) \quad (\rho(\phi_t, |\nabla\phi|))_t + \nabla \cdot (\rho(\phi_t, |\nabla\phi|)\nabla\phi) = 0.$$

Henceforth we omit the arguments of ρ . Moreover, we eliminate A with the substitution

$$A \leftarrow 0, \quad \phi(t, \vec{x}) \leftarrow \phi(t, \vec{x}) - tA$$

(so that $\phi_t \leftarrow \phi_t - A$). Hence we use

$$(2.7) \quad \rho = \pi^{-1}\left(-\phi_t - \frac{1}{2}|\nabla\phi|^2\right)$$

from now on.

Using $c^2 = p_\rho$ and

$$(2.8) \quad (\pi^{-1})' = (\pi_\rho)^{-1} = \left(\frac{p_\rho}{\rho}\right)^{-1} = \frac{\rho}{c^2},$$

the equation can also be written in nondivergence form:

$$(2.9) \quad \phi_{tt} + 2\nabla\phi_t \cdot \nabla\phi + \sum_{i,j=1}^d \phi_i\phi_j\phi_{ij} - c^2\Delta\phi = 0.$$

Equation (2.9) is hyperbolic (as long as $c > 0$). For polytropic pressure law, the local sound speed c is given by

$$(2.10) \quad c^2 = c_0^2 + (\gamma - 1)\left(-\phi_t - \frac{1}{2}|\nabla\phi|^2\right).$$

³We consider simply connected domains; otherwise ϕ could be multivalued.

2.2 Self-Similar Potential Flow

Our initial data is self-similar: it is constant along rays emanating from $\vec{x} = (0, 0)$. Our domain $\mathbb{C}W$ is self-similar, too: it is a union of rays emanating from $(t, x, y) = (0, 0, 0)$. In any such situation it is expected—and confirmed by numerical results—that the solution is self-similar as well, i.e., that ρ and \vec{v} are constant along rays $\vec{x} = t\vec{\xi}$ emanating from the origin. Self-similarity corresponds to the ansatz

$$(2.11) \quad \phi(t, \vec{x}) := t\psi(\vec{\xi}), \quad \vec{\xi} := t^{-1}\vec{x}.$$

Clearly, $\phi \in C^{0,1}(\Omega)$ if and only if $\psi \in C^{0,1}(\mathbb{C}W)$. This choice yields

$$\begin{aligned} \vec{v}(t, \vec{x}) &= \nabla\phi(t, \vec{x}) = \nabla\psi(t^{-1}\vec{x}), \\ \rho(t, \vec{x}) &= \pi^{-1}\left(-\phi_t - \frac{1}{2}|\nabla\phi|^2\right) = \pi^{-1}\left(-\psi + \vec{\xi} \cdot \nabla\psi - \frac{1}{2}|\nabla\psi|^2\right). \end{aligned}$$

The expression for ρ can be made more pleasant (and independent of $\vec{\xi}$) by using

$$\chi(\vec{\xi}) := \psi(\vec{\xi}) - \frac{1}{2}|\vec{\xi}|^2;$$

this yields

$$(2.12) \quad \rho = \pi^{-1}\left(-\chi - \frac{1}{2}|\nabla\chi|^2\right).$$

We call $\nabla\chi = \nabla\psi - \vec{\xi}$ the *pseudovelocity*.

Equation (2.6) then reduces to

$$(2.13) \quad \nabla \cdot (\rho\nabla\chi) + 2\rho = 0$$

(or $+d\rho$ in d dimensions), which holds in a distributional sense. For smooth solutions we obtain the nondivergence form

$$(2.14) \quad \begin{aligned} (c^2I - \nabla\chi\nabla\chi^T) : \nabla^2\chi &= (c^2 - \chi_{\xi\xi}^2)\chi_{\xi\xi\xi} - 2\chi_{\xi\xi}\chi_{\eta\xi} + (c^2 - \chi_{\eta\xi}^2)\chi_{\eta\xi} \\ &= |\nabla\chi|^2 - 2c^2. \end{aligned}$$

Another convenient form is

$$(2.15) \quad \begin{aligned} (c^2I - \nabla\chi\nabla\chi^T) : \nabla^2\psi &= (c^2 - \chi_{\xi\xi}^2)\psi_{\xi\xi\xi} - 2\chi_{\xi\xi}\chi_{\eta\xi}\psi_{\xi\eta} + (c^2 - \chi_{\eta\xi}^2)\psi_{\eta\xi\xi} \\ &= 0. \end{aligned}$$

Here, (2.10) for polytropic pressure law yields

$$(2.16) \quad c^2 = c_0^2 + (\gamma - 1)\left(-\chi - \frac{1}{2}|\nabla\chi|^2\right).$$

Remark 2.1. Equation (2.13) inherits a number of symmetries from (2.1)–(2.2):

- (1) It is invariant under rotation.
- (2) It is invariant under reflection.

- (3) It is invariant under translation in $\vec{\xi}$, which is not as trivial as translation in \vec{x} : it corresponds to the Galilean transformation $\vec{v} \leftarrow \vec{v} + \vec{v}_0$ and $\vec{x} \leftarrow \vec{x} - \vec{v}_0 t$ (with constant $\vec{v}_0 \in \mathbb{R}^d$) in (t, \vec{x}) coordinates. This is sometimes called *change of inertial frame*.

Equation (2.14) is a PDE of mixed type. The type is determined by the (*local*) *pseudo-Mach number*

$$(2.17) \quad L := \frac{|\nabla\chi|}{c},$$

with $0 \leq L < 1$ for elliptic (pseudosubsonic), $L = 1$ for parabolic (pseudosonic), and $L > 1$ for hyperbolic (pseudosupersonic) regions.

The pseudo-Mach number L can be interpreted in a way analogous to the Mach number M : consider a *steady* solution of the *unsteady* potential flow equation. Loosely speaking, in an $M < 1$ (subsonic) region, a small localized disturbance will be propagated in all directions, whereas in an $M > 1$ region it is propagated only in the *Mach cone*. $L < 1$ and $L > 1$ are analogous, except that we study the propagation of disturbances in the unsteady potential flow equation written in $(t, \vec{x}/t)$ -coordinates, rather than (t, \vec{x}) .

There is no strong relation between $M < 1$ and $L < 1$: consider two constant-state (hence steady and self-similar) flows, one with zero velocity ($M = 0$ constant), the other with supersonic velocity \vec{v} ($M > 1$ constant). Each flow has $L = 0$ in the point $\vec{\xi} = 0$ and $\vec{\xi} = \vec{v}$, respectively, and $L \uparrow \infty$ as $\vec{\xi} \uparrow \infty$, so there are examples for each of the four cases $M, L < 1$, $M < 1 < L$, $L < 1 < M$, and $1 < L, M$.

While velocity \vec{v} is motion relative to space coordinates \vec{x} , pseudovelocity

$$\vec{z} := \nabla\chi$$

is motion relative to similarity coordinates $\vec{\xi}$ at time $t = 1$.

The simplest class of solutions of (2.14) are the *constant-state solutions*: ψ affine in $\vec{\xi}$; hence \vec{v} , ρ , and c are constant. They are elliptic in a circle centered in $\vec{\xi} = \vec{v}$ with radius c , parabolic on the boundary of that circle and hyperbolic outside.

CONVENTION 2.2 If we study a function called (e.g.) $\tilde{\chi}$, then $\tilde{\psi}$, $\tilde{\rho}$, \tilde{L} , etc., will refer to the quantities computed from it as ψ , ρ , and L are computed from χ (e.g., $\tilde{\psi} = \tilde{\chi} + \frac{1}{2}|\tilde{\xi}|^2$). We will tacitly use this notation from now on.

2.3 Shock Conditions

Consider a ball U and a simple smooth curve S so that $U = U^u \cup S \cup U^d$ where U^u and U^d are open and connected, and S, U^u , and U^d disjoint. Consider $\chi : U \rightarrow \mathbb{R}$ so that $\chi = \chi^{u,d}$ in $U^{u,d}$ where $\chi^{u,d} \in C^2(\overline{U^{u,d}})$.

χ is a weak solution of (2.13) if and only if it is a strong solution in each point of U_- and U_+ and if it satisfies the following conditions in each point of S :

$$(2.18) \quad \chi^u = \chi^d,$$

$$(2.19) \quad \vec{n} \cdot (\rho^u \nabla \chi^u - \rho^d \nabla \chi^d) = 0.$$

Here \vec{n} is a normal to S .

Equations (2.18) and (2.19) are the *Rankine-Hugoniot* conditions for self-similar potential flow shocks. They do not depend on $\vec{\xi}$ or on the shock speed explicitly; these quantities are hidden by the use of χ rather than ψ . The Rankine-Hugoniot conditions are derived in the same way as those for the full Euler equations (see [15, sec. 3.4.1]).

Note that (2.18) is equivalent to

$$(2.20) \quad \psi^u = \psi^d.$$

Taking the tangential derivative of (2.18) and (2.20), respectively, yields

$$(2.21) \quad \frac{\partial \chi^u}{\partial t} = \frac{\partial \chi^d}{\partial t},$$

$$(2.22) \quad \frac{\partial \psi^u}{\partial t} = \frac{\partial \psi^d}{\partial t}.$$

The shock relations imply that the tangential velocity is continuous across shocks.

Define $(z_u^x, z_u^y) := \vec{z}_u := \nabla \chi^u$ and $(v_u^x, v_u^y) := \vec{v}_u := \nabla \psi^u$. Abbreviate $z_u^t := \vec{z}_u \cdot \vec{t}$ and $z_u^n := \vec{z}_u \cdot \vec{n}$, and do the same for v instead of z . The same notation is used with subscript d instead of u . We can restate the shock relations as

$$(2.23) \quad \rho_u z_u^n = \rho_d z_d^n,$$

$$(2.24) \quad z_u^t = z_d^t.$$

Using the last relation, we often write z^t without distinction.

The *shock speed* is $\sigma = \vec{\xi} \cdot \vec{n}$, where $\vec{\xi}$ is any point on the shock. A shock is *steady* in a point if its tangent passes through the origin. We can restate (2.23) as

$$\rho_u v_u^n - \rho_d v_d^n = \sigma(\rho_u - \rho_d),$$

which is a more familiar form.

We focus on $\rho_u, \rho_d > 0$ from now on, which will be the case in all circumstances. If $\rho_u = \rho_d$ in a point, we say the shock *vanishes*; in this case $z_d^n = z_u^n$ in that point, by (2.24). In all other cases z_d^n and z_u^n must have the same sign by (2.24); we fix \vec{n} so that $z_d^n, z_u^n > 0$. This means the normal points *downstream*. The shock is *admissible* if and only if $\rho_u \leq \rho_d$, which is equivalent to $z_u^n \geq z_d^n$.

A shock is called *pseudonormal* in a point $\vec{\xi}$ if $z^t = 0$ there. For $\vec{\xi} = 0$, this means that the shock is *normal* ($v^t = 0$), but for $\vec{\xi} \neq 0$ normal and pseudonormal are not always equivalent.

It is good to keep in mind that for a *straight* shock, ρ_d and \vec{v}_d are constant if ρ_u and \vec{v}_u are. Obviously \vec{z}_d may vary in this case.

Remark 2.3. The Rankine-Hugoniot conditions for the original isentropic Euler equations cannot be used for potential flow: even if the flow on one side of a shock is irrotational, the flow on the other side has nonzero vorticity for curved shocks.

2.4 Pseudonormal Shocks

Here we study the consequences and the solutions of the shock relations in a pseudonormal point. We state all results for pseudovelocities \vec{z} and for pseudo-Mach numbers L because moving shocks are ubiquitous in this article. For better intuition the reader may bear in mind that $\vec{z} = \vec{v}$ and $L = M$ if the shock is steady, i.e., passes through the origin. In fact, by Remark 2.1, weak and entropy solutions of self-similar potential flow are invariant under translation and rotation, so we may always consider translating the shock so that it becomes steady, which does not change \vec{z} , ρ , and L , whereas \vec{v} is changed only by a constant vector. Hence the behavior of arbitrary shocks is entirely determined by those of steady shocks.

$L^n := z^n/c$ and $L^t := z^t/c$ will be referred to as the normal and tangential pseudo-Mach numbers, respectively. Equations (2.18) and (2.12) imply

$$(2.25) \quad \rho_d = \pi^{-1} \left(\pi(\rho_u) + \frac{|z_u|^2}{2} - \frac{|z_d|^2}{2} \right).$$

It is apparent that (2.25) reduces to

$$(2.26) \quad \rho_d = \pi^{-1} \left(\pi(\rho_u) + \frac{(z_u^n)^2}{2} - \frac{(z_d^n)^2}{2} \right).$$

Combined with (2.24) there is a direct relation between normal velocities, independent of the tangential velocities.

For polytropic pressure laws we may use a rather convenient simplification: there is an explicit relation connecting L_u^n and L_d^n , independent of ρ_u and \vec{z}_u .

LEMMA 2.4 *For $L_u^n, L_d^n > 0$, (2.26) and (2.23) are equivalent to*

$$(2.27) \quad g(L_u^n) = g(L_d^n),$$

$$(2.28) \quad g(x) = \begin{cases} \left(x^2 + \frac{2}{\gamma-1}\right) x^{\frac{2(1-\gamma)}{\gamma+1}}, & \gamma > 1, \\ x^2 - 2 \log x, & \gamma = 1. \end{cases}$$

Note that

$$(2.29) \quad \frac{\partial g}{\partial x} = \frac{4}{\gamma + 1} (x - x^{-1}) x^{-2\frac{\gamma-1}{\gamma+1}},$$

$$(2.30) \quad \frac{\partial^2 g}{\partial x^2} = \frac{4}{(\gamma + 1)^2} ((3 - \gamma) + (3\gamma - 1)x^{-2}) x^{-2\frac{\gamma-1}{\gamma+1}}.$$

$$(2.31) \quad g(x) \sim \begin{cases} x^{-2\frac{\gamma-1}{\gamma+1}}, & \gamma > 1, \\ -\log x, & \gamma = 1, \end{cases} \quad \text{as } x \downarrow 0,$$

$$(2.32) \quad g(x) \sim x^{\frac{4}{\gamma+1}} \quad \text{as } x \uparrow \infty,$$

Moreover,

$$(2.33) \quad \frac{c_u}{c_d} = \left(\frac{L_d^n}{L_u^n} \right)^{\frac{\gamma-1}{\gamma+1}}$$

as well as

$$(2.34) \quad \frac{\rho_u}{\rho_d} = \left(\frac{L_d^n}{L_u^n} \right)^{\frac{2}{\gamma+1}}.$$

PROOF: Equation (2.23) can be written

$$(2.35) \quad \begin{aligned} \rho_u L_u^n c_u &= \rho_d L_d^n c_d \\ \Rightarrow (\rho_u)^{(\gamma+1)/2} L_u^n &= \rho_d^{(\gamma+1)/2} L_d^n; \end{aligned}$$

this yields (2.34), which yields (2.33).

Consider $\gamma > 1$, so that $c^2 = c_0^2 + (\gamma - 1)\pi(\rho)$. Equation (2.26) can be transformed to

$$\begin{aligned} \frac{(z_u^n)^2}{2} + \frac{c_u^2}{\gamma - 1} &= \frac{(z_d^n)^2}{2} + \frac{c_d^2}{\gamma - 1} \\ \Leftrightarrow \left((L_u^n)^2 + \frac{2}{\gamma - 1} \right) c_u^2 &= \left((L_d^n)^2 + \frac{2}{\gamma - 1} \right) c_d^2. \end{aligned}$$

Substitute (2.33) to obtain (2.27).

For $\gamma = 1$, (2.26) is

$$\begin{aligned} c_0^2 \log \frac{\rho_d}{\rho_0} + \frac{(z_d^n)^2}{2} &= c_0^2 \log \frac{\rho_u}{\rho_0} + \frac{(z_u^n)^2}{2} \\ \Rightarrow \frac{\rho_d}{\rho_u} &= \exp \frac{(L_u^n)^2 - (L_d^n)^2}{2}; \end{aligned}$$

using (2.34) we obtain (2.27). □

PROPOSITION 2.5 *There exists an analytic, strictly decreasing function $L_d^n = L_d^n(L_u^n)$, which is its own inverse, so that the shock relation (2.27) is solved for all $L_u^n \in (0, \infty)$. For $L_u^n \neq 1$, the only other solution of (2.27) is the trivial one: $L_d^n = L_u^n$. For $L_u^n = 1$, both coincide.*

$$(2.36) \quad L_d^n \begin{cases} \uparrow \infty, & L_u^n \downarrow 0, \\ = 1, & L_u^n = 1, \\ \sim (L_u^n)^{\frac{-2}{\gamma-1}}, & L_u^n \uparrow \infty, \gamma > 1, \\ \sim \exp\left(-\frac{(L_u^n)^2}{2}\right), & L_u^n \uparrow \infty, \gamma = 1. \end{cases}$$

The resulting shock is admissible if and only if $L_u^n \geq 1$.

$$(2.37) \quad \frac{\partial L_d^n}{\partial L_u^n} = \frac{L_u^n - 1/L_u^n}{L_d^n - 1/L_d^n} \left(\frac{L_u^n}{L_d^n} \right)^{-2\frac{\gamma-1}{\gamma+1}} < 0,$$

$$(2.38) \quad \frac{\partial L_d^n}{\partial L_u^n} \Big|_{L_u^n=L_d^n(=1)} = -1,$$

For $L_u^n > 1 > L_d^n$, we have $\rho_d > \rho_u$ and $z_d^n < z_u^n$. ρ_d and c_d are strictly increasing in L_u^n for ρ_u fixed. For $\gamma > 1$, $c_d > c_u$ as well, and c_d is strictly increasing in L_u^n for ρ_u fixed.

PROOF: Assume $L_u^n > 1$. From (2.29) it is obvious that $\frac{\partial g}{\partial x}(x) > 0$ for $x > 1$. Hence $g(L_u^n) > g(1)$; moreover, (2.27) cannot have more than one solution L_d^n in $[1, \infty]$ for fixed $L_u^n \geq 1$; in fact, $L_d^n = L_u^n$ is the unique solution. For $x < 1$, $\frac{\partial g}{\partial x}(x) < 0$, so (2.27) cannot have more than one solution L_d^n in $(0, 1)$. It must have one, though, because $g(0+) = +\infty > g(L_u^n) > g(1)$.

For $L_u^n < 1$, the existence of a nontrivial solution $L_d^n \in (1, \infty)$ is obtained from the previous case by analogous arguments. For $L_u^n = 1$, the sign of $\partial g/\partial x$ rules out any other solutions. Since (2.27) is symmetric in L_u^n and L_d^n , it is obvious that $L_u^n \mapsto L_d^n$ is its own inverse.

The trivial solution branch $L_d^n = L_u^n$ is obviously smooth; by the implicit function theorem the nontrivial branch is analytic away from $L_u^n = 1$. In $L_u^n = 1$ there is a degeneracy that has to be analyzed by inspecting the Hessian of $h(L_u^n, L_d^n) := g(L_u^n) - g(L_d^n) = 0$:

$$A := \begin{bmatrix} \frac{\partial^2 h}{(\partial L_u^n)^2} & \frac{\partial^2 h}{\partial L_u^n \partial L_d^n} \\ \frac{\partial^2 h}{\partial L_u^n \partial L_d^n} & \frac{\partial^2 h}{(\partial L_d^n)^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2}(L_u^n) & 0 \\ 0 & -\frac{\partial^2 g}{\partial x^2}(L_d^n) \end{bmatrix} \Big|_{L_u^n=L_d^n=1} \stackrel{(2.30)}{=} \begin{bmatrix} \frac{8}{\gamma+1} & 0 \\ 0 & \frac{-8}{\gamma+1} \end{bmatrix}.$$

A is an invertible indefinite matrix; the solutions of $\vec{w}^\top A \vec{w} = 0$ are $\vec{w} = (1, 1)$ and $\vec{w} = (1, -1)$. The classical Morse lemma (see e.g., [29, lemma 12.19]) shows that in a small neighborhood of $(L_u^n, L_d^n) = (1, 1)$, the solution of (2.27) form two analytic curves that intersect in $(1, 1)$ with tangents $(1, 1)$ (trivial branch) and $(1, -1)$ (nontrivial branch). The latter yields (2.38).

The remainder of (2.36) follows from the asymptotics of g (see (2.31) and (2.32)).

From (2.34) and (2.37) we see that $\rho_d > \rho_u$ for $L_d^n < 1 < L_u^n$ and that ρ_d is an increasing function of L_u^n for ρ_u held fixed. Clearly the same applies to c_d if $\gamma > 1$, and (because of (2.23)) $z_d^n < z_u^n$. This means the shock is admissible for $L_u^n > 1$. □

Remark 2.6. In the remaining arguments we always assume that the shock is admissible (or vanishing) and ignore the branch $L_d^n = L_u^n$.

PROPOSITION 2.7 *If we hold c_u and ρ_u fixed,*

$$(2.39) \quad \frac{z_u^n}{z_d^n} \cdot \frac{\partial z_d^n}{\partial z_u^n} < \frac{\gamma - 1}{\gamma + 1} < 1.$$

In particular,

$$(2.40) \quad \frac{\partial z_d^n}{\partial z_u^n} < \frac{\gamma - 1}{\gamma + 1}.$$

Therefore

$$(2.41) \quad \frac{z_d^n}{c_u} - 1 < \frac{\gamma - 1}{\gamma + 1} \left(\frac{z_u^n}{c_u} - 1 \right).$$

Remark 2.8. Inequality (2.40) is not very tight (we can show < 0 for $\gamma < 3$), but sufficient for our purposes.

PROOF: We use (2.33):

$$\frac{\partial c_d}{\partial L_u^n} = c_u \frac{\partial}{\partial L_u^n} \left(\frac{L_u^n}{L_d^n} \right)^{\frac{\gamma-1}{\gamma+1}} = \frac{c_u}{L_d^n} \left(\frac{L_u^n}{L_d^n} \right)^{\frac{-2}{\gamma+1}} \frac{\gamma-1}{\gamma+1} \left(1 - \frac{L_u^n}{L_d^n} \frac{\partial L_d^n}{\partial L_u^n} \right)$$

so

$$\begin{aligned} \frac{\partial z_d^n}{\partial z_u^n} &= c_u^{-1} \frac{\partial z_d^n}{\partial L_u^n} = \frac{L_d^n}{c_u} \frac{\partial c_d}{\partial L_u^n} + \frac{c_d}{c_u} \frac{\partial L_d^n}{\partial L_u^n} \\ &\stackrel{(2.33)}{=} \left(\frac{L_u^n}{L_d^n} \right)^{\frac{-2}{\gamma+1}} \left(\frac{2}{\gamma+1} \cdot \frac{L_u^n}{L_d^n} \cdot \frac{\partial L_d^n}{\partial L_u^n} + \frac{\gamma-1}{\gamma+1} \right). \end{aligned}$$

Then

$$\begin{aligned} \frac{z_u^n}{z_d^n} \frac{\partial z_d^n}{\partial z_u^n} &= \frac{\rho_d}{\rho_u} \left(\frac{L_u^n}{L_d^n} \right)^{\frac{-2}{\gamma+1}} \left(\frac{2}{\gamma+1} \cdot \frac{L_u^n}{L_d^n} \cdot \frac{\partial L_d^n}{\partial L_u^n} + \frac{\gamma-1}{\gamma+1} \right) \\ &\stackrel{(2.34)}{=} \underbrace{\frac{2}{\gamma+1}}_{>0} \cdot \underbrace{\frac{L_u^n}{L_d^n}}_{>0} \cdot \underbrace{\frac{\partial L_d^n}{\partial L_u^n}}_{<0} + \frac{\gamma-1}{\gamma+1} < \frac{\gamma-1}{\gamma+1}. \end{aligned}$$

Integrating (2.40) from $z_d^n = z_u^n = c_u$ for a vanishing shock, we obtain (2.41). \square

PROPOSITION 2.9 *Consider a shock with velocity $\sigma := \vec{\xi} \cdot \vec{n}$. Our convention $z_u^n > 0$ requires $\sigma < v_u^n$. Vary σ while holding \vec{n} and \vec{v}_u fixed. Then:*

$$(2.42) \quad \frac{\partial v_d^n}{\partial \sigma} \geq 1 - \frac{\partial z_d^n}{\partial z_u^n} > 0,$$

$$(2.43) \quad \frac{\partial \rho_d}{\partial \sigma} < 0.$$

PROOF: For moving shocks, $v_d^n = z_d^n + \sigma$ and $z_u^n = v_u^n - \sigma$, so

$$\frac{\partial v_d^n}{\partial \sigma} = 1 + \frac{\partial z_d^n}{\partial \sigma} = 1 - \frac{\partial z_d^n}{\partial z_u^n} \stackrel{(2.40)}{>} 1 - \frac{\gamma - 1}{\gamma + 1} = \frac{2}{\gamma + 1} > 0$$

and

$$\frac{\partial \rho_d}{\partial \sigma} = -\frac{\partial \rho_d}{\partial z_u^n} = -\frac{\partial \rho_d}{\partial L_u^n} c_u^{-1} \stackrel{(2.34)}{<} \stackrel{(2.37)}{0}.$$

□

2.5 Shock Polar

Here we prove only the results needed for our purposes.

PROPOSITION 2.10 Consider a fixed point on a shock with upstream density ρ_u and pseudovelocity \vec{z}_u held fixed while we vary the normal. Define $\beta := \angle(\vec{z}_u, \vec{n})$. ρ_d is strictly decreasing in $|\beta|$, whereas L_d and $|\vec{z}_d|$ are strictly increasing. c_d is strictly decreasing for $\gamma > 1$, and constant otherwise. Moreover,

$$(2.44) \quad (\partial_\beta \vec{v}_d) \cdot \vec{n} = (\partial_\beta \vec{z}_d) \cdot \vec{n} = z^t \left(\frac{\partial z_d^n}{\partial z_u^n} - 1 \right),$$

$$(2.45) \quad (\partial_\beta \vec{v}_d) \cdot \vec{t} = (\partial_\beta \vec{z}_d) \cdot \vec{t} = z_d^n - z_u^n.$$

If $\vec{z}_u = (z_u^x, 0)$ with $z_u^x > 0$, then z_d^x is increasing in $|\beta|$.

PROOF:

$$(2.46) \quad \begin{aligned} \partial_\beta \vec{z}_d &= \partial_\beta (z_d^n \vec{n} + z^t \vec{t}) = \frac{\partial z_d^n}{\partial z_u^n} \partial_\beta z_u^n \vec{n} + z_d^n \vec{t} - z^t \vec{n} + \partial_\beta z^t \vec{t} \\ &= z^t \left(\frac{\partial z_d^n}{\partial z_u^n} - 1 \right) \vec{n} + (z_d^n - z_u^n) \vec{t}. \end{aligned}$$

This is (2.44) and (2.45), by using that $\vec{\xi}$ is fixed. For $\beta > 0$, $z^t = -|\vec{z}_u| \sin \beta$ is strictly decreasing (and negative).

$$\begin{aligned} \frac{1}{2} \partial_\beta (|\vec{z}_d|^2) &= \frac{1}{2} \partial_\beta ((z_d^n)^2 + (z^t)^2) = z_d^n \frac{\partial z_d^n}{\partial z_u^n} \partial_\beta z_u^n + z^t \partial_\beta z^t \\ &= z^t \left(z_d^n \frac{\partial z_d^n}{\partial z_u^n} - \underbrace{z_u^n}_{\geq z_u^n} \right) \geq \underbrace{z^t}_{<0} \underbrace{z_d^n}_{>0} \left(\underbrace{\frac{\partial z_d^n}{\partial z_u^n} - 1}_{\leq \frac{\gamma-1}{\gamma+1}} \right) > 0 \end{aligned}$$

by (2.40), so $|\vec{z}_d|$ is strictly increasing. Then by (2.25)

$$\rho_d = \pi^{-1} \left(\pi(\rho_u) + \frac{1}{2} (|\vec{z}_u|^2 - |\vec{z}_d|^2) \right)$$

is strictly decreasing, as is c_d (except constant for $\gamma = 1$), so $L_d = |\vec{z}_d|/c_d$ is increasing.

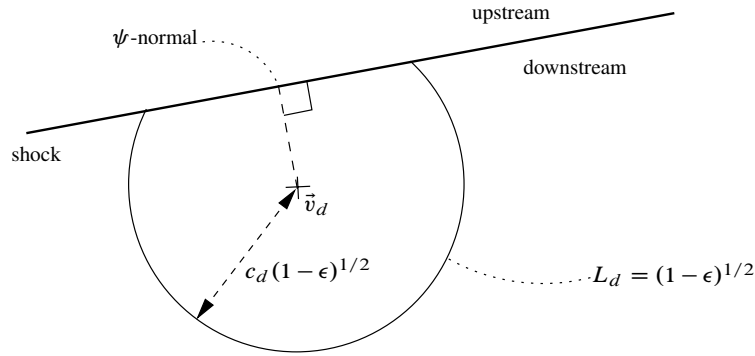


FIGURE 2.1. L -values downstream of a shock.

For $\beta > 0$, (2.46) yields

$$\partial_\beta(z_d^x) = \underbrace{z^t}_{<0} \underbrace{\left(\frac{\partial z_d^n}{\partial z_u^n} - 1\right)}_{<0} \underbrace{\cos \beta}_{>0} - \underbrace{(z_d^n - z_u^n)}_{<0} \underbrace{\sin \beta}_{>0} > 0$$

by (2.40).

$\beta < 0$ is analogous by symmetry. □

2.6 Shock-Parabolic Corners with Fixed Vertical Downstream Velocity

The corners of our elliptic region (see Figure 4.1) are points on shocks where $L_d = 1$ (or $L_d = \sqrt{1 - \epsilon}$ if regularized); on the other hand, $v_d^y = 0$ (if the x -direction is tangential to the wall). We study such shocks in detail.

PROPOSITION 2.11 Consider a straight shock (see Figure 2.1) passing through a point $\vec{\xi}$. The shock is pseudonormal in the point $\vec{\xi}_M = \vec{\xi} + \vec{t} \cdot (\vec{v}_d - \vec{\xi})\vec{t}$ and pseudo-oblique in every other point. Vector $\vec{\xi}_M$ is the closest point on the shock both to \vec{v}_u and to \vec{v}_d . The circle with center \vec{v}_u and radius c_u does not intersect the shock. The downstream flow has $L < \sqrt{1 - \epsilon}$ inside the circle with radius $c_d\sqrt{1 - \epsilon}$ and center \vec{v}_d , $L = \sqrt{1 - \epsilon}$ on the circle and $L > \sqrt{1 - \epsilon}$ outside. If $L_d^n < \sqrt{1 - \epsilon}$, then the circle intersects the shock in the two points, $\vec{\xi}_M \pm c_d\sqrt{1 - \epsilon - (L_d^n)^2} \cdot \vec{t}$.

PROOF: This is straightforward to check. $\vec{\xi}_M \cdot \vec{t} = \vec{v}_d \cdot \vec{t} = \vec{v}_u \cdot \vec{t}$, so $|\vec{v}_u - \vec{\xi}_M| = |(\vec{v}_u - \vec{\xi}_M) \cdot \vec{n}| = |z_u^n| > c_u$. □

PROPOSITION 2.12 Consider a straight shock with $v_u^x = 0$, $v_u^y < 0$, and downstream normal $\vec{n} = (\sin \beta, -\cos \beta)$ through $\vec{\xi} = (0, \eta)$ (see Figure 2.2). For every

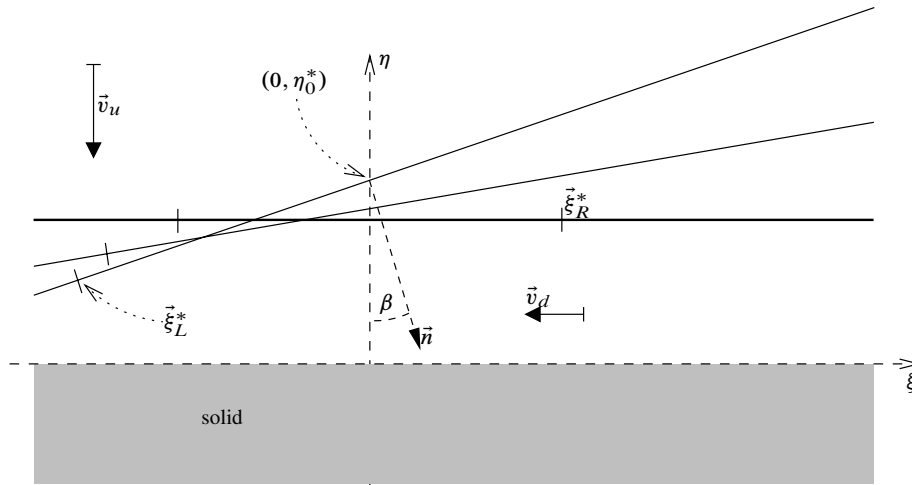


FIGURE 2.2. Changing shock normals while keeping \vec{v}_d horizontal. Vector $\vec{\xi}_L^*$ is the left $L_d^2 = 1 - \epsilon$ point for each shock.

$\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ there is a unique $\eta = \eta_0^* \in \mathbb{R}$ so that $v_d^y = 0$. η_0^* and the corresponding downstream data are analytic functions of β ; η_0^* is strictly increasing in $|\beta|$.

For the shock passing through $(0, \eta_0^*)$, let $\vec{\xi}_L^*$ and $\vec{\xi}_R^*$ be the two points with $L_d = \sqrt{1 - \epsilon}$, as given by Proposition 2.11. These points are analytic functions of β . L_u^n , ρ_d , and z_u^n are increasing functions of β ; v_d^x and L_d^n are decreasing functions of β . For $\beta \in [0, \frac{\pi}{2})$, η_L^* is a strictly decreasing function of β with range $(\underline{\eta}_L^*, \bar{\eta}_0^*]$, where $\bar{\eta}_0^*$ is the η_0^* for $\beta = 0$, and $\underline{\eta}_L^*$ is some negative constant.

PROOF: First, regard everything as a function of β and η , with \vec{v}_u and ρ_u held fixed and the shock held as passing through $(0, \eta)$. In $(0, \eta)$:

$$(2.47) \quad z_u^n = z_u^x n^x + z_u^y n^y = \underbrace{(v_u^x)}_{=0} - \underbrace{\xi}_{=0} \sin \beta - (v_u^y - \eta) \cos \beta = (\eta - v_u^y) \cos \beta,$$

$$\partial_\eta z_u^n = \cos \beta,$$

$$(2.48) \quad v_d^y = v_u^y + (v_d^y - v_u^y) = v_u^y + (v_d^n - v_u^n) n^y = v_u^y + (z_d^n - z_u^n) n^y = v_u^y + (z_u^n - z_d^n) \cos \beta,$$

⁴ All of these are independent of the location along the (straight) shock.

$$\begin{aligned} \partial_\eta v_d^y &= \left(1 - \frac{\partial z_d^n}{\partial z_u^n}\right) \partial_\eta z_u^n \cos \beta \\ &= \left(1 - \frac{\partial z_d^n}{\partial z_u^n}\right) \cos^2 \beta \stackrel{(2.40)}{>} \frac{2}{\gamma + 1} \cos^2 \beta > 0. \end{aligned}$$

v_d^y is an increasing function of η , so for fixed β there can be at most one η with $v_d^y = 0$. For $\eta = v_u^y + c_u / \cos \beta$ we have $L_u^n = 1$ in the point $(0, \eta)$, so $v_d^y = v_u^y < 0$ there; on the other hand (2.48) has uniformly lower-bounded right-hand side (for every fixed $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$), so $v_d^y \uparrow +\infty$ if we take $\eta \uparrow +\infty$. Therefore there is exactly one solution $\eta = \eta_0^*$ for each β .

Now consider $\beta \in (0, \frac{\pi}{2})$ first (so $z^t = (v_u^y - \eta) \sin \beta < 0$); the case $\beta < 0$ is symmetric.

$$\begin{aligned} \partial_\beta v_d^y &= \left(1 - \frac{\partial z_d^n}{\partial z_u^n}\right) \partial_\beta z_u^n \cos \beta - (z_u^n - z_d^n) \sin \beta \\ &= \left(1 - \frac{\partial z_d^n}{\partial z_u^n}\right) z^t \cos \beta - (z_u^n - z_d^n) \sin \beta \\ \partial_\beta \eta_0^* &= -\frac{\partial_\beta v_d^y}{\partial_\eta v_d^y} = \underbrace{\frac{1}{\cos \beta}}_{>0} \left(\underbrace{-z^t}_{>0} + \underbrace{\frac{z_u^n - z_d^n}{1 - \partial z_d^n / \partial z_u^n}}_{>0} \underbrace{\tan \beta}_{>0} \right) > 0. \end{aligned}$$

For the remainder of the proof, fix $\eta = \eta_0^*$ and consider everything a function of β . Consider $\beta \in (0, \frac{\pi}{2})$ increasing (the other case is symmetric). Then

$$\begin{aligned} z_u^n &= (\vec{v}_u - \vec{\xi}) \cdot \vec{n}, \\ \partial_\beta z_u^n &= \underbrace{(\vec{v}_u - \vec{\xi}) \cdot \vec{t}}_{=z^t} - \partial_\beta \eta_0^* \underbrace{n^y}_{=-\cos \beta} = \frac{z_u^n - z_d^n}{1 - \partial z_d^n / \partial z_u^n} \tan \beta > 0. \end{aligned}$$

Therefore L_u^n and ρ_d are also strictly increasing, whereas L_d^n is strictly decreasing.

$$\begin{aligned} v_d^x &= \underbrace{v_u^x}_{=0} + v_d^x - v_u^x \stackrel{v_d^t=v_u^t}{=} (v_d^n - v_u^n) n^x \stackrel{v_d^t=v_u^t}{=} \underbrace{(v_d^y - v_u^y)}_{=0} n^x / n^y \\ &= \underbrace{v_u^y}_{<0} \tan \beta < 0, \\ \partial_\beta v_d^x &= \frac{v_u^y}{\cos^2 \beta} < 0. \end{aligned}$$

Obviously v_d^x is strictly decreasing.

$$\begin{aligned} \eta_M &= v_u^y - z_u^n n^y = v_u^y + z_u^n \cos \beta, \\ \partial_\beta \eta_M &= -z_u^n \sin \beta + \partial_\beta z_u^n \cos \beta = \left(-z_u^n + \frac{z_u^n - z_d^n}{1 - \partial z_d^n / \partial z_u^n} \right) \sin \beta \\ (2.49) \qquad &= \frac{z_u^n \partial z_d^n / \partial z_u^n - z_d^n}{1 - \partial z_d^n / \partial z_u^n} \sin \beta \stackrel{(2.39)}{<} \stackrel{(2.40)}{0}. \end{aligned}$$

So η_M is strictly decreasing.

$$\eta_L^* = \eta_M - c_d \sin \beta \sqrt{1 - \epsilon - (L_d^n)^2};$$

c_d is increasing and L_d^n strictly decreasing, so η_L^* is strictly decreasing. Moreover,

$$\begin{aligned} \eta_L^* &= \underbrace{v_d^y}_{=0} + z_d^n \cos \beta - c_d \sin \beta \sqrt{1 - \epsilon - (L_d^n)^2} \\ &= (L_d^n \cos \beta - \sin \beta \sqrt{1 - \epsilon - (L_d^n)^2}) c_d. \end{aligned}$$

L_d^n is $< 1 - \epsilon$ for $\beta = 0$ and decreasing in $\beta > 0$, so it is uniformly bounded above away from $1 - \epsilon$ as $\beta \uparrow \frac{\pi}{2}$. For β large enough the η_L^* expression is negative. Thus η_L^* covers an interval $(\underline{\eta}_L^*, \bar{\eta}_0^*]$ for some $\underline{\eta}_L^* < 0$. □

3 Maximum Principles

In this section we derive many a priori estimates for smooth elliptic regions and for smooth shocks separating elliptic and constant-state hyperbolic regions.

3.1 Common Techniques for Extremum Principles

LEMMA 3.1 *Let $m \geq 1$. If a_0, \dots, a_m define a positive semidefinite tensor, i.e.,*

$$\sum_{j=0}^m \binom{m}{j} a_j \xi^j \eta^{m-j} \geq 0 \quad \forall \xi, \eta \in \mathbb{R},$$

and if

$$(3.1) \qquad Aa_k + Ba_{k+1} + Ca_{k+2} = 0 \quad \forall k \in \{0, \dots, m-2\}$$

with constants $A, B, C \in \mathbb{R}$ so that $4AC > B^2$, then $a_0 = \dots = a_m = 0$.

PROOF: $4AC > B^2$ means that $A + Bz + Cz^2 = 0$ has two roots $z, \bar{z} \in \mathbb{C} - \mathbb{R}$. Then

$$a_k = \Re(az^k) \quad (k = 0, \dots, m)$$

where $a \in \mathbb{C}$ is some linear combination of a_0 and a_1 , so

$$\begin{aligned} 0 \leq \sum_{k=0}^m \binom{m}{k} \Re(a z^k) \xi^k \eta^{m-k} &= \Re \left(a \sum_{k=0}^m \binom{m}{k} z^k \xi^k \eta^{m-k} \right) \\ &= \Re(a(z\xi + \eta)^m) \end{aligned}$$

for all $\vec{\xi} \in \mathbb{R}^2$. Since $\Im z \neq 0$, $(\xi, \eta) \mapsto (z\xi + \eta)^m$ is onto \mathbb{C} . The inequality cannot be true unless $a = 0$, so $a_0 = \dots = a_m = 0$. \square

LEMMA 3.2 *Let $m \geq 3$, $\vec{b} = (b_1, b_2) \in \mathbb{R}^2 - \{0\}$, and $\vec{n} \in \mathbb{R}^2 - \{0\}$. If $a_0, \dots, a_m \in \mathbb{R}$ satisfy*

$$(3.2) \quad \sum_{k=0}^{m-1} \binom{m-1}{k} (b_1 a_k + b_2 a_{k+1}) \xi^k \eta^{m-1-k} \geq 0 \quad \forall \vec{\xi} \in \mathbb{R}^2, \vec{\xi} \cdot \vec{n} > 0,$$

as well as (3.1) with constants $A, B, C \in \mathbb{R}$ so that $4AC > B^2$, then

$$a_k = 0 \quad \forall k \in \{0, \dots, m\}.$$

PROOF: $4AC > B^2$ means that $A + Bz + Cz^2 = 0$ has two roots $z, \bar{z} \in \mathbb{C} - \mathbb{R}$. The general solution of (3.1) is

$$a_k = \Re(a z^k) \quad (k = 0, \dots, m),$$

where $a \in \mathbb{C}$ is some linear combination of a_0 and a_1 . Substitute this into (3.2):

$$\begin{aligned} 0 &\leq \sum_{k=0}^{m-1} \binom{m-1}{k} (b_1 \Re(a z^k) + b_2 \Re(a z^{k+1})) \xi^k \eta^{m-1-k} \\ &= \Re \left(a(b_1 + b_2 z) \sum_{k=0}^{m-1} \binom{m-1}{k} z^k \xi^k \eta^{m-1-k} \right) \\ (3.3) \quad &= \Re(a(b_1 + b_2 z)(z\xi + \eta)^{m-1}) \quad \forall \vec{\xi} \in \mathbb{R}^2, \vec{\xi} \cdot \vec{n} \geq 0. \end{aligned}$$

We may use $\vec{\xi} \cdot \vec{n} \geq 0$ by continuity, which defines a closed half-plane of \mathbb{R}^2 . $z \in \mathbb{C} - \mathbb{R}$, so the range of $(\xi, \eta) \mapsto \xi + z\eta$ is a closed half-plane of \mathbb{C} . The range of $(\xi, \eta) \mapsto (\xi + z\eta)^{m-1}$ is all of \mathbb{C} because $m - 1 \geq 2$. Moreover, $b_1 + b_2 z \neq 0$ because either $b_2 = 0$, and then $b_1 + b_2 z = b_1 \neq 0$, or $b_2 \neq 0$, and then $\Im(b_1 + b_2 z) = b_2 \Im(z) \neq 0$. Thus the range of $(\xi, \eta) \mapsto a(b_1 + b_2 z)(z\xi + \eta)^{m-1}$ is all of \mathbb{C} , contradicting (3.3), unless $a = 0$, which means $a_k = 0$ for all $k = 0, \dots, m$. \square

LEMMA 3.3 *Consider an open set U and a point $\vec{\xi}_0 \in \bar{U}$. Assume that there is a $\vec{n} \neq 0$ (quasi an inner normal) so that for every $\vec{\xi}$ with $\vec{n} \cdot \vec{\xi} > 0$, there is a $\delta > 0$ with*

$$\{\vec{\xi}_0 + t\vec{\xi} : t \in (0, \delta)\} \subset U.$$

Let ψ be an analytic solution of (2.15) in \bar{U} , so that $L < 1$, $\rho > 0$, and $D^2\psi = 0$ in $\vec{\xi}_0$. Then

$$f = f\left(\psi, \nabla\psi, \underbrace{\chi + \frac{1}{2}|\nabla\chi|^2}_a\right)$$

(with f a C^∞ function of its arguments) cannot have an extremum in $\vec{\xi}_0$ unless ψ is linear or

$$(3.4) \quad \frac{\partial f}{\partial(\nabla\psi)} + \frac{\partial f}{\partial a} \nabla\chi = 0 \quad \text{in } \vec{\xi}_0.$$

PROOF: We use dot notation (such as $\dot{=}$) to indicate relations holding only in $\vec{\xi}_0$. We show by complete induction over $k = 3, 4, \dots$ that $D^k\psi \dot{=} 0$.

Induction step ($3, \dots, k-1 \rightarrow k \geq 3$): for $j = 0, \dots, k-2$ take $\partial_1^j \partial_2^{k-2-j}$ of the equation. This yields

$$(3.5) \quad (c^2 I - \nabla\chi^2) : \nabla^2 \partial_1^j \partial_2^{k-2-j} \psi \dot{=} 0$$

because all other terms contain at least one component of $D^2\psi, \dots, D^{k-1}\psi$ as a factor, and hence vanish.

We may exploit that $D^j\chi = D^j\psi$ for $j \geq 3$.

$$\nabla^2\left(\chi + \frac{1}{2}|\nabla\chi|^2\right) = \nabla^2\chi \nabla^2\psi + \sum_{i=1}^2 \partial_i\chi \nabla^2\partial_i\chi \dot{=} \sum_{i=1}^2 \partial_i\chi \nabla^2\partial_i\psi,$$

and for multiindices α with $3 \leq |\alpha| < k$,

$$\partial^\alpha\left(\chi + \frac{1}{2}|\nabla\chi|^2\right) \dot{=} \sum_{i=1}^2 \partial_i\chi \partial_i \partial^\alpha\psi,$$

so for all $2 \leq |\alpha| < k$,

$$\partial^\alpha\left(\chi + \frac{1}{2}|\nabla\chi|^2\right) \dot{=} \nabla\chi \cdot \nabla\partial^\alpha\psi.$$

Thus for $j = 0, \dots, k-1$,

$$\begin{aligned} \partial_1^j \partial_2^{k-1-j}(f) &= \frac{\partial f}{\partial(\nabla\psi)} \cdot \nabla \partial_1^j \partial_2^{k-1-j} \psi + \frac{\partial f}{\partial a} \nabla\chi \cdot \nabla \partial_1^j \partial_2^{k-1-j} \psi \\ &\quad + \text{terms with } D^2\psi, \dots, D^{k-1}\psi \text{ components as factor} \\ &= \left(\frac{\partial f}{\partial(\nabla\psi)} + \frac{\partial f}{\partial a} \nabla\chi\right) \cdot \nabla \partial_1^j \partial_2^{k-1-j} \psi \end{aligned}$$

because $D^2\psi, \dots, D^{k-1}\psi \dot{=} 0$.

In a similar way we obtain $D^\ell(f) \stackrel{\bullet}{=} 0$ for $\ell = 2, \dots, k - 2$, and $D(f) = 0$ already by assumption. Therefore the $(k - 1)$ th-order minimum conditions for f apply: for all $\vec{\xi} = (\xi, \eta) \in \mathbb{R}^2$ with $\vec{\xi} \cdot \vec{n} > 0$,

$$\begin{aligned}
 (3.6) \quad & 0 \leq \sum_{j=0}^{k-1} \binom{k-1}{j} \partial_1^j \partial_2^{k-1-j} (f) \xi^j \eta^{k-1-j} \\
 & = \sum_{j=0}^{k-1} \binom{k-1}{j} \underbrace{\left(\frac{\partial f}{\partial(\nabla\psi)} + \frac{\partial f}{\partial a} \nabla\chi \right)}_{=: \vec{b}} \cdot \nabla \partial_1^j \partial_2^{k-1-j} \psi \cdot \xi^j \eta^{k-1-j}.
 \end{aligned}$$

Applying Lemma 3.2 to (3.5) and (3.6), with $a_j = \partial^j \partial^{k-1-j} \psi$, and using $L < 1$ yields $D^k \psi \stackrel{\bullet}{=} 0$. Note that $\vec{b} \neq 0$ if and only if (3.4) is not satisfied. The induction step is complete.

We have shown that $D^k \psi \stackrel{\bullet}{=} 0$ for all $k \geq 2$. Since ψ is analytic, it must be linear, which represents constant density and velocity. □

Remark 3.4. Lemma 3.3 applies trivially to the interior case $\vec{\xi}_0 \in U$: any $\vec{n} \neq 0$ will do.

3.2 Density in the Interior

PROPOSITION 3.5 *Let χ be an analytic solution of (2.14) in an open connected domain Ω . Assume that $L < 1$ in Ω and that ρ is positive and not constant. Then ρ does not have maxima in points where $\nabla\chi \neq 0$, and it does not have minima anywhere.*

Remark 3.6. Proposition 3.5 trivially implies corresponding results for variables like p and c that are strictly monotone functions ρ (except for c in the isothermal case, where it is constant).

PROOF OF PROPOSITION 3.5: The first-order condition for a critical point is

$$0 \stackrel{\bullet}{=} \nabla(\rho) = \nabla \left(\pi^{-1} \left(-\chi - \frac{1}{2} |\nabla\chi|^2 \right) \right) = -\frac{\rho}{c^2} \nabla^2 \psi \nabla\chi.$$

If $\nabla\chi \neq 0$, then combined with the PDE (2.15) we obtain $D^2 \psi \stackrel{\bullet}{=} 0$. Now we can apply Lemma 3.3 to show that ψ is actually a constant-state solution. In applying the lemma we choose $f = f(\psi, \nabla\psi, a) = f(a)$ only, using $f_a \neq 0$ and $\nabla\chi \neq 0$ so that (3.4) is false.

If $\nabla\chi \stackrel{\bullet}{=} 0$, then $\nabla(\rho) \stackrel{\bullet}{=} 0$ is trivially satisfied. We need to study the second-order condition for a minimum, which implies in particular

$$\begin{aligned} 0 \stackrel{\bullet}{\leq} \Delta(\rho) &= (\pi^{-1})'(-\Delta\chi - |\nabla^2\chi|^2 - \underbrace{\nabla\chi}_{\stackrel{\bullet}{=}0} \cdot \nabla\Delta\chi) \\ &\quad + (\pi^{-1})'' \left| \underbrace{\nabla\chi}_{\stackrel{\bullet}{=}0} + \nabla^2\chi \right| \underbrace{\nabla\chi}_{\stackrel{\bullet}{=}0} \Big|^2 \\ &\stackrel{\bullet}{=} \rho c^{-2}(\Delta\psi - |\nabla^2\psi|^2). \end{aligned}$$

Equation (2.15) reduces to $\Delta\psi \stackrel{\bullet}{=} 0$, so

$$(3.7) \quad 0 \stackrel{\bullet}{\leq} -\rho c^{-2}|\nabla^2\psi|^2.$$

Since $\rho, c > 0$ this implies $\nabla^2\psi \stackrel{\bullet}{=} 0$. Then $\nabla^2(\rho) \stackrel{\bullet}{=} 0$ as well.

We show for $k = 3, 4, \dots$ by induction that $D^k(\rho) \stackrel{\bullet}{=} 0$ and $D^k\psi \stackrel{\bullet}{=} 0$ as well. Induction step ($2, \dots, k-1 \rightarrow k \geq 3$): a minimum of $\rho = \pi^{-1}(-\chi - \frac{1}{2}|\nabla\chi|^2)$ is the same as a maximum of $\chi + \frac{1}{2}|\nabla\chi|^2$. For $j = 0, \dots, k$:

$$\begin{aligned} &\partial_1^j \partial_2^{k-j} \left(\chi + \frac{1}{2}|\nabla\chi|^2 \right) \\ &= \partial_1^j \partial_2^{k-j} \chi + \underbrace{\nabla\chi}_{\stackrel{\bullet}{=}0} \cdot \nabla \partial_1^j \partial_2^{k-j} \chi + j \partial_1 \nabla\chi \cdot \partial_1^{j-1} \partial_2^{k-j} \nabla\chi \\ &\quad + (k-j) \partial_2 \nabla\chi \cdot \partial_1^j \partial_2^{k-j-1} \nabla\chi \\ &\quad + \text{terms with components of } D^3\psi, \dots, D^{k-1}\psi \text{ as factor} \\ &\stackrel{\bullet}{=} \partial_1^j \partial_2^{k-j} \chi - j \partial_1^{j-1} \partial_2^{k-j} \partial_1 \chi - (k-j) \partial_1^j \partial_2^{k-1-j} \partial_2 \chi \\ &= (1-k) \partial_1^j \partial_2^{k-j} \psi. \end{aligned}$$

Here we used that $\partial_1 \nabla\chi \stackrel{\bullet}{=} (-1, 0)$ and $\partial_2 \nabla\chi \stackrel{\bullet}{=} (0, -1)$ because $\nabla^2\psi \stackrel{\bullet}{=} 0$. The induction assumption, $D^j\psi \stackrel{\bullet}{=} 0$ for $j = 2, \dots, k-1$, eliminates the other terms.

Since $D^j(\chi + \frac{1}{2}|\nabla\chi|^2) \stackrel{\bullet}{=} 0$ for $j = 1, \dots, k - 1$, a maximum requires that $D^k(\chi + \frac{1}{2}|\nabla\chi|^2) \stackrel{\bullet}{\geq} 0$ (i.e., is a negative semidefinite tensor), so $0 \stackrel{\bullet}{\geq} D^k(\chi + \frac{1}{2}|\nabla\chi|^2) \stackrel{\bullet}{=} (1 - k)D^k\psi$, so $D^k\psi \stackrel{\bullet}{\geq} 0$. Taking $k - 2$ derivatives of the equation yields (for $\nabla\chi \stackrel{\bullet}{=} 0$)

$$\partial_1^j \partial_2^{k-2-j} \Delta\psi = 0 \quad (j = 0, \dots, k - 2).$$

Lemma 3.1 implies that $D^k\psi \stackrel{\bullet}{=} 0$. The induction step is complete.

Again we have shown that $D^k\psi \stackrel{\bullet}{=} 0$ for all $k \geq 2$. Therefore ψ , which is analytic, must be linear. □

Remark 3.7. Note that the proof fails for *maxima* in $\nabla\chi$: in that case \leq in (3.7) turns into \geq , which does not yield sufficient information. Indeed, there are counterexamples.

3.3 Velocity Components in the Interior

PROPOSITION 3.8 *Let χ be an analytic solution of (2.14) in an open connected domain Ω , with $\rho > 0$ and $L < 1$ in Ω . For any $\vec{w} \in \mathbb{R}^2 - \{0\}$, the velocity component $\vec{w} \cdot \nabla\psi$ does not have a maximum or a minimum in Ω unless χ is a constant-state solution in Ω .*

PROOF: Assume that ψ_1 has a minimum in some point. Then $\nabla\psi_1 \stackrel{\bullet}{=} 0$; using equation (2.15) yields $D^2\psi \stackrel{\bullet}{=} 0$ because $L < 1$ implies $c^2 - \chi_2^2 > 0$. Now we apply Lemma 3.3 to obtain that ψ must be a constant-state solution.

Any other \vec{w} can be treated by rotating around the origin so that $\vec{w} \cdot \nabla\psi$ becomes ψ_1 (see Remark 2.1). □

3.4 Velocity Components on the Wall

PROPOSITION 3.9 *Consider a point $\vec{\xi}_0$ on a straight line I , let $r > 0$, $U := B_r(\vec{\xi}_0)$, $\Gamma := I \cap U$, and U^+ one of the two connected components of $U - I$. Consider a solution ψ of (2.14) that is analytic in $\overline{U^+}$ and satisfies the slip condition $\chi_n = 0$ on Γ . Assume that $L < 1$ in $\vec{\xi}_0$.*

For any \vec{w} ,

$$\nabla\psi(\vec{\xi}_0) \cdot \vec{w} = \inf_{\overline{U^+}} \nabla\psi \cdot \vec{w}$$

is not possible unless ψ is a constant-state solution or \vec{w} is normal to Γ .

PROOF: Assume there is an extremum point on Γ . We may assume (by rotation and translation) that Γ is a piece of the horizontal axis, that the extremum point is the origin, and that U_+ is contained in the upper half-plane; then $\vec{w} = (w^1, w^2)$ with $w^1 \neq 0$ (not normal). A tangential derivative of the boundary condition $\psi_2 = 0$ implies $\psi_{12} = 0$ on Γ . A minimum requires

$$0 \stackrel{\bullet}{=} (\vec{w}\nabla\psi)_1 = w^1\psi_{11} + w^2\psi_{12} = w^1\psi_{11} \quad \Rightarrow \quad \psi_{11} \stackrel{\bullet}{=} 0.$$

Equation (2.15) yields that $D^2\psi \stackrel{\bullet}{=} 0$. Now the result is delivered by Lemma 3.3. \square

3.5 Velocity Components at Shocks

PROPOSITION 3.10 *Consider disjoint open connected domains Ω and Ω^h and a simple analytic curve $S \subset \overline{\Omega} \cap \overline{\Omega}^h$ (excluding the endpoints). Consider a constant-state (linear) potential ψ^h of (2.18) in Ω^h . Let χ be an analytic solution of (2.14) in $\Omega \cup S$. Let the shock relations (2.18) and (2.19) be satisfied on S and assume the shock is admissible. Assume that χ satisfies $\rho > 0$ and $L < 1$ in $\Omega \cup S$.*

Let $\vec{w} \neq 0$. Assume that $\vec{w} \cdot \nabla\psi$ has a local maximum (with respect to $\Omega \cup S$) in $\vec{\xi} \in S$. Then either S is straight and ψ is constant-state in Ω , or

$$(3.8) \quad \vec{w} \cdot \left((1 - c^{-2}\chi_n^2)\vec{t} + \chi_t \left(\frac{1}{\chi_n^h} + c^{-2}\chi_n \right) \vec{n} \right) = 0$$

and

$$(3.9) \quad \text{sgn } \kappa = \text{sgn } w^n \neq 0$$

where κ is the curvature of S in $\vec{\xi}$ ($\kappa > 0$ for Ω locally convex).

PROOF: We use a dot to indicate relations that hold only in the hypothetical extremum point. By $L < 1$, Ω must be downstream and Ω^h is upstream. Without loss of generality, rotate around $\vec{\xi}_0$ until $\vec{n} \stackrel{\bullet}{=} (0, -1)$. In this setting $\partial_1 \stackrel{\bullet}{=} \partial_t$ and $-\partial_2 \stackrel{\bullet}{=} \partial_n$, and $\chi_1 \stackrel{\bullet}{=} \chi_t = \chi_t^h \stackrel{\bullet}{=} \chi_1^h$ by (2.21). Use horizontal translation (Remark 2.1) so that $\psi_1^h \stackrel{\bullet}{=} 0$ and therefore $\psi_1 \stackrel{\bullet}{=} 0$; this adds a constant vector to velocities, while leaving ρ and L unchanged, so no generality is lost. Let the shock be parametrized by $\xi \mapsto (\xi, s(\xi))$ locally; then $s_1 \stackrel{\bullet}{=} 0$.

Take ∂_{tt} of (2.20):

$$(3.10) \quad \psi_{11} + (\psi_2 - \psi_2^h)s_{11} \stackrel{\bullet}{=} 0.$$

Take ∂_t of (2.19):

$$\begin{aligned} 0 &= \partial_t \left(\left(\pi^{-1} \left(-\chi - \frac{1}{2} |\nabla\chi|^2 \right) \nabla\chi - \rho^h \nabla\chi^h \right) \cdot \vec{n} \right) \\ &= \vec{n} \cdot \partial_t \left(\pi^{-1} \left(-\chi - \frac{1}{2} |\nabla\chi|^2 \right) \nabla\chi - \rho^h \nabla\chi^h \right) \\ &\quad + \left(\pi^{-1} \left(-\chi - \frac{1}{2} |\nabla\chi|^2 \right) \nabla\chi - \rho^h \nabla\chi^h \right) \cdot (\vec{n})_t \\ &\stackrel{\bullet}{=} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cdot \partial_1 \left(\pi^{-1} \left(-\chi - \frac{1}{2} |\nabla\chi|^2 \right) \nabla\chi - \rho^h \nabla\chi^h \right) \\ &\quad + \left(\pi^{-1} \left(-\chi - \frac{1}{2} |\nabla\chi|^2 \right) \nabla\chi - \rho^h \nabla\chi^h \right) \cdot \begin{bmatrix} s_{11} \\ 0 \end{bmatrix} \stackrel{\bullet}{=} \end{aligned}$$

$$\begin{aligned}
 &\doteq \rho c^{-2}(\chi_1 + \chi_1\chi_{11} + \chi_2\chi_{12})\chi_2 - \rho\chi_{12} + \rho^h\chi_{12}^h + (\rho\chi_1 - \rho^h\chi_1^h)s_{11} \\
 (3.11) \quad &\doteq \rho\left(- (1 - c^{-2}\chi_2^2)\psi_{12} + c^{-2}\chi_1\chi_2\psi_{11} + \chi_1\left(1 - \frac{\chi_2}{\chi_2^h}\right)s_{11}\right).
 \end{aligned}$$

Combining these results with the equation and $(\vec{w} \cdot \nabla\psi)_t = 0$, we get the system

$$(3.12) \quad \begin{bmatrix} w^1 & w^2 & 0 & 0 \\ 1 & 0 & 0 & \chi_2 - \chi_2^h \\ c^2 - \chi_1^2 & -2\chi_1\chi_2 & c^2 - \chi_2^2 & 0 \\ -c^{-2}\chi_1\chi_2 & 1 - c^{-2}\chi_2^2 & 0 & \left(\frac{\chi_2}{\chi_2^h} - 1\right)\chi_1 \end{bmatrix} \begin{bmatrix} \psi_{11} \\ \psi_{12} \\ \psi_{22} \\ s_{11} \end{bmatrix} = 0.$$

The determinant of the system matrix is

$$\begin{aligned}
 \det &= -(c^2 - \chi_2^2)\left(w^2\frac{\chi_1}{\chi_2^h}(\chi_2 - \chi_2^h) \right. \\
 &\quad \left. + (\chi_2 - \chi_2^h)(w^1(1 - c^{-2}\chi_2^2) + w^2c^{-2}\chi_1\chi_2)\right) \\
 (3.13) \quad &= -\underbrace{(c^2 - \chi_2^2)}_{>0}\underbrace{(\chi_2 - \chi_2^h)}_{>0}\vec{w} \cdot \begin{bmatrix} 1 - c^{-2}\chi_2^2 \\ \chi_1\left(\frac{1}{\chi_2^h} + c^{-2}\chi_2\right) \end{bmatrix}
 \end{aligned}$$

The determinant is 0 if and only if the final scalar product is 0. The latter condition can be written (3.8) if we return to original coordinates.

If the determinant is nonzero, then $D^2\psi \doteq 0$ and $s_{11} \doteq 0$ is the only solution. If the determinant is 0, but $s_{11} \doteq 0$, still $D^2\psi \doteq 0$. In either case we can invoke Lemma 3.3 to get that ψ is a constant-state solution; then (2.20) shows that the shock is straight.

Now assume the determinant is 0 and $s_{11} \neq 0$. By row 2 of the system, this implies $\psi_{11} \neq 0$. Then by row 1 and $\vec{w} \neq 0$, necessarily $w^2 \neq 0$.

By solving rows 1, 2, and 3 of the system for $\nabla^2\psi$ as a function of s_{11} and then substituting the result into $(\vec{w} \cdot \nabla\psi)_2$, we obtain

$$\begin{aligned}
 (\vec{w} \cdot \nabla\psi)_2 &= \frac{(\chi_2 - \chi_2^h)((c^2 - \chi_2^2)w_1^2 + 2\chi_1\chi_2w^1w^2 + (c^2 - \chi_1^2)w_2^2)}{w^2(c^2 - \chi_2^2)}s_{11} \\
 &= \frac{(\chi_2 - \chi_2^h)(\vec{w}^\perp)^\top(c^2I - \nabla\chi^2)\vec{w}^\perp}{c^2 - \chi_2^2} \cdot \frac{s_{11}}{w_2}.
 \end{aligned}$$

All factors in the coefficient of s_{11}/w_2 are positive; s_{11} and $w_2 \neq 0$ are already known, so $(\vec{w} \cdot \nabla\psi)_2 \neq 0$. A maximum requires $(\vec{w} \cdot \nabla\psi)_2 > 0$, so $\text{sgn } \kappa = -\text{sgn } s_{11} = -\text{sgn } w^2 = \text{sgn } w^n \neq 0$. \square

3.6 Pseudo-Mach Number at Shocks

PROPOSITION 3.11 *Consider the setting in the first paragraph of the statement of Proposition 3.10. (We do not need analyticity here.)*

Let $\delta_{L\rho} > 0$ be such that

$$(3.14) \quad \frac{\rho}{\rho^h} \in [\delta_{L\rho}, 1 - \delta_{L\rho}] \quad \text{on } S.$$

Let $b \in C^1(\Omega \cup S)$. There is a $\delta_{LS} > 0$ (depending continuously and only on $\delta_{L\rho}, \gamma, c^h, \rho^h$) with the following property: $L^2 + b$ cannot attain a local (with respect to $\Omega \cup S$) maximum in a point on S where $L^2 \in [1 - \delta_{LS}, 1)$ and $|\nabla b| \leq \delta_{LS}$.

PROOF: We use the same notation and simplifications as explained at the start of the proof of Proposition 3.10.

From $\rho/\rho^h \leq 1 - \delta_{L\rho} < 1$ we obtain

$$(3.15) \quad \frac{\chi_2^2}{c^2}, \frac{\chi_2}{\chi_2^h} \leq 1 - C_s$$

for some constant $C_s = C_s(\delta_{L\rho}, \gamma) > 0$.

$$(3.16) \quad \begin{aligned} & \partial_1(L^2 + b) \\ &= \partial_1\left(\frac{|\nabla\chi|^2}{c^2}\right) + b_1 \\ &= \partial_1\left(\frac{|\nabla\chi|^2}{c_0^2 + (1-\gamma)(\chi + \frac{1}{2}|\nabla\chi|^2)}\right) + b_1 \\ &= c^{-2}((2 + (\gamma - 1)L^2)(\chi_1\chi_{11} + \chi_2\chi_{12}) + (\gamma - 1)L^2\chi_1) + b_1 \\ &= c^{-2}((2 + (\gamma - 1)L^2)(\chi_1\psi_{11} + \chi_2\psi_{12}) - 2\chi_1) + b_1 \end{aligned}$$

and analogously

$$(3.17) \quad \partial_2(L^2 + b) = c^{-2}((2 + (\gamma - 1)L^2)(\chi_1\psi_{12} + \chi_2\psi_{22}) - 2\chi_2) + b_2.$$

Combining $\partial_1(L^2) \stackrel{\bullet}{=} 0$ with (2.15), (3.10), and (3.11), we have a linear system

$$\begin{bmatrix} \chi_1 & \chi_2 & 0 & 0 \\ 1 & 0 & 0 & \chi_2 - \chi_2^h \\ c^2 - \chi_1^2 & -2\chi_1\chi_2 & c^2 - \chi_2^2 & 0 \\ -\frac{\chi_1\chi_2}{c^2} & 1 - \frac{\chi_2^2}{c^2} & 0 & (\frac{\chi_2}{\chi_2^h} - 1)\chi_1 \end{bmatrix} \begin{bmatrix} \psi_{11} \\ \psi_{12} \\ \psi_{22} \\ s_{11} \end{bmatrix} \stackrel{\bullet}{=} \begin{bmatrix} \frac{2\chi_1 - c^2 b_1}{2 + (\gamma - 1)L^2} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

First consider $\nabla b = 0$ and $L = 1$, i.e.,

$$\chi_1 = \text{sgn } \chi_1 \cdot \sqrt{c^2 - \chi_2^2}.$$

Then the inverse of the system matrix has entries polynomial in c , χ_2 , χ_2^h , and $\text{sgn } \chi_1$ divided by a common denominator

$$c^2(c^2 - \chi_2^2)^{3/2}((\chi_2^h)^2 - \chi_2^2).$$

This denominator is bounded below away from 0 by $c^7 \delta_D$ where δ_D depends continuously and only on $\delta_{L\rho}$ and γ . The rest of the inverse matrix is bounded by some constant depending only on ρ^h , c^h , γ , and $\delta_{L\rho}$.

Solving for $D^2\psi$ and substituting the result yields

$$\partial_2(L^2 + b) = \frac{2}{\chi_2^h + \chi_2} < 0,$$

so clearly a maximum of $L^2 + b$ is not possible.

For $\nabla b \neq 0$ and $L < 1$ we use that the inverse has been bounded away from 0, so that small perturbations are possible. Thus, there is a δ_{LS} , depending only on $\delta_{L\rho}$, γ , ρ^h , and c^h , so that no maximum is possible if $|\nabla b| \leq \delta_{LS}$ and $L^2 \geq 1 - \delta_{LS}$. \square

Remark 3.12. We could assume $L \leq \bar{L} < 1$, which by itself would imply $\rho/\rho^h \leq C(\bar{L}) < 1$. However, this is not sufficient as δ_{LS} would depend on \bar{L} , so Proposition 3.11 would be void. It is necessary to obtain uniform shock strength bounds separately; only then can L be controlled.

3.7 Density at Shocks

PROPOSITION 3.13 *Consider the setting in the first paragraph of Proposition 3.10. If ρ has a local extremum with respect to $\Omega \cup S$ in $\vec{\xi}_0$, then one of the following alternatives must hold:*

- (1) χ is a constant-state solution in $\Omega \cup S$, and S is straight.
- (2) The shock is pseudonormal in $\vec{\xi}_0$. In a local minimum, S has curvature $\kappa > 0$ (as before, $\kappa > 0$ for Ω locally strictly convex in that point).

In the latter case, make the stronger assumption that ρ has a global minimum with respect to $\bar{\Omega}$ in $\vec{\xi}_0$. Assume for technical convenience that the shock tangents differ by no more than an angle $< \frac{\pi}{2}$ from the one in $\vec{\xi}_0$. Let S^ be the tangent to S in $(\xi_0, s(\xi_0))$. Then S^* does not meet S anywhere else.*

PROOF: We use the same notation and simplifications as explained at the start of the proof of Proposition 3.10.

A ρ extremum requires

$$\begin{aligned} 0 \stackrel{\bullet}{=} \partial_t(\rho) \stackrel{\bullet}{=} \partial_1 \left(\pi^{-1} \left(-\chi - \frac{1}{2} |\nabla \chi|^2 \right) \right) &= -\frac{\rho}{c^2} (\chi_1 + \chi_1 \chi_{11} + \chi_2 \chi_{12}) \\ &\Rightarrow \chi_1 \psi_{11} + \chi_2 \psi_{12} \stackrel{\bullet}{=} 0. \end{aligned}$$

We combine this with the now-familiar equations (2.15), (3.10), and (3.11). The resulting linear system is

$$\begin{bmatrix} \chi_1 & \chi_2 & 0 & 0 \\ 1 & 0 & 0 & \chi_2 - \chi_2^h \\ c^2 - \chi_1^2 & -2\chi_1\chi_2 & c^2 - \chi_2^2 & 0 \\ -c^{-2}\chi_1\chi_2 & 1 - c^{-2}\chi_2^2 & 0 & (\frac{\chi_2}{\chi_2^h} - 1)\chi_1 \end{bmatrix} \begin{bmatrix} \psi_{11} \\ \psi_{12} \\ \psi_{22} \\ s_{11} \end{bmatrix} = 0.$$

The determinant is

$$\begin{aligned} &= -(c^2 - \chi_2^2)(\chi_2 - \chi_2^h)\nabla\chi \cdot \begin{bmatrix} 1 - c^{-2}\chi_2^2 \\ \frac{\chi_1}{\chi_2^h} + c^{-2}\chi_1\chi_2 \end{bmatrix} \\ &= -\underbrace{(c^2 - \chi_2^2)}_{>0} \underbrace{(\chi_2 - \chi_2^h)}_{>0} \underbrace{\left(1 + \frac{\chi_2}{\chi_2^h}\right)}_{>0} \cdot \chi_1. \end{aligned}$$

It is nonzero if and only if $\chi_1 \neq 0$. In that case, $\nabla^2\psi \stackrel{\bullet}{=} 0$ and $s_{11} \stackrel{\bullet}{=} 0$. Now Lemma 3.3 yields that the shock is straight and the solution constant-state. (The lemma applies because ρ is a strictly decreasing function of $\chi + \frac{1}{2}|\nabla\chi|^2$ alone, and (3.4) is satisfied because $\chi_n \neq 0$, hence $\nabla\chi \neq 0$ at any shock).

If $\chi_1 \stackrel{\bullet}{=} 0$ but $s_{11} \stackrel{\bullet}{=} 0$, then the equations imply that $\nabla^2\psi \stackrel{\bullet}{=} 0$, so the lemma still applies. This concludes the proof of the first part.

For part two we may assume that all of S is parametrized by s , because the shock tangents cannot become vertical by assumption. Consider the remaining case $\chi_1 \stackrel{\bullet}{=} 0$ and $s_{11} \neq 0$. $\chi_1 \stackrel{\bullet}{=} 0$ and $s_1 \stackrel{\bullet}{=} 0$ imply $\xi \stackrel{\bullet}{=} 0$. Here it is sufficient to argue without the interior. So we may exploit that at the shock downstream, ρ_d is an increasing function of z_u^n (by (2.34) and (2.37) for ρ_u held fixed). Geometrically, z_u^n in a point $\vec{\xi}$ on the shock is the distance of \vec{v}_u to the shock tangent through $\vec{\xi}$. A pseudonormal point is actually the closest point on the tangent to \vec{v}_u (see Figure 3.3). Obviously the tangents through nearby points are *closer* to \vec{v}_u if $s_{11} > 0$ (see Figure 3.1), which contradicts the assumption that ρ is a local (in fact, global) minimum. Therefore $s_{11} \leq 0$; $s_{11} \stackrel{\bullet}{=} 0$ has already been excluded, so $s_{11} < 0$.

Let $\xi \mapsto s^*(\xi)$ parametrize S . Assume that S^* meets S somewhere else, e.g., that $s(\xi_1) = s^*(\xi_1)$ for some $\xi_1 < \xi_0$ (see Figure 3.2). By the mean value theorem there must be a $\xi_2 \in (\xi_1, \xi_0)$ so that $s_1(\xi_2) = (s(\xi_1) - s(\xi_0))/(\xi_1 - \xi_0) = s_1(\xi_0)$. Clearly we may choose ξ_2 maximal with this property, so that $s_1(\xi) \neq s_1(\xi_0)$ for $\xi \in (\xi_2, \xi_0)$. But $s_{11} < 0$, so $s_1(\xi) > s_1(\xi_0)$ for $\xi \in (\xi_2, \xi_0)$. This implies

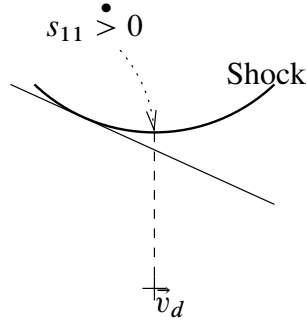


FIGURE 3.1. Tangent distance argument.

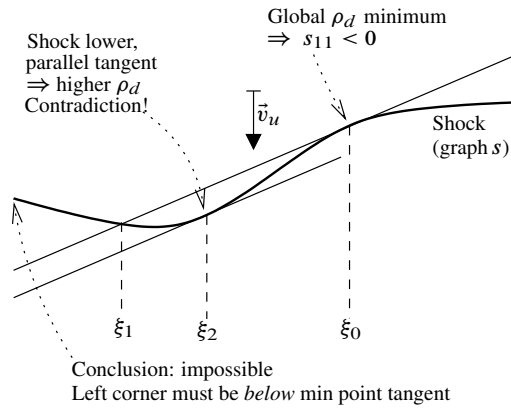


FIGURE 3.2. Shock below global density minimum tangent.

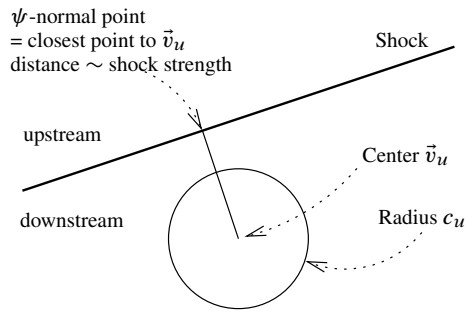


FIGURE 3.3. Shock strength: distance of tangent from upstream velocity.

$s(\xi_2) < s^*(\xi_0)$: the shock tangent through $(\xi_2, s(\xi_2))$ is parallel to the one in ξ_0 , but lower. That means the shock strength is smaller, so the downstream density is

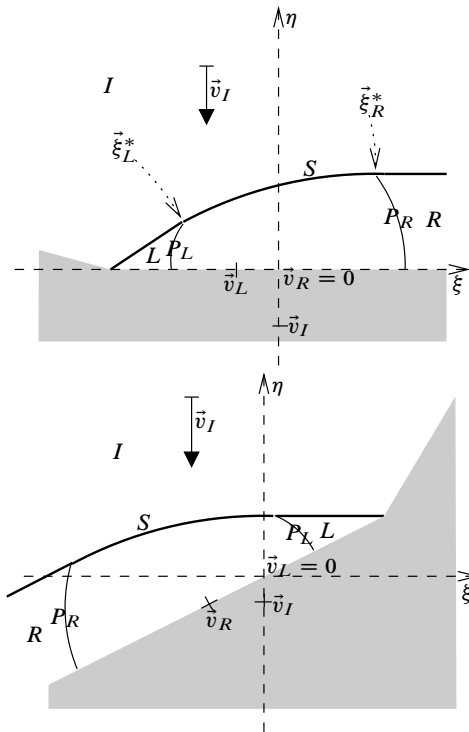


FIGURE 4.1. Left: standard coordinates; right: L picture.

lower (by (2.43)). This is a contradiction, because we assumed that ρ has a global minimum in $\vec{\xi}_0$.

Therefore $s(\xi) > s(\xi_0)$ for all $\xi < \xi_0$ (on S). For $\xi > \xi_0$ the arguments are symmetric. \square

4 Construction of the Flow

4.1 Problems

Our solution, as observed in numerics (see Figure 1.4), has the structure in Figure 4.1. The upstream region, labeled I , is a constant-state hyperbolic region. The shock has three parts: a straight shock emanating from the tip, with a constant-state hyperbolic region labeled L below; a curved shock S with a nontrivial elliptic region below; and a straight shock parallel to the wall, with another constant-state hyperbolic region (R) below. The L and R regions are separated from the elliptic region by parabolic arcs P_L and P_R , respectively, with radius c_L and c_R , respectively, centered in \vec{v}_L and \vec{v}_R , respectively.

Several difficulties complicate the problem: the equation is nonlinear (quasi-linear, divergence form), with coefficients depending on χ and $\nabla\chi$. The boundary

conditions (except on the wall) are fully nonlinear; i.e., they are nonlinear in $\nabla\chi$ as well, which makes compactness hard to obtain. Moreover, the boundary conditions linearize to oblique derivative conditions where the χ and χ_n coefficients have *opposite* signs—the most complicated case. The equation and boundary conditions have singularities: for example, we have to avoid the vacuum ($\rho = 0$). The shock is a *free* boundary, forming two corners with the arcs.⁵

Most significantly, the equation is of *mixed type*: if χ and $\nabla\chi$ are not sufficiently controlled, points in a supposedly elliptic region could be parabolic or hyperbolic. While the elliptic region is uniformly elliptic at the shock, it is *degenerate* elliptic at the parabolic arcs P_L and P_R . Moreover, it appears from numerics that $\nabla\chi$ is normal on the parabolic arcs, meaning they are *characteristic* (in the Cauchy-Kovalevskaya sense), so loss of regularity has to be expected. The linearization of the degenerate problem is useless in this case; e.g., although we expect finite gradients (= velocities) in the nonlinear problem (and observe them in numerics), it can be checked for simple examples that the linear equation has solutions with gradient that is infinite at parabolic arcs.

For many of these obstacles there are theoretical tools in the literature; many have been addressed in other contexts. However, the nonlinear characteristic degeneracy seems unprecedented. There is little theory on degenerate elliptic equations; most of it can be found in [27]. A large part of the theory was motivated by the linear *Tricomi equation*, which arises from steady potential flow via the *hodograph transform* (see [15, sec. 4.4.3.a]). The hodograph transform applies to quasi-linear equations whose nondivergence form coefficients depend only on the gradient of the solution; the original equation is converted into a linear equation. Unfortunately, the coefficients of self-similar potential flow also depend on the solution itself. It is not clear whether a modified hodograph transform can be developed for self-similar potential flow.

Nonlinear degenerate elliptic equations, other than steady potential flow, have not been explored much, apart from steady potential flow, which can often be reduced to a linear equation. The combination of nonlinearity and degeneracy is particularly difficult: the solution can be characteristic degenerate, noncharacteristic degenerate, elliptic, or hyperbolic in each point of the arcs; each case is qualitatively very different. Precise bounds on the solution have to be established before we even know which case occurs in which point.

4.2 Approach

In this paper we construct a weak solution, so only χ -continuity and conservation are shown across the parabolic arcs. Most of the effort is concerned with the elliptic region.

⁵The wall-arc corners can be removed by reflection across the wall.

Symmetries

Due to Remark 2.1, we can change our coordinate system in several ways.

Starting in original coordinates (see Figure 1.4), we translate $\vec{\xi} \leftarrow \vec{\xi} - \vec{v}_R$ and $\vec{v} \leftarrow \vec{v} - \vec{v}_R$, then rotate so that the right shock is horizontal (and in the upper half-plane); see Figure 4.1. Now $\vec{v}_R = 0$, $\vec{v}_L = (v_L^x, 0)$ with $v_L^x \leq 0$, and $\vec{v}_I = (0, v_I^y)$ with $v_I^y < 0$. Moreover, ψ is *constant* in the R region, and P_R is centered in the origin, which will be rather convenient. We use this as the standard picture, as it is the most convenient for discussing the elliptic region, which consumes most of our efforts.

For another choice, which we will call the “ L picture”, we start in original coordinates, translate $\vec{\xi} \leftarrow \vec{\xi} - \vec{v}_L$ and $\vec{v} \leftarrow \vec{v} - \vec{v}_L$, then rotate so that the left shock is horizontal (and in the upper half-plane), then reflect everything across the vertical coordinate axis; see Figure 4.1. In this picture $\vec{v}_L = 0$ whereas $\vec{v}_R = (v_R^x, 0)$ with $v_R^x \leq 0$, and $\vec{v}_I = (0, v_I^y)$ with $v_I^y < 0$ (which need not have the same value as before). Now ψ is constant in the L region, and P_L is centered in the origin.

Extremum Principles

As for most nonlinear elliptic problems, maximum principles are the key technique. In Section 3 we have developed many that apply to self-similar potential flow in general. Their interior versions are comparable to the classical strong maximum principle [17, theorem 3.5] or to maximum principles for gradients of special quasi-linear equations [17, sec. 15.1]. In addition, we have several extremum principles at the shock (Propositions 3.10, 3.11, and 3.13) ruling out local (with respect to the domain) extrema of certain variables at the shock.

The general proof technique for extremum principles is to combine the equation with the first- and second-order conditions for an extremum to obtain a contradiction. At the shock we also include the boundary conditions. In many cases it is necessary to include first and higher derivatives of equation and boundary conditions as well as higher-order extremum conditions. In some borderline cases (notably density and velocity) it is necessary to consider *all* derivatives and to assume that the solution is analytic (see the proofs of Propositions 3.5, 3.8, 3.10, and 3.13).

Regularization and Ellipticity

As we have mentioned, our elliptic region is nonlinear and characteristic degenerate; there are very few results for such problems. The type of our PDE is governed by the pseudo-Mach number L . We avoid the degeneracy by *regularization*: we consider slightly different elliptic regions, with $L = \sqrt{1 - \epsilon}$ on the parabolic arcs rather than $L = 1$ (see Figure 4.2). We need L uniformly bounded above away from 1 so that our regularized problems are uniformly elliptic and standard theory can be applied.

A basic problem of our constructive approach is that it is infeasible if the flow pattern has a complicated structure. For example, if the elliptic region could, upon

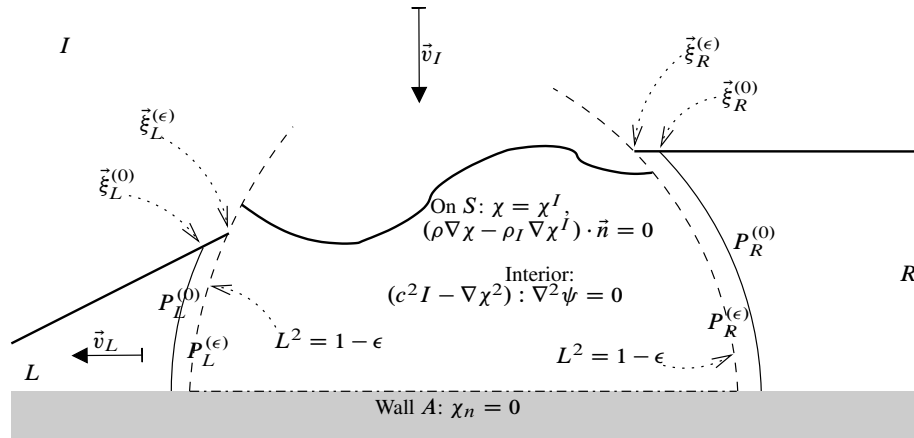


FIGURE 4.2. The solution in the elliptic region and its delimiting shock S are unknown. The parabolic arcs (solid) are modified to quasi-parabolic arcs (dashed) with boundary condition $L = \sqrt{1 - \epsilon}$, which is slightly elliptic.

perturbation, develop a (potentially infinite) number of parabolic and hyperbolic bubbles in its interior, only abstract functional-analytic methods have a chance to succeed. Fortunately, we have shown in [12] that the pseudo-Mach number L cannot have maxima in the interior of an elliptic region or at a straight wall. On the arcs we have imposed the value of L . It remains to control L at the shock. This result (Proposition 3.11) is less general: we can rule out L maxima close to 1 at the shock if the shock has uniform strength. This condition needs to be verified with separate methods (see below).

All combined, we can be sure that our elliptic region stays uniformly elliptic under perturbation. We obtain, after many additional steps described below, one solution $\chi^{(\epsilon)}$ for each sufficiently small $\epsilon > 0$. To obtain a solution of the original degenerate problem, it is necessary to obtain estimates that are uniform in $\epsilon \downarrow 0$.

As discussed in [12], the maximum principle for L can be strengthened to a maximum principle for $L + b$ where b is a small, smooth positive function that is 0 on the arcs. As a consequence, L is uniformly in $\epsilon \downarrow 0$ bounded above away from 1, in subsets of the elliptic region that have positive distance from the arcs. We obtain uniform regularity in all $C^{k,\alpha}$ norms (in fact, analyticity) in each such subset. Hence we have compactness in each subset, so we can find a converging subsequence. By a diagonalization argument we have a subsequence that converges everywhere away from the arcs, in any norm (see Section 4.16).

Another of the many benefits of the ellipticity principle: $L \leq 1 - \epsilon \leq 1$ implies a uniform bound on the gradient $|\nabla\chi|$ (see (4.46) and the following paragraph), which is a basic ingredient of all higher regularity estimates for nonlinear elliptic equations (see the discussion in [17, sec. 11.3]). Moreover, it shows that $\chi^{(\epsilon)}$ is

uniformly Lipschitz, so $\chi^{(0)}$ is Lipschitz as well, including the arcs. This is not quite sufficient for the boundary condition $L = 1$ to be well-defined, but we only seek a weak solution.

Iteration

For any nonlinear elliptic problem, there are several ways of turning a priori estimates into an existence proof (cf. [17, chap. 11, sec. 17.2]). The *method of continuity* is straightforward: verify that the linearizations of the problem are isomorphisms between suitable spaces, apply the inverse function theorem to obtain small perturbations, then use the a priori estimates to show that we can repeat small perturbations indefinitely.

A major drawback is that the linearizations are entirely determined by the nonlinear problem. In comparison, *fixed-point iteration methods* are more flexible: there is a large variety of maps whose fixed points solve the nonlinear problem. In our case this flexibility is vital: the Fréchet derivative of $L^2 = 1 - \epsilon$ is an oblique derivative condition where $\delta\chi$ and $\delta\chi_n$ ($\delta\chi$ being the variation of χ) have opposite sign. In this case, maximum principles fail, but they are needed to verify uniqueness (the other option, energy methods, seems impractical for our complicated problem).

Another (nonfatal) drawback is that linearizations of a free-boundary problem involve a complicated coordinate transform. Iteration methods can alternate solving a fixed-boundary problem with adjusting the free boundary (see, e.g., [3, 4]), which is easier.

Fixed-point methods can be subdivided into applications of the Schauder fixed-point theorem (or its generalizations), and of Leray-Schauder degree theory (see [9, 29, 30]). The Schauder fixed-point theorem requires showing that the iteration maps a closed ball (or homeomorphic image thereof) into itself. This corresponds to certain estimates for all elements of the ball, most of which are not fixed points (only approximately perhaps). These new estimates are particular to the chosen iteration, which is unattractive and perhaps difficult.⁶ Moreover, every time details of the iteration are changed, all the a priori estimates have to be checked and changed; this takes an excessive amount of time for complicated problems like the present one.

In addition, while many function sets in nonlinear elliptic problems are defined by a single constraint $\|u\|_{C^{2,\alpha}} \leq M < \infty$, so that the set is obviously a closed ball, our function set $\overline{\mathcal{F}}$ is defined by more than 20 different constraints (see Definition 4.4). For many choices $\overline{\mathcal{F}}$ has nontrivial topology; it is rather difficult to check whether an intersection of many sets is homeomorphic to a ball (unless convexity can be used).

⁶However, in [4] the authors have successfully applied the Schauder fixed-point theorem to solve a steady potential flow problem.

Ultimately, the Schauder fixed-point theorem is a means of showing that a particular iteration has nonzero Leray-Schauder degree. However, there are other ways of computing a degree that turn out to be simpler in our case (see Section 4.14).

The iteration has two steps. The first step solves a fixed-boundary elliptic problem for a function $\hat{\chi}$; in this step we use a modification of the second shock condition (2.19); the data of this problem depends on χ (argument of the iteration map). In the second step the shock is adjusted (and $\hat{\chi}$ mapped to a new domain) to satisfy the first shock condition (2.18).

As mentioned, we have some flexibility in choosing the first-step elliptic problem. An obvious constraint is that fixed points of the iteration, $\chi = \hat{\chi}$, must solve the original problem. This is satisfied by replacing every occurrence of χ in the original equation and boundary conditions either by χ (old iterate) or by $\hat{\chi}$ (new iterate after step 1).

For the arc boundary condition (4.48), we choose

$$\frac{1}{2}|\nabla\hat{\chi}|^2 + \frac{(1-\epsilon)((\gamma-1)\chi - c_0^2)}{2 + (\gamma-1)(1-\epsilon)} = 0$$

for the iteration, where χ is the old and $\hat{\chi}$ the new solution. Linearization with respect to $\hat{\chi}$ no longer has a zeroth-order term! Hence the opposite-sign problem does not apply, and we can apply the Hopf lemma to show that the linearizations of the nonlinear elliptic problems arising in the iteration are isomorphisms. We use the same technique for the interior equation, but keep $\hat{\chi}$ in the ρ function in the shock condition (2.19): we need at least one occurrence of $\hat{\chi}$ to be able to use maximum principles to achieve uniqueness in various contexts. Obviously if $\hat{\chi}$ does not appear anywhere in the elliptic problem defining the iteration, then the next iterate is not uniquely determined: any constant can be added.

To apply Leray-Schauder degree theory, we argue that we are solving a continuum of problems, the simplest one being the “unperturbed” problem (see Figure 4.9) with $\gamma = 1$. This problem is simple enough to verify that its Leray-Schauder degree is ± 1 , in particular nonzero (see Section 4.14). All other problems have the same degree due to the combined a priori estimates (no fixed points on the boundary of the iteration domain) and continuity.

The opposite-sign difficulty resurfaces in two aspects: first, to prove degree $\neq 0$ we have to show uniqueness of the unperturbed problem (see Proposition 4.27). For $\gamma > 1$ this seems difficult, but for $\gamma = 1$ the χ coefficient is 0 (see above). Moreover, it is necessary to show that the linearization of the iteration does not have eigenvalue 1. Again $\gamma > 1$ has unpleasant boundary conditions, but $\gamma = 1$ is amenable. Thus solving the isothermal and isentropic problems in a single paper is a *necessity* rather than a choice.

Unfortunately, it is necessary to show that the iteration is compact, which requires somewhat stronger regularity estimates than the method of continuity. In fact, Gilbarg and Trudinger state [17, p. 482]: “Even for the quasilinear [equation]

case [with fully nonlinear boundary conditions], the fixed point methods . . . are not appropriate, since it is not in general possible to construct a *compact* operator.”

Recent results, many of which are due to Gary Lieberman, have alleviated this problem, at least for our case. Some unpleasant choices are necessary, however: by replacing the occurrences of $\nabla\chi$ by a clever balance of $\nabla\chi$ and $\nabla\hat{\chi}$, we could obtain a *linear* elliptic problem as part the iteration. But if $\nabla\chi$ occurs in any of the boundary conditions, then the new solution $\hat{\chi}$ cannot be expected to be smoother than χ at the boundary. While we may use χ and $\nabla\chi$ in the interior coefficients, we must use $\nabla\hat{\chi}$ for the boundary conditions, which stay nonlinear.

Now our general approach has a hierarchy of three elliptic problems: the original nonlinear problem, a modified nonlinear problem as part of the iteration, and its linearization. It appears that we have to start all over showing a priori estimates for the modified nonlinear problem. A trick avoids this:⁷ if the linearizations evaluated at $\chi = \hat{\chi}$ are isomorphisms, then the nonlinear problems are local isomorphisms. Thus we restrict the set of χ so that the next iterate $\hat{\chi} = \mathcal{K}(\chi)$ is *close* to χ . Existence and uniqueness of $\hat{\chi}$ as well as its continuous dependence on χ then follow by linearization around χ ; large perturbations are not necessary.

Boundary of the Function Set

Both method of continuity and degree theory have a common feature: the bulk of the effort goes in showing that the problem does not have solutions on the boundary of the function set. For function sets defined by inequality constraints, we have to show that for a solution of the original problem (a fixed point of the iteration), each inequality is in fact a strict ($<$ or $>$) inequality. In this step we are allowed to use the nonstrict (\leq or \geq) forms: we are moving from one well-behaved (smooth, elliptic, no vacuum, . . .) solution to another without having to show any results for arbitrary weak solutions with no prior information. In particular, we do not have to verify the strict inequalities in any particular order.

Parabolic Arcs and Corners

The core of the solution—and its most difficult part—is the treatment of the parabolic arcs: to obtain a weak solution in the limit, we have to verify the shock conditions are satisfied across the arcs. To this end we show that $\chi^{(\epsilon)}$ and $\nabla\chi^{(\epsilon)}$ are almost (up to $O(\epsilon^{1/2})$ jumps) continuous across the arcs; in the $\epsilon \downarrow 0$ limit continuity and mass conservation for $\chi^{(0)}$ are implied.

χ and $\nabla\chi$ continuity corresponds to two boundary conditions,

$$\lim_{\text{hyperbolic}} \chi = \lim_{\text{elliptic}} \chi, \quad \lim_{\text{hyperbolic}} \chi_n = \lim_{\text{elliptic}} \chi_n$$

⁷ Another, less elegant trick is to use the a priori estimates of the original problem to cut off the coefficients of the second problem so that they grow linearly and avoid all singularities like vacuum or parabolicity. The cutoff terms and their derivatives would make the linearization unnecessarily complicated.

(continuity for χ_t is implied by continuity for χ). However, with two boundary conditions the problem is *overdetermined*, at least in the regularized case. Hence we can impose only one boundary condition. A trick is needed to verify the second condition—approximately for the regularized solutions and exactly (albeit weakly) in the degenerate limit.

In each point on the arcs, there are three components of $\nabla^2\chi$, but only two relations constraining them—equation (2.14) and the tangential derivative of the boundary condition, $L^2 = 1 - \epsilon$ (see (4.49)). Our key observation is that $L^2 < 1 - \epsilon$ inside the elliptic region (as we have already shown previously), whereas $L^2 = 1 - \epsilon$ on the arcs: L has a global maximum in each arc point. Therefore $(L^2)_n \geq 0$ on the arcs, which provides an additional inequality. Thus we obtain “ $2\frac{1}{2}$ ” relations; solving them yields one explicit inequality for each second derivative, with the right-hand side depending only on χ and $\nabla\chi$. Only the twice tangential derivative matters (see (4.53)).

In the shock-arc corners χ is C^1 (Proposition 4.13, using Appendix A), so we may combine the boundary conditions (one on P , two on the shock, but with unknown shock tangent) to represent χ and $\nabla\chi$ as functions of the corner location. If we assume that the corners have precisely the “expected” location, then we know the corner values of χ and $\nabla\chi$; in particular $\chi_t = 0$, where ∂_t is the counterclockwise tangential derivative along P , $c = c_L$ in the upper left corner, and $c = c_R$ in the upper right corner. However, our analysis is greatly complicated by having *free* corners. In the other endpoint of each arc the wall boundary condition $\chi_\eta = 0$ fixes $\chi_t = 0$. Then the ordinary differential inequalities mentioned above yield *tight* bounds for χ and $\nabla\chi$ on the arcs: they have the desired values, up to $O(\epsilon)$. This fortunate circumstance is the key to our solution.⁸

We first discuss isothermal flow, which is comparatively simple. In this case the inequality for χ_t on P_R has the form

$$(4.1) \quad \chi_{tt} \geq f(\chi_t) = -\epsilon + O(\chi_t^2)$$

(see (4.66)), with the O -term constant independently of ϵ . In the wall-arc corner $\chi_t = 0$, so integration to the arc-shock corner yields

$$(4.2) \quad \chi_t \geq -O(\epsilon)$$

along the entire arc.

In Section 4.9 we compute the values of χ , $\nabla\chi$, and dependent quantities in the right shock-arc corner as the corner location varies along the arc. Formula (4.78) shows that χ_t decreases uniformly as the corner moves down from its expected location. This contradicts $\chi_t \geq -O(\epsilon)$; the second-derivative inequalities already imply a lower bound for each corner (see Proposition 4.17) to within $O(\epsilon^{1/2})$ of their expected location.

⁸It is worth noting that if, for whatever reason, we had $(L^2)_n \geq a$ for some bounded (smooth) function a , the trick would still work. The tightness does not stem from $a = 0$ (in fact, 0 is far from optimal), but rather from the characteristic degeneracy.

However, an entirely different argument (see Proposition 4.22) is needed to bound the right shock-arc corner *above*: ψ must attain a global minimum in the domain. It can never attain a local minimum in the interior, on the wall, or on the left arc. For *isothermal* flow we have $\psi_n \geq 0$ on the right arc, so the Hopf lemma excludes a minimum there as well. Hence there has to be a global minimum on the shock or in the right shock-arc corner. Essential observation: if that corner is above its expected location, then $\psi_2 > 0$ in it, as well as in any hypothetical global minimum on the shock, so we obtain a contradiction.

Having obtained upper and lower bounds on the corner, we know that $\chi_t = O(\epsilon^{1/2})$ in it. Integrating (4.1) again, but in the opposite direction, we obtain

$$(4.3) \quad \chi_t \leq -O(\epsilon^{1/2}).$$

Combined with (4.2) we have $\chi_t = O(\epsilon^{1/2})$ on the arcs. By integration we also control χ ; the boundary condition $|\nabla\chi|^2 = c^2$ yields control over χ_n .

The arguments for the left arc are analogous, with a few modifications.

For *isentropic* flow the arguments are similar, but much more complicated, because the sound speed c can vary. Instead of considering a single-variable ordinary differential inequality, we have to combine it with an ODE for c (see (4.53)). To make the system more tractable, it is linearized ((4.56) and (4.57)) and restated in polar variables ((4.58) and (4.59)). A delicate analysis (see the proofs of Propositions 4.16 and 4.17) again establishes a lower bound for the right shock-arc corner, as well as a lower bound for χ_t . For an upper bound on the corner we adapt the isothermal argument, considering minima of $\psi + a\xi$ instead of ψ for some small $a > 0$. This is necessary because for isentropic flow $\psi_n \geq -O(\epsilon^{1/2})$ rather than ≥ 0 on the right arc. More delicate analysis (see the proof of Proposition 4.22 and its references) shows that a can be chosen so large that $(\psi + a\xi)_n \geq 0$, but so small that still $(\psi + a\xi)_\eta > 0$ in the right shock-arc corner and in any hypothetical minimum point on the shock. Having bounded the corner above, an upper bound for χ_t on the arc is obtained as well.

An entirely different approach to parabolic arcs can be found in [2].

Shock Control

It is clear that we have to expect degeneracy at the parabolic arcs. However, there can be degeneracy on the elliptic side of a hyperbolic-elliptic shock, too, which we have to rule out. Moreover, we need some control over the shock location and shape.

To prove that the shock has uniform strength, we show that the density on the elliptic side is uniformly bounded below away from ρ_I . Proposition 3.5 rules out density minima in the interior of the elliptic region or at the wall (Remark 2.1). The analysis described above controls the density on the arc up to $O(\epsilon^{1/2})$ (see (4.60)). Finally, Proposition 3.13 shows that density cannot have local minima at the shock *except* in a pseudonormal point. The shock curvature must be positive (elliptic region locally convex) in such a point. Moreover, if we have a *global*

minimum in this point, then the rest of the shock, including the shock-arc corners, must be below the shock tangent in that point; otherwise there would be another shock tangent parallel to this one, but lower, meaning lower density, which is a contradiction (see Proposition 4.23 and Figure 4.8 for a detailed argument).

Hence we can have density minima at the shock, but only in pseudonormal points and *above* the line connecting the shock-arc corners. In such a point, the density is $> \rho_I$ if and only if the connecting line does not meet the circle with center \vec{v}_I and radius c_I . (Otherwise the shock vanishes or violates entropy.) This is precisely (1.1). Unfortunately, the cases where (1.1) is violated are not covered in the present paper.

Having established uniform shock strength, Proposition 3.11 shows that L cannot have local maxima (with respect to the domain) close to 1 at the shock. This is a necessary ingredient for L control in the elliptic region; the ellipticity must be uniform at the shock (away from the corners). Hence the shock is analytic except perhaps in the corners.

To control the shock tangents, ρ and L arguments are not sufficient. It is necessary to control some components of the velocity vector. More precisely, we control the horizontal (in standard coordinates) velocity v^x , as well as horizontal velocity in the L picture, which is $v^x + \alpha v^y$ in the standard picture, for some $\alpha > 0$. We show that v^x cannot have maxima in the interior (Proposition 3.8), at the wall (Proposition 3.9), or at the shock (Proposition 3.10). In the latter case, as for density, there are exceptional cases (see (3.8)) where the shock curvature (see (3.9)) is needed to rule out maxima. On the arcs we control velocity (like everything else) up to $O(\epsilon^{1/2})$ (see (4.61)).

All combined, we have that v^x must be between $v_L^x < 0$ and $v_R^x = 0$, up to $O(\epsilon^{1/2})$, and an analogous result in L -coordinates (see (4.33) and (4.34)). This yields a slew of additional information (Proposition 4.24): bounds on the shock normals (4.35), the distance of shock to wall (4.12), the shock-arc corner angles (4.14), and the vertical velocity (4.18).

Note that the vertical velocity *can* have minima at the shock, as can be observed in numerics (bottom of Figure 1.5). Other velocity directions can have extrema as well; we would be able to cover all cases of the theorem, even those violating (1.1), if we had perfect velocity control.

As a convenient side effect we have ruled out vacuum or negative densities, which is another type of singularity affecting self-similar potential flow.

4.3 Parameter Set

We consider $\gamma \in [1, \infty)$, $\rho_I \in (0, \infty)$, and $c_I \in (0, \infty)$. In addition, we use $M_I^y \in (-\infty, 0)$, which defines $\vec{v}_I = (0, M_I^y c_I)$ (in *standard* coordinates; see Figure 4.1). M_I^y is the upstream Mach number normal to the downstream wall. ρ_I

and \vec{v}_I define a potential ψ^I for the I region:

$$\psi^I(\vec{\xi}) = -\pi(\rho_I) - \frac{|\vec{v}_I|^2}{2} + \vec{v}_I \cdot \vec{\xi}.$$

Given upstream data $\vec{v}_u = \vec{v}_I$ and $\rho_u = \rho_I$, Proposition 2.12 defines a horizontal shock (R shock) with $\vec{v}_R := \vec{v}_d = 0$. Let η_R^* be its height; we choose η_R^* as a parameter that determines \vec{v}_I rather than vice versa, so that we may regard η_R^* defined as independent of γ (of course, \vec{v}_I and M_I^y now depend on γ).

Let $\epsilon \in (0, \bar{\epsilon}]$ (for some suitable small $\bar{\epsilon} > 0$, which will be fixed later). Of the two $L_d = \sqrt{1 - \epsilon}$ points for this shock, as defined by Proposition 2.11, let $\vec{\xi}_R^*$ be the one farther from the origin (see Figure 4.1). Set $\vec{n}_R = (0, -1)$.

Starting in the R shock, Proposition 2.12 yields a family of shocks with $v_d^y = 0$. Each has two $L_d = \sqrt{1 - \epsilon}$ points; let $\vec{\xi}_L^*$ be the one closer to the origin. We focus on choices $\eta_L^* \in (0, \eta_R^*]$. We call this shock the L shock. Let \vec{n}_L be the corresponding downstream normal. $\vec{\xi}_L^*$ and $\vec{\xi}_R^*$ will be called the *expected* corner locations (although they are most likely not the true locations except for $\epsilon = 0$).

Let c_C, ρ_C , and \vec{v}_C be the downstream data of the C shock ($C \in \{L, R\}$). Note that $v_L^y = 0$, but $v_L^x < 0$ for $\eta_L^* < \eta_R^*$.

Define $\vec{\xi}_{BR} := (c_R, 0)$ and $\vec{\xi}_{BL} = (v_L^x - c_L, 0)$. Let $A = (\xi_{BL}, \xi_{BR}) \times \{0\}$; we will call A the *wall* (it is only the elliptic portion of the wall). Let P_R^* be a circular arc centered in \vec{v}_R passing counterclockwise from $\vec{\xi}_{BR}$ to $\vec{\xi}_R^*$; let P_L^* be the arc centered in \vec{v}_L passing counterclockwise from $\vec{\xi}_L^*$ to $\vec{\xi}_{BL}$, both *excluding* the endpoints. P_C has radius $\sqrt{1 - \epsilon} \cdot c_C$ (for $C = L, R$).

There is a $\delta_{nt} > 0$ so that, for any unit vector \vec{t}_S from \vec{n}_R^\perp to \vec{n}_L^\perp (counterclockwise) and any unit tangent \vec{t}_P of P_L^* or P_R^* ,

$$(4.4) \quad |\vec{t}_S \times \vec{t}_P| \geq \delta_{nt} > 0.$$

We choose extended arcs $\hat{P}_{L,R}$ that overshoot $\vec{\xi}_{L,R}^*$ by an angle $\delta_{\hat{p}} > 0$, which we choose continuous in γ and η_L^* so that

$$(4.5) \quad |\vec{t}_S \times \vec{t}_P| \geq \frac{\delta_{nt}}{2}$$

for the same \vec{t}_S , but unit tangents \vec{t}_P of the *extended* arcs $\hat{P}_{L,R}$; i.e., the possible shock tangents (restricted in (4.35) below) and arc tangents are uniformly not collinear.

$P_{L,R}^*$, $\hat{P}_{L,R}$, and later $P_{L,R}$ are called *quasi-parabolic arcs* (or *parabolic arcs*, by abuse of terminology, or short *arcs*). (Of course, these arcs are *circular*; “parabolic” refers to the expected type of the PDE at these arcs.) We use $P_{L,R}^{*(\epsilon)}$ and $\hat{P}_{L,R}^{(\epsilon)}$ to identify the arcs for a particular choice of ϵ .

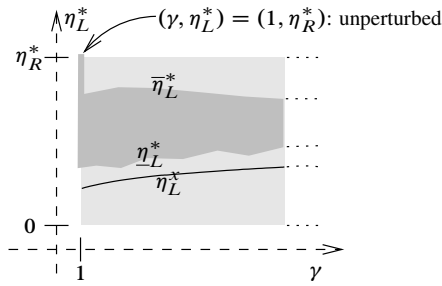


FIGURE 4.3. η_L^* below η_L^x violate (1.1). For $\gamma > 1$ some distance from η_R^* is needed for technical reasons. The Leray-Schauder degree argument uses parameters in the dark shaded set Λ , which is path-connected for small $\epsilon > 0$.

Definitions 4.1, 4.5, and 4.4 use many constants and other objects that will be fixed later on. In all of these cases, an upper (or lower) bound for each constant is found. Whenever we say “for sufficiently small constants,” for example, we mean that bounds for them are adjusted. To rule out circularity, it is necessary to specify which bounds may depend on the values of which other bounds. In the following list, bounds on a constant may only depend on bounds of other constants *before* them.

$$(4.6) \quad \delta_{\hat{p}}, C_L, C_\eta, \delta_{SA}, \delta_{Cc}, \delta_{P\sigma}, \delta_{Pn}, \delta_d, \delta_\rho, \delta_{Lb}, C_{Pt}, C_{vx}, C_{vL}, C_{Sn}, \delta_{vy}, \delta_o, C_d, \epsilon, C_c, r_I, \alpha, \beta.$$

The constants $C_c, r_I, \alpha,$ and β may depend on ϵ itself, not just on an upper bound; r_I may also depend on ψ . The reader may convince himself that the following discussion respects this order.

The parameters γ and η_L^* used in Leray-Schauder degree arguments will be restricted to compact sets below so that any constant that can be chosen continuous in them might as well be taken independent of them. Dependence on other parameters like ρ_I will not be pointed out explicitly.

Constants δ_γ as well as $\alpha, \beta - 1, r_I,$ and ϵ are meant to be small and positive; constants C_γ are meant to be large and finite.

DEFINITION 4.1 For the purposes of degree theory we define a restricted parameter set (see Figure 4.3)

$$\Lambda := \{ \lambda = (\gamma, \eta_L^*) : \gamma \in [1, \bar{\gamma}], \eta_L^* \in [\underline{\eta}_L^*(\gamma), \bar{\eta}_L^*(\gamma)] \}$$

with

$$\bar{\gamma} \in [1, \infty), \quad 0 < \underline{\eta}_L^*(\gamma) < \bar{\eta}_L^*(\gamma) \leq \eta_R^*,$$

where

$$(4.7) \quad \bar{\eta}_L^*(\gamma) = \begin{cases} \eta_R^*, & \gamma = 1, \\ \eta_R^* - C_\eta \epsilon^{1/2}, & \gamma > 1, \end{cases}$$

where C_η (to be determined in Proposition 4.22) may depend on γ but not on ϵ .

Moreover, we restrict $\underline{\eta}_L^*(\gamma)$ so that (1.1) is satisfied.

LEMMA 4.2 *For sufficiently small ϵ , with bound depending on C_η , there is an $\eta_L^x(\gamma) \in [0, \bar{\eta}_L^*(\gamma))$, continuous in γ , so that (1.1) is satisfied for all $\eta_L^* \in (\eta_L^x(\gamma), \eta_R^*]$, but never for $\eta_L^* \in (0, \eta_L^x(\gamma)]$.*

PROOF: Equation (1.1) is satisfied for $\eta_L^* = \eta_R^*$ (see Figure 1.6), by Proposition 2.11: in this case L shock and R shock coincide, and the R shock never intersects the circle with center $\vec{v}_u = \vec{v}_I$ and radius $c_u = c_I$.

Clearly the distance to that circle is strictly decreasing as η_L^* decreases. By continuity there must be an η_L^x , depending continuously on γ , so that for $\eta_L^* = \eta_L^x$ the L shock touches the circle. For all smaller $\eta_L^* > 0$ the circle is intersected, because $\eta_L^*, \eta_R^* > 0$ does not allow the L shock to pass below the circle.

Since the L shock tangent and location depends continuously on γ , η_L^x must also be continuous in it. □

LEMMA 4.3 *For sufficiently small ϵ , with bound depending on C_η , for all $\underline{\eta}_L^*(\gamma)$ (continuous in γ) that satisfy*

$$(4.8) \quad \eta_L^x(\gamma) < \underline{\eta}_L^*(\gamma) < \bar{\eta}_L^*(\gamma),$$

Λ is path-connected and contains the point $(\gamma, \eta_L^*) = (1, \eta_R^*)$.

PROOF: The curve $\gamma \mapsto \eta_R^* - C_\eta \epsilon^{1/2}$ is continuous and contained in Λ . It intersects every line $\{\gamma\} \times [\underline{\eta}_L^*, \bar{\eta}_L^*]$, including the one for $\gamma = 1$, which is $(\underline{\eta}_L^*, \eta_R^*]$. Λ is the union of all these lines, so the proof is complete. □

4.4 Function Set and Iteration

DEFINITION 4.4 Let $U \subset \mathbb{R}^n$ be open, nonempty, and bounded with ∂U uniformly Lipschitz. Let $F \subset \partial U$. For $k \in \mathbb{N}_0$, $\alpha \in [0, 1]$, and $\beta \in (-\infty, k + \alpha]$, we define the *weighted Hölder space* $\mathcal{C}_\beta^{k,\alpha}(U, F)$ as the set of $u \in \mathcal{C}^{k,\alpha}(\bar{U} - F)$ so that

$$\|u\|_{\mathcal{C}_\beta^{k,\alpha}(U,F)} := \sup_{r>0} r^{k+\alpha-\beta} \|u\|_{\mathcal{C}^{k,\alpha}(\bar{U}-B_r(F))}$$

is finite.

DEFINITION 4.5 For sufficiently small $\delta_{\hat{P}} > 0$, there is a function $b \in \mathcal{C}^2(\mathbb{R}^2)$ with $b, |\nabla b| \leq 1$ so that $b = 0$ on $\hat{P}_L^{(0)}$ and $\hat{P}_R^{(0)}$, $b > 0$ elsewhere, b even in η . From now on we fix a particular b .

PROOF: The construction is straightforward. $\delta_{\hat{P}}$ is taken so small that $\hat{P}_L^{(0)}$ and $\hat{P}_R^{(0)}$ are well separated. That is possible because $\xi_L^{*(0)} < \xi_R^{*(0)}$. □

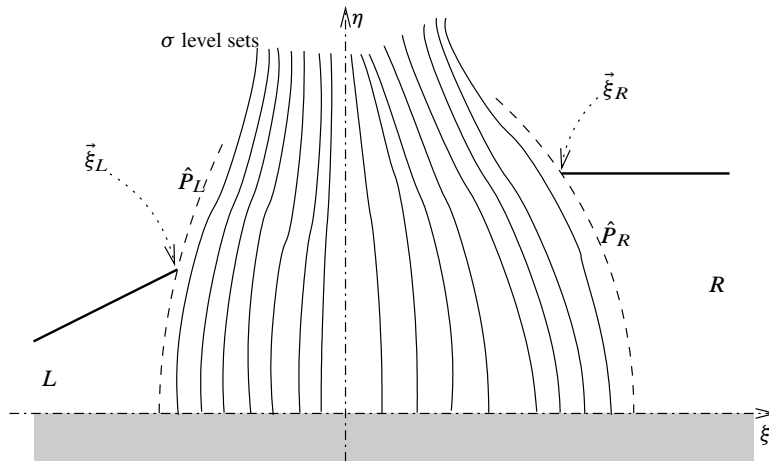


FIGURE 4.4. Onion coordinates. (ξ, η) plane shown; solid curves are σ level sets.

Onion Coordinates

To define a function subset in a fixed Banach space, we need to map the domain with its free shock boundary to a fixed square. Define a C^∞ change to coordinates $(\sigma, \eta) \in \mathbb{R}^2$ (see Figure 4.4) so that:

- η is preserved and $\sigma = \sigma(\xi, \eta)$, a C^∞ function with a C^∞ function $\xi = \xi(\sigma, \eta)$ inverting it.
- $\hat{P}_L^{(\epsilon)}$ maps to a subset of $\{\sigma = 0\}$.
- $\hat{P}_R^{(\epsilon)}$ maps to a subset of $\{\sigma = 1\}$.
- $\vec{\xi}_{BL}^{(\epsilon)}$ maps to $(0, 0)$.
- $\vec{\xi}_{BR}^{(\epsilon)}$ maps to $(1, 0)$.
- $A^{(\epsilon)}$ maps to $\{\sigma \in (0, 1), \eta = 0\}$ precisely.
- Let $T \subset \mathbb{S}^1$ be the set of unit tangents of $\hat{P}_L^{(\epsilon)}$ and $\hat{P}_R^{(\epsilon)}$. For some constant $\delta_{P\sigma} > 0$ depending only and continuously on $\lambda \in \Lambda$, require that for every unit tangent \vec{t} of a σ level set,

$$(4.9) \quad d(\vec{t}, T) \leq \delta_{P\sigma}.$$

We require that the change of coordinates depends continuously (in C^∞) on $\lambda \in \Lambda$. The construction is straightforward.

Here and in what follows, we will use the weighted Hölder spaces $C_\beta^{2,\alpha}(\bar{U})$, as in Definition 4.4. The domain U is either $[0, 1]^2$ with $F = \{(0, 1), (1, 1)\}$ or $\bar{\Omega}$ with $F = \{\vec{\xi}_L, \vec{\xi}_R\}$ (to be defined). For the shock we use $U = [0, 1]$ with $F = \{0, 1\}$, or $U = [\xi_L, \xi_R]$ with $F = \{\xi_L, \xi_R\}$; for the arcs only the upper endpoints are in F and for the wall A we have $F = \emptyset$. We omit F , as it will be clear from

the context. $C_\beta^{2,\alpha}$ are Banach spaces so that standard functional analysis applies. Moreover, $C_\beta^{2,\alpha}(\overline{\Omega})$ is continuously embedded in $C^1(\overline{\Omega})$, so we have C^1 regularity in the corners as well, which is crucial. The values for $\beta \in (1, 2)$ and $\alpha \in (0, \beta - 1]$ will be determined later.

Free Boundary Fit

Let us define $\overline{\mathcal{F}}$ be the set of functions $\psi \in C_\beta^{2,\alpha}([0, 1]^2)$ that satisfy all of the many conditions explained below. Require

$$(4.10) \quad \|\psi\|_{C_\beta^{2,\alpha}([0,1]^2)} \leq C_C(\epsilon).$$

For all $\sigma \in [0, 1]$ define

$$(4.11) \quad s(\sigma) := \frac{\psi(\sigma, 1) - \psi^I(0, 0)}{v_I^y};$$

it satisfies $\psi(\sigma, 1) = \psi^I(\xi(\sigma, \eta), \eta)$ with $\eta = s(\sigma)$. We define another coordinate transform by first mapping $(\sigma, \zeta) \in [0, 1]$ to (σ, η) with $\eta = \zeta s(\sigma)$, and then mapping to $\vec{\xi}$ with the previous coordinate transform.

Let $\vec{\xi}_L$ and $\vec{\xi}_R$ be the $\vec{\xi}$ -coordinates for the (σ, ζ) -plane points $(0, 1)$ and $(1, 1)$. Let S be the $\vec{\xi}$ -plane curve for $(0, 1) \times \{1\}$. Define P_L and P_R to be the images of $\{0\} \times (0, 1)$ and $\{1\} \times (0, 1)$, respectively. Finally, let Ω be the image of $(0, 1) \times (0, 1)$.

We require shock-wall separation:

$$(4.12) \quad \max_{\sigma \in [0,1]} s(\sigma) \geq \delta_{SA} > 0$$

and that

$$(4.13) \quad \xi = \xi(\sigma, \zeta) \text{ is strictly increasing in } \sigma \text{ for } \zeta = 1.$$

For corners close to the target, we require

$$(4.14) \quad |\eta_L - \eta_L^*|, |\eta_R - \eta_R^*| \leq \epsilon^{1/2},$$

and we require ϵ to be so small that $\vec{\xi}_C \in \hat{P}_C$ ($C = L, R$); i.e., $\vec{\xi}_C$ may not be higher than the upper endpoint of \hat{P}_C . For later use we define $\eta_{L,R}^\pm := \eta_{L,R}^* \pm \epsilon^{1/2}$ and let $\vec{\xi}_{L,R}^\pm$ be so that $\vec{\xi}_{L,R}^\pm \in \hat{P}_{L,R}$.

For corner cones, we require

$$(4.15) \quad \sup_{\vec{\xi}, \vec{\xi}' \in \overline{\Omega}} \angle(\vec{\xi} - \vec{\xi}_C, \vec{\xi}' - \vec{\xi}_C) \leq \pi - \delta_{Cc} \quad (C \in \{L, R\}).$$

(As discussed in the introduction, $\angle(\vec{x}, \vec{y})$ is the counterclockwise angle from \vec{x} to \vec{y} .)

Inequality (4.12) ensures that the map from (σ, ζ) to $\vec{\xi}$ is a well-defined change of coordinates, uniformly nondegenerate (depending on δ_{SA} and C_C), with regularity that is either $C_\beta^{2,\alpha}([0, 1]^2)$ or $C_\beta^{2,\alpha}(\overline{\Omega})$. Since the step from (σ, ζ) to (σ, η) uses ψ , the entire coordinate change is as smooth as ψ . If we prove higher regularity for ψ either in (σ, ζ) – or (ξ, η) –coordinates, we immediately obtain the same higher regularity for the coordinate transform and in the respective other coordinates.

It is clear now that $\partial\Omega$ is the union of the disjoint sets $S, P_L, P_R, A, \{\vec{\xi}_R\}, \{\vec{\xi}_L\}, \{\vec{\xi}_{BL}\},$ and $\{\vec{\xi}_{BR}\}$. By (4.12), Ω is a simply connected set.

Equality (4.13) ensures that s can be defined as a function of ξ , which is the way we use it from now on.

Iteration

For shock strength/density, we require that

$$(4.16) \quad -\chi - \frac{1}{2}|\nabla\chi|^2 > 0,$$

so that ρ is well-defined, and that

$$(4.17) \quad \min_{\overline{\Omega}} \rho \geq \rho_I + \delta_\rho.$$

The vertical velocity is

$$(4.18) \quad \psi_\eta \geq v_I^y + \delta_{vy} \quad \text{in } \overline{\Omega}.$$

For the pseudo-Mach number bound, we require

$$(4.19) \quad L^2 \leq 1 - \delta_{Lb} \cdot b \quad \text{in } \overline{\Omega}.$$

(Note that L is well-defined because by (4.17) $\rho > 0$, so $c > 0$.) $b = 0$ on $\hat{P}_L^{(0)}$ and $\hat{P}_R^{(0)}$, which have distance $\geq \frac{\epsilon}{3}$ (for sufficiently small ϵ) from $\overline{\Omega}$, so (4.19) implies

$$(4.20) \quad L^2 \leq 1 - \frac{1}{3}|\nabla b|_{L^\infty} \delta_{Lb} \cdot \epsilon \leq 1 - \frac{1}{3} \delta_{Lb} \epsilon \quad \text{in } \overline{\Omega}.$$

We also require that there be a function $\hat{\psi} \in C_\beta^{2,\alpha}(\overline{\Omega})$ with the following properties:

- (1) ψ close to $\hat{\psi}$:

$$(4.21) \quad \|\psi - \hat{\psi}\|_{C_\beta^{2,\alpha}([0,1]^2)} \leq r_I(\psi)$$

where $r_I \in C(\overline{\mathcal{F}}; (0, \infty))$ is a continuous function to be determined later. Here and later we regard $\hat{\psi}$ as defined on $[0, 1]^2$ instead of $\overline{\Omega}$ via the coordinate transform from $\vec{\xi}$ to (σ, ζ) defined by ψ (see above).

(2) We require r_I to be so small that

$$(4.22) \quad -\hat{\chi} - \frac{1}{2}|\nabla \hat{\chi}|^2 > 0,$$

$$(4.23) \quad \nabla \hat{\psi} \neq \bar{v}_I,$$

so that in particular $\hat{\rho}$ is well-defined and positive.

(3) Moreover, we require r_I to be so small that

$$(4.24) \quad \left(c_0^2 + (1 - \gamma) \left(\chi + \frac{1}{2}|\nabla \hat{\chi}|^2 \right) \right) I - \nabla \hat{\chi}^2 > 0;$$

i.e., the left-hand side is a (symmetric) positive definite matrix.

(4) Let $\mathcal{L} = \mathcal{L}(\psi, \hat{\psi})$ be defined as

$$(4.25) \quad \left(\left(\left(c_0^2 + (1 - \gamma) \left(\chi + \frac{1}{2}|\nabla \hat{\chi}|^2 \right) \right) I - \nabla \hat{\chi}^2 \right) : \nabla^2 \hat{\psi}, \right.$$

$$(4.26) \quad \left. \frac{|\nabla \hat{\chi}|^2}{2} + \frac{(1 - \epsilon)((\gamma - 1)\chi + c_0^2)}{2 + (1 - \epsilon)(\gamma - 1)}, \right.$$

$$(4.27) \quad \left. (\hat{\rho} \nabla \hat{\chi} - \rho_I \nabla \chi^I) \cdot \frac{\bar{v}_I - \nabla \hat{\psi}}{|\bar{v}_I - \nabla \hat{\psi}|}, \right.$$

$$(4.28) \quad \left. \hat{\psi}_\eta \right).$$

(4.27) is well-defined by (4.22) and (4.23). The other components have no singularities.

$\nabla \psi \in C_{\beta-1}^{1,\alpha}$, so $|\nabla \chi|^2 \in C_{\beta-1}^{1,\alpha}$, so

$$\left(\left(c_0^2 + (1 - \gamma) \left(\chi + \frac{1}{2}|\nabla \hat{\chi}|^2 \right) \right) I - \nabla \hat{\chi}^2 \right) \in C_{\beta-1}^{1,\alpha} \hookrightarrow C^{0,\beta-1} \hookrightarrow C^{0,\alpha}$$

(note $\alpha \leq \beta - 1$ as required above), and $\nabla^2 \psi \in C_{\beta-2}^{0,\alpha}$, so (4.25) is $\in C_{\beta-2}^{0,\alpha}$. In the same way we check that (4.26), (4.27), and (4.28) are $C_{\beta-1}^{1,\alpha}$.

Hence we may take the codomain of \mathcal{L} to be the Banach space

$$Y := C_{\beta-2}^{0,\alpha}(\bar{\Omega}) \times C_{\beta-1}^{1,\alpha}(\bar{S}) \times C_{\beta-1}^{1,\alpha}(\bar{P}_L) \times C_{\beta-1}^{1,\alpha}(\bar{P}_R) \times C_{\beta-1}^{1,\alpha}(\bar{A}).$$

Alternatively, if we consider the pullback to (σ, ζ) -coordinates, as defined by ψ above, we may consider

$$C_{\beta-2}^{0,\alpha}([0, 1]^2) \times C_{\beta-1}^{1,\alpha}[0, 1] \times C_{\beta-1}^{1,\alpha}[0, 1] \times C_{\beta-1}^{1,\alpha}[0, 1] \times C_{\beta-1}^{1,\alpha}[0, 1].$$

In the same way we can discuss $\hat{\psi}$ either in $C_{\beta}^{2,\alpha}(\bar{\Omega})$ or in $C_{\beta}^{2,\alpha}([0, 1]^2)$.

With these topologies, clearly \mathcal{L} is a smooth function of ψ and $\hat{\psi}$.

Most importantly, we require

$$(4.29) \quad \mathcal{L}(\psi, \hat{\psi}) = 0.$$

Other Bounds

We require

$$(4.30) \quad \|\psi\|_{C^{0,1}(\bar{\Omega})} \leq C_L$$

where C_L may not depend on ϵ .

χ_t and χ_n on the parabolic arcs satisfy

$$(4.31) \quad \max_{P_L \cup P_R} c^{-1} \left| \frac{\partial \chi}{\partial t} \right| \leq C_{Pt} \epsilon^{1/2},$$

$$(4.32) \quad \max_{P_L \cup P_R} c^{-1} \frac{\partial \chi}{\partial n} \leq -\delta_{Pn}.$$

Here δ_{Pt} and δ_{Pn} may depend *only* on λ , but not on ϵ (or ψ).

The horizontal velocity satisfies

$$(4.33) \quad \max_{\bar{\Omega}} \psi_{\xi} \leq C_{vx} \epsilon^{1/2}.$$

Moreover,

$$(4.34) \quad \max_{\bar{\Omega}} \nabla \psi \times \vec{n}_L \geq \vec{v}_L \times \vec{n}_L + C_{vL} \epsilon^{1/2}.$$

For the shock normal, let $N \subset \mathbb{S}^1$ (unit circle) be the set from \vec{n}_R to \vec{n}_L (counterclockwise). We require

$$(4.35) \quad \sup_S d(\vec{n}, N) \leq C_{Sn} \epsilon^{1/2}.$$

Set $\Sigma^1 := P_L$, $\Sigma_2 := S$, $\Sigma_3 := P_R$, and $\Sigma_4 := A$. Write the components (4.26), (4.27), and (4.28) of \mathcal{L} as

$$g^i(\vec{\xi}, \hat{\chi}(\vec{\xi}), \underbrace{\nabla \hat{\chi}(\vec{\xi})}_{=: \vec{p}}) \quad (i = 1, \dots, 4),$$

where the $\vec{\xi}$ -dependence includes the dependence on $\chi(\vec{\xi})$ and $\nabla \chi(\vec{\xi})$.

g^2 has some singularities, but not on the set of $\vec{\xi}$, χ , and $\nabla \chi$ so that (4.18) and (4.17) (as well as (4.22) and (4.23)) are satisfied. That set is simply connected, so we can modify g^2 on its complement and extend it smoothly to $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2$. The modification is chosen to depend smoothly on λ .

We also require uniform obliqueness:⁹

$$(4.36) \quad |g_{\vec{p}}^i \cdot \vec{n}| \geq \delta_o |g_{\vec{p}}^i| \quad \forall \vec{\xi} \in \Sigma^i.$$

⁹In various articles Lieberman uses the term “uniformly oblique,” probably with “oblique” in the sense of “not tangential.” We adopt this terminology instead of the more common but less useful sense “not normal” (see, e.g., [17] or [28]).

Regarding functional independence in upper corners, for $i, j = 1, 2$ and for $i, j = 2, 3$, set

$$G := \begin{bmatrix} g_{p^1}^i & g_{p^1}^j \\ g_{p^2}^i & g_{p^2}^j \end{bmatrix},$$

regard it as a function of $\vec{\xi}$ (including the dependence on $\nabla \hat{\chi}(\xi)$) and require

$$(4.37) \quad \|G\|, \|G^{-1}\| \leq C_d \quad \text{in } B_{\delta_d}(\vec{\xi}_C) \cap \bar{\Omega}, C = L, R.$$

Let $\bar{\mathcal{F}}$ be the set of admissible functions so that all of these conditions are satisfied.¹⁰ Define \mathcal{F} to be the set of admissible functions such that all of these conditions are satisfied with *strict* inequalities, i.e., replace \leq and \geq by $<$ and $>$, “increasing” by “strictly increasing,” etc.

Remark 4.6. If $\hat{\psi} = \psi$, then (4.25), (4.27), (4.26), and (4.28) and the definition of S yield

$$\begin{aligned} (c^2 I - \nabla \chi^2) : \nabla^2 \psi &= 0 && \text{in } \bar{\Omega}, \\ \chi_\eta &= 0 && \text{on } \bar{A}, \\ \chi^I &= \chi, \\ (\rho \nabla \chi - \rho_I \nabla \chi^I) \cdot \vec{n} &= 0 && \text{on } \bar{S}, \\ L &= \sqrt{1 - \epsilon} && \text{on } \bar{P}_L \cup \bar{P}_R \end{aligned}$$

(we may take closures by regularity (4.10)).

Remark 4.7. In any point on \bar{A} , we can use even reflection (see Remark 2.1) of ψ across A to obtain a point in the interior or (in the bottom corners) a point at a quasi-parabolic arc with the interior equation applying inside. $\psi_\eta = 0$ on A , for even reflection of ψ , implies that the solution is C^1 across \bar{A} ; then necessarily it is also $C^{2,\alpha}$ (away from the shock-arc corners).

For ψ standard regularity theory immediately yields that the solution is analytic in the bigger domain near \bar{A} . The same technique applied to $\hat{\psi}$ and to solutions $\hat{\psi}$ of linearized equations (here ψ , $\hat{\psi}$, and $\hat{\psi}$ are reflected) yields $C^{2,\alpha}$ regularity (away from the shock-arc corners).

PROPOSITION 4.8 *For sufficiently small ϵ (with bound depending only on C_{Pt}) and r_I (depending continuously and only on ψ, δ_{vy}), for all $\psi \in \bar{\mathcal{F}}$, $\mathcal{L}(\psi, \hat{\psi}')$ is well-defined for $\hat{\psi}'$ near ψ , and the Fréchet derivative $\partial \mathcal{L} / \partial \hat{\psi}'(\psi, \psi)$ (of \mathcal{L} with respect to its second argument $\hat{\psi}'$, evaluated at $\hat{\psi}' = \psi$) is a linear isomorphism of $C_\beta^{2,\alpha}$ onto Y .*

¹⁰The notation $\bar{\mathcal{F}}$ does not necessarily imply that $\bar{\mathcal{F}}$ is the closure of \mathcal{F} .

PROOF: (4.27) is the only part of \mathcal{L} with a singularity. (4.17) and (4.23) guarantee that it is well-defined and smooth for $\hat{\psi} = \psi$. For $\hat{\psi}$ in a sufficiently small neighborhood of ψ (i.e., take r_I small), it stays well-defined.

Let $\hat{\psi} \in \mathcal{C}_\beta^{2,\alpha}(\bar{\Omega})$; it is meant to be the first variation of $\hat{\psi}$ (and, at the same time, $\hat{\chi}$). $\partial\mathcal{L}/\partial\hat{\psi}(\psi, \psi)\hat{\psi}$ is a tuple of functions in Y . Its first component (see (4.25)) is of type

$$(4.38) \quad (c^2I - \nabla\chi^2) : \nabla^2\hat{\psi} + \vec{b} \cdot \nabla\hat{\psi}$$

where \vec{b} is some vector field.

We linearize (4.27): here we can use that we linearize at $\hat{\psi} = \psi$, so

$$|\vec{v}_I - \nabla\psi|^{-1}(\vec{v}_I - \nabla\psi) = \vec{n}$$

because $\psi = \psi^I$ defines the shock (see (4.11) in Definition 4.4). The result is

$$(4.39) \quad \rho(1 - c^{-2}\chi_n^2)\hat{\psi}_n - \rho c^{-2}\chi_n\chi_t\hat{\psi}_t - \rho c^{-2}\hat{\psi} - (\rho\nabla\chi - \rho_I\nabla\chi^I)|\vec{v}_I - \nabla\psi|^{-1} \underbrace{\left(1 - \underbrace{\left(\frac{\vec{v}_I - \nabla\psi}{|\vec{v}_I - \nabla\psi|}\right)^2}_{=\vec{n}}\right)}_{=\vec{t}\vec{t}^\top} \nabla\hat{\psi}.$$

The coefficient $\rho(1 - c^{-2}\chi_n^2)$ of $\hat{\psi}_n$, which is positive by (4.17) and (4.20), has the opposite sign to the nonzero coefficient $-\rho c^{-2}$ of $\hat{\psi}$, which is negative by (4.17).

The other components linearize to

$$(4.40) \quad \nabla\chi \cdot \nabla\hat{\psi}, \quad (\text{parabolic})$$

$$(4.41) \quad \hat{\chi}_n. \quad (\text{wall})$$

Note that the coefficient vectors $g_{\vec{p}}^i$ of $\nabla\hat{\psi}$ are the same as in (4.36) and (4.37).

To prove the proposition, we apply a theorem of Lieberman [24, theorem 1.4]. Our $C_\beta^{k,\alpha}(\Omega; F)$ spaces correspond to his spaces $H_{k+\alpha}^{(-\beta)}$ if $\alpha > 0$ and $F \subset \partial\Omega$ is finite and contains all points where $\partial\Omega$ is not C^2 —as is the case here. We check the preconditions:

- (1) Σ_i are C^2 curves (except perhaps for the endpoints), meeting in single points (corners). In each corner the two curves meet at an angle $0 < \theta_{ij} \leq \theta_0$, with $\theta_0 < \pi$. ($0 < \theta_{ij}$ is trivial from (4.35)). The lower bound is obvious; the upper bound is obvious for corners with A and supplied by (4.15) for corners with S .
- (2) The equation is uniformly elliptic, by (4.20).
- (3) All boundary operators are uniformly oblique, by (4.36).
- (4) Condition (1.17) in [24] with $\beta_i \neq \beta_j$ is equivalent to (4.37).

- (5) Finally, the *only* appearance of $\hat{\psi}$ is in (4.39), where its coefficient is nonzero and has the opposite sign to the coefficient of $\hat{\psi}_n$ (note that \vec{n} is the *inward* normal here). (This observation allows us to apply maximum principles to obtain uniqueness.) This means (1.19) in [24] is satisfied.
- (6) The bottom corners are not a concern: by Remark 4.7 we can use even reflection across A .
- (7) The other preconditions are technical and easy to verify.

Theorem 1.4 in [24] yields that $\partial\mathcal{L}/\partial\hat{\chi}$ is an isomorphism on $\mathcal{C}_\beta^{2,\alpha}$ onto Y if we choose $\alpha \in (0, 1)$ and $\beta \in (1, 2)$ sufficiently small, depending on the constants $\delta_o, C_d, \delta_{Lb}, \delta_d,$ and δ_{Cc} . □

PROPOSITION 4.9 *r_I can be chosen so that $\hat{\psi}$ is unique and depends continuously on λ and on $\psi \in \overline{\mathcal{F}}$ (both ψ and $\hat{\psi}$ in the $\mathcal{C}_\beta^{2,\alpha}([0, 1]^2)$ topology).*

PROOF: We use the subscript λ for \mathcal{L} and $\overline{\mathcal{F}}$ here to indicate their dependence on it.

Take $r_I := 1$ first. $\mathcal{L}_\lambda(\psi, \hat{\psi}')$ is well-defined for all $\psi \in \overline{\mathcal{F}}_\lambda$ and $\hat{\psi}' = \psi$, as well as sufficiently small perturbations of $\hat{\psi}'$, by (4.16) and (4.20). It is easy to check that there is an $r_2 > 0$, depending continuously on ψ and λ , so that $\hat{\psi}' \in B_{r_2}(\psi) \mapsto \mathcal{L}_\lambda(\psi, \hat{\psi}')$ is well-defined and C^1 . Take $r_I \leftarrow \min\{r_I, r_2\}$. This may shrink $\overline{\mathcal{F}}_\lambda$, but the properties of \mathcal{L} are not changed.

Consider a particular $\lambda \in \Lambda$ and a corresponding $\psi \in \overline{\mathcal{F}}_\lambda$. By Proposition 4.8 and the inverse/implicit function theorem for Banach spaces, there is an $r \in (0, r_I]$ so that $\hat{\psi}' \in B_r(\psi) \mapsto \mathcal{L}_{\lambda'}(\psi', \hat{\psi}')$ is a diffeomorphism for every $\psi' \in B_r(\psi)$ and $\lambda' \in \Lambda \cap B_r(\lambda)$.

Let r_3 be the supremum of all r with this property. $(\psi, \lambda) \mapsto r_3(\psi, \lambda)$ is continuous: for any $\psi'' \in B_r(\psi)$ and $\lambda'' \in B_r(\lambda)$, set $r'' := r - \max\{|\psi - \psi''|, |\lambda - \lambda''|\}$. Then $B_{r''}(\psi'') \subset B_r(\psi)$ and $B_{r''}(\lambda'') \subset B_r(\lambda)$, so $\hat{\psi}' \in B_{r''}(\psi'') \mapsto \mathcal{L}_{\lambda'}(\psi', \hat{\psi}')$ is a diffeomorphism for all $\psi' \in B_{r''}(\psi'')$ and $\lambda' \in \Lambda \cap B_{r''}(\lambda'')$. Therefore

$$r_3(\psi'', \lambda'') \geq r_3(\psi, \lambda) - \max\{|\psi - \psi''|, |\lambda - \lambda''|\}.$$

On the other hand, if $\max\{|\psi - \psi''|, |\lambda - \lambda''|\} < \frac{1}{2}r_3(\psi, \lambda)$, then $r_3(\psi'', \lambda'') \geq r_3(\psi, \lambda)/2$, so $\psi \in B_{r_3(\psi'', \lambda'')}(\psi'')$ and $\lambda \in B_{r_3(\psi'', \lambda'')}(\lambda'')$, so we may apply the same argument with roles reversed to obtain

$$r_3(\psi, \lambda) \geq r_3(\psi'', \lambda'') - \max\{|\psi - \psi''|, |\lambda - \lambda''|\}.$$

Clearly r_3 is continuous.

For $r = r_3$, the property need not hold, but we take $r_I \leftarrow \min\{r_I, \frac{1}{2}r_3\}$. This may shrink $\overline{\mathcal{F}}$ more, but again the properties of \mathcal{L} and r_2 above are not changed. With this choice, $\hat{\psi}$ from Definition 4.4 must be unique (determined by (4.21) and (4.29)). It is also clear from the properties above that $(\psi, \lambda) \mapsto \hat{\psi}$ is a continuous (in fact, C^1) map. □

PROPOSITION 4.10 For δ_{Tt} , ϵ , and r_I sufficiently small, for all continuous paths $t \in [0, 1] \mapsto \lambda(t)$ in Λ , $\bigcup_{t \in (0,1)} (\{t\} \times \mathcal{F}_{\lambda(t)})$ is open and $\bigcup_{t \in [0,1]} (\{t\} \times \overline{\mathcal{F}}_{\lambda(t)})$ is closed in $[0, 1] \times C_{\beta}^{2,\alpha}([0, 1]^2)$.¹¹

PROOF: All conditions on ψ in Definition 4.4 are inequalities that can be made scalar by taking a suitable supremum or infimum. Then their sides are continuous under $C_{\beta}^{2,\alpha}([0, 1]^2)$ changes to ψ , and therefore $\hat{\psi}$. (Most inequalities need only $C^1([0, 1]^2)$.)

(1) To show closedness, consider sequences (t_n, ψ_n) in $\bigcup_{t \in [0,1]} (\{t\} \times \overline{\mathcal{F}}_{\lambda(t)})$ that converge to a limit (t, ψ) . Let $\hat{\psi}_n$ be associated to ψ_n as in Definition 4.4. By continuity (Proposition 4.9), $(\hat{\psi}_n)$ converges to a limit $\hat{\psi}$ as well. By continuity of \mathcal{L} in ψ , $\hat{\psi}$, and λ , we have $\mathcal{L}_{\lambda(t)}(\psi, \hat{\psi}) = 0$ as well.

By (4.11) for s_n and ψ_n instead of s and ψ , (s_n) converges in $C_{\beta}^{2,\alpha}[0, 1]$ as well to a limit s that satisfies (4.11) itself.

Most conditions on ψ are nonstrict and continuous inequalities, so they are still satisfied by ψ . We check the strict inequalities explicitly and in order:

- (4.13) is implied by (4.35) and the nondegeneracy of the $(\sigma, \zeta) \mapsto (\xi, \eta)$ coordinate change.
- (4.16) is implied by (4.17).
- By choosing r_I sufficiently small, (4.22) is implied by (4.21), (4.23) by (4.18), and (4.24) by (4.20).

All inequalities are satisfied, so $\psi \in \overline{\mathcal{F}}$.

(2) For openness, the proof is the same by using that all inequalities are strict now, by definition of \mathcal{F} , hence preserved by sufficiently small perturbations. \square

DEFINITION 4.11 Define $\mathcal{K} : \overline{\mathcal{F}} \rightarrow C_{\beta}^{2,\alpha}([0, 1]^2)$ to map ψ into $\hat{\psi}$ as given in Definition 4.4, but pulled back to (σ, ζ) -coordinates and the $[0, 1]^2$ domain (see Definition 4.4) with the coordinate transform defined by ψ .

4.5 Regularity and Compactness

To obtain regularity at the free boundary, we need a kludge: a transformation to a fixed boundary problem. With some further advances in elliptic theory this step should become obsolete.

Remark 4.12. Using the (σ, η) -coordinates from Definition 4.4, define the coordinate transformation from $\bar{\xi}$ to $x = (\sigma, \eta)$ and then to $y = (\sigma, \mu)$ where

$$\mu := \psi(\sigma, \eta) - \psi^I(\sigma, \eta) = \psi(\sigma, \eta) - \psi^I(0, 0) - v_I^y \eta.$$

$\psi = \psi^I$ on the shock, so $\mu = 0$ there; $\mu > 0$ in Ω (by (4.18)).

¹¹We make no statement about $\overline{\mathcal{F}}$ being the closure of \mathcal{F} . It certainly contains the closure, but it could be bigger, for example, if one of the inequalities in Definition 4.4 becomes nonstrict in the interior without being violated.

We check that this transformation satisfies all conditions of Proposition B.1 and Remark B.2, where we take $u = \psi$. The $\vec{\xi}$ -to- x part is trivial, since it is independent of ψ . We study the second part:

$$y_x = \frac{\partial(\sigma, \mu)}{\partial(\sigma, \eta)} = \begin{bmatrix} 1 & 0 \\ 0 & -v_I^y \end{bmatrix}, \quad y_u = \frac{\partial(\sigma, \mu)}{\partial\psi} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u_x = [\psi_\sigma \quad \psi_\eta].$$

Obviously y_x is uniformly invertible, and

$$y_x + y_u u_x = \begin{bmatrix} 1 & 0 \\ \psi_\sigma & \psi_\eta - v_I^y \end{bmatrix}$$

is uniformly invertible by (4.18). The norms are bounded by constants depending only on δ_{vy} (to bound $\psi_\eta - v_I^y$ below away from 0) and C_L (to bound ψ and $\nabla\psi$).

All conditions of Proposition B.1 and Remark B.2 are satisfied, so we obtain the analogues of (4.20), (4.36), and (4.37): the coordinate transformation yields a uniformly elliptic equation with uniformly oblique boundary conditions that are uniformly functionally independent (as functions of $\nabla\psi$) near the corner. The constant for each property grows at most by a factor depending continuously and only on δ_{vy} and C_L .

PROPOSITION 4.13 *For sufficiently small $\beta \in (1, 2)$ and $\alpha \in (0, \beta - 1)$, depending only on $C_d, \delta_{Lb} \cdot \epsilon, \delta_o, C_L,$ and δ_{vy} :*

(1)

$$(4.42) \quad \|s\|_{C^{0,1}[\xi_L, \xi_R]} \leq C_{sL}$$

and

$$(4.43) \quad \|s\|_{C_\beta^{2,\alpha}[\xi_L, \xi_R]} \leq C_s$$

for $C_{sL} = C_{sL}(C_L, \delta_{vy})$ and $C_s = C_s(C_C, \delta_{vy})$.

(2) For a fixed point ψ of \mathcal{K} :

(a) (4.30) is strict for sufficiently large C_L .

(b) (4.10) is strict for sufficiently large $C_C = C_C(C_d, \delta_{Lb} \cdot \epsilon, C_L, \delta_o, \delta_{vy}, \delta_d)$.

(c) For $K \Subset \bar{\Omega} - \{\vec{\xi}_L, \vec{\xi}_R\}$ and all $k \geq 0, \alpha' \in (0, 1)$,

$$(4.44) \quad \|\psi\|_{C^{k,\alpha'}(K)} \leq C_{CK}$$

where $C_{CK} = C_{CK}(d, C_L, \delta_o, \delta_{vy})$ is decreasing in $d := d(K, \hat{P}_L^{(0)} \cup \hat{P}_R^{(0)})$ and not dependent on ϵ .

(d) ψ is analytic in $\bar{\Omega} - \{\vec{\xi}_L, \vec{\xi}_R\}$.

(3) For sufficiently small $r_I > 0$, depending continuously and only on ψ , there are $\delta_\alpha, \delta_\beta > 0$ so that for all $\psi \in \mathcal{F}$,

$$(4.45) \quad \|\hat{\psi}\|_{C_{\beta+\delta_\beta}^{2,\alpha+\delta_\alpha}(\bar{\Omega})} \leq C_{\mathcal{K}}.$$

Here $C_{\mathcal{K}}, \delta_\alpha,$ and δ_β depend only on $C_d, \delta_{Lb} \cdot \epsilon, \delta_o, C_L,$ and δ_{vy} .

PROOF:

(1) The shock is implicitly defined by

$$\psi(\xi, s(\xi)) - \psi^I(\xi, s(\xi)) = 0 \quad (\xi \in [\xi_L, \xi_R])$$

(compare Definition 4.4 and (4.11)). The derivative with respect to s of the left-hand side is $\psi_\eta - v_I^y$. (4.18) bounds this away from 0. The implicit function theorem yields the derivative part of (4.42); bounds on s itself are supplied by integration and the $\vec{\xi}_R^*$ part of (4.14) (for example). The implicit function theorem also yields (4.43) from (4.10). Henceforth we use without further mention that estimates on χ automatically yield corresponding estimates on s .

(2) Now consider a fixed point $\psi = \mathcal{K}(\psi)$.

(a) (4.20) implies $L \leq 1$, which can be rewritten

$$(4.46) \quad |\nabla\chi|^2 \leq \frac{2(c_0^2 + (1 - \gamma)\chi)}{\gamma + 1}.$$

To control χ , we use that $\chi(\vec{\xi}_R) = \chi^I(\vec{\xi}_R)$, which is bounded by (4.14). Using the differential inequality (4.46) vertically downwards from $\vec{\xi}_R$ to A , then along A , and finally along any vertical line upward to S , we achieve uniform bounds on χ and $|\nabla\chi|$ in all of $\bar{\Omega}$. This is (4.30), which is strict if we take C_L large enough.

(b) By (4.20) the equation

$$(c^2 I - \nabla\chi^2) : \nabla^2\psi = 0$$

is uniformly elliptic. Standard De Giorgi–Nash–Moser and Schauder theory [17] converts (4.30) into interior $C^{k,\alpha}$ estimates for any $k \geq 2$ and $\alpha \in (0, 1)$.

At the shock and near the shock-parabolic corners we first apply the coordinate transform from Remark 4.12. This is necessary because we have only a $C^{0,1}$ bound of S at this point; using $C^{1,\beta}$ regularity would cause circularity. The transform yields a new problem with fixed boundary (both arc and shock are mapped into straight line segments). The uniform ellipticity (4.20), the uniform obliqueness of the boundary conditions (4.36), and their uniform functional independence in the corner (4.37) are preserved by the transform.

It is also crucial that the boundary conditions themselves be smooth. For this purpose we have combined the two shock conditions (2.18) and (2.19) so that the boundary operator in (4.27) is smooth in $\vec{\xi}$, $\hat{\chi}(\xi)$, and $\nabla\hat{\chi}(\xi)$.¹² This also requires a bound of ρ below away from 0, by (4.17), as well as a bound of $\nabla\psi$ away from \vec{v}_I , by (4.18).

¹²If we had used (2.19) by itself, it would contain \vec{n} for which we have only an L^∞ bound at this point.

At the boundaries away from the corners, we use [22], which yields $C^{2,\alpha}$ regularity for any $\alpha \in (0, 1)$. Near the corners we apply Proposition A.1. It yields $C_\beta^{2,\alpha}$ regularity for some range of $\alpha \in (0, 1)$ and $\beta \in (1, 2)$; we fix α and β below.

We obtain (4.10), with $C_C = C_C(C_d, \delta_{Lb} \cdot \epsilon, C_L, \delta_o, \delta_{vy}, \delta_d)$. δ_{vy} is due to the transform from Remark 4.12. Moreover, $C_S = C_S(C_C, \delta_{vy})$, as discussed above.

- (c) In every $K \Subset \Omega$ with positive distance to P_L and P_R , we use that (4.19) provides

$$\sup_K L^2 \leq 1 - \delta_{Lb} \sup_K b$$

where the right-hand side is < 1 and *independent* of ϵ (by Definition 4.5). Thus the equation is uniformly elliptic in K , with ellipticity constant depending on $d(K, \overline{P}_L^{(0)} \cup \overline{P}_R^{(0)})$, but independent of ϵ . Since the obliqueness constant δ_o in (4.36) was already independent of ϵ , the same arguments as before yield uniform-in- ϵ regularity in each K for any Hölder norm. This is (4.44).

- (d) For elliptic problems with analytic coefficients, analyticity of the solution is classical (e.g., [26, theorem 6.7.6']) in the case of fixed boundaries. To deal with a free boundary, we use the transform from Remark 4.12 again. The theorem just cited yields analyticity for the new problem. The inverse coordinate transformation is defined in terms of the new coordinates and the solution of the new problem, so it is analytic as well. Then ψ itself is analytic in $\overline{\Omega} - \{\vec{\xi}_L, \vec{\xi}_R\}$.

- (3) For a general $\psi \in \overline{\mathcal{F}}$, not necessarily a fixed point of \mathcal{K} , ψ and $\hat{\psi}$ in \mathcal{L} are different. The boundary conditions are oblique derivative conditions:¹³

$$g^k(\vec{\xi}, \hat{\psi}(\vec{\xi}), \nabla \hat{\psi}(\vec{\xi})) = 0.$$

Each g^k is $C_\beta^{2,\alpha}$ in $\vec{\xi}$ (from χ in (4.26)),¹⁴ in particular $C^{1,\beta-1}$.

Moreover, each g^k is C^∞ in $\hat{\psi}(\vec{\xi})$ and $\nabla \hat{\psi}(\vec{\xi})$. For (4.27) this requires that $\hat{\rho}$ be uniformly bounded below away from 0, and $\nabla \hat{\psi}$ uniformly bounded away from \vec{v}_I . This is guaranteed by (4.17) and (4.18) (bounding ρ and $\nabla \psi$, respectively) combined with (4.21) (bounding $|\hat{\rho} - \rho|$ and $|\nabla \psi - \nabla \hat{\psi}|$, respectively), if r_I is chosen small enough (depending only and continuously on ψ and δ_{vy}).

¹³For a *Dirichlet* boundary condition, $C^{1,\beta}$ data would yield $C^{1,\beta}$, but not $C^{1,\beta+\delta_\beta}$ regularity. But for a *derivative* condition we can gain one order.

¹⁴Here it is crucial that $\nabla \psi$ not appear *anywhere* in (4.26), (4.27), and (4.28); otherwise g^k would only be $C_\beta^{1,\alpha}$ in $\vec{\xi}$, which is not enough to gain regularity.

The boundaries Γ^k are A, P_L , and P_R , which are perfectly smooth, and S has a $C_\beta^{2,\alpha}$ bound (4.43), implying $C^{1,\beta}$.¹⁵

Uniform ellipticity, obliqueness, and functional independence still hold by (4.20), (4.36), and (4.37), combined with (4.21), for r_I sufficiently small(er).

All combined, Proposition A.1 and [23, cor. 1.4] (away from the corners) yield $C_\lambda^{2,\kappa}$ regularity. Here, κ depends on β only, whereas λ does not depend on α or β at all. Therefore we may pick $\beta = 1 + \lambda/2$ and then $\alpha = \kappa/2$ for the resulting κ . With $\delta_\beta = \lambda/2$ and $\delta_\alpha = \kappa/2$, (4.45) is satisfied for suitable C_κ . \square

PROPOSITION 4.14 *For sufficiently small $r_I > 0$ (depending continuously and only on ψ and δ_{vy}), \mathcal{K} is a continuous and compact function of ψ and λ on $\bigcup_{\lambda \in \Lambda} \overline{\mathcal{F}}_\lambda$.*

PROOF: We have already shown in Proposition 4.8 that $\hat{\psi}$ is a continuous function of ψ . Therefore \mathcal{K} depends continuously on λ and ψ .

Moreover, \mathcal{K} is compact: by (4.45) in Proposition 4.13 the range of $\hat{\psi}$ is a bounded subset of $C_{\beta+\delta_\beta}^{2,\alpha+\delta_\alpha}(\overline{\Omega})$, which is precompact in $C_\beta^{2,\alpha}(\overline{\Omega})$. Pullback to (σ, ζ) -coordinates is continuous in the latter topology, so the image under it is still pre-compact.¹⁶ Altogether \mathcal{K} is a compact map. \square

4.6 Pseudo-Mach Number Control

PROPOSITION 4.15 *For ϵ and δ_{Lb} sufficiently small, with bounds depending only on δ_ρ , if $\psi \in \overline{\mathcal{F}}$ is a fixed point of \mathcal{K} , then (4.19) is strict and*

$$(4.47) \quad L^2 < 1 - \epsilon \quad \text{in } \overline{\Omega} - \overline{P}_L - \overline{P}_R.$$

PROOF: We have

$$d(\overline{\Omega}, \hat{P}_L^{(0)} \cup \hat{P}_R^{(0)}) \geq \frac{1}{3} \cdot \epsilon$$

for ϵ small enough. Remember from Definition 4.5 that $b = 0$ on $\hat{P}_L^{(0)} \cup \hat{P}_R^{(0)}$. Therefore, on $P_L \cup P_R$,

$$L^2 = 1 - \epsilon < 1 - \underbrace{\|b\|_{C^{0,1}}}_{\leq 1} \cdot d(P_L \cup P_R, \hat{P}_L^{(0)} \cup \hat{P}_R^{(0)}) \leq 1 - \delta_{Lb} \cdot b,$$

e.g., for $\delta_{Lb} \leq 1$.

On the shock, we may use (4.17) combined with Proposition 3.11 to rule out that $L^2 + \delta_{Lb} \cdot b$ has a maximum in a point where $L < 1$ and $L \geq 1 - \delta_{LS}$. Here δ_{Lb} has to be chosen so that $|\delta_{Lb} \nabla b| \leq \delta_{LS}$ is satisfied. (Now δ_{Lb} depends

¹⁵Here we can apply Proposition A.1 without the coordinate transform from Remark 4.12, because the shock is already known to be $C^{1,\beta}$.

¹⁶We may actually lose regularity because the pullback is defined by ψ , hence only $C_\beta^{2,\alpha}$, but we do retain compactness.

continuously on δ_ρ as well.) Then $L^2 + \delta_{Lb} \cdot b < 1$ as well if, again, δ_{Lb} is small enough.

In addition, we can choose δ_{Lb} so small that $\delta_{Lb} \cdot b$ satisfies the preconditions of theorem 1 and theorem 2 in [12] (where it is called b). Note that b is even in η by Definition 4.5, so $\partial b / \partial n = 0$ on the wall. Let $\delta_{L\Omega}$ be the δ from those theorems (it depends only and continuously on λ). Then $L^2 + \delta_{Lb} \cdot b$ cannot have a maximum in a point where $L^2 \geq 1 - \delta_{L\Omega}$. If, again, δ_{Lb} is chosen sufficiently small (no new dependencies), then $L^2 + \delta_{Lb} \cdot b < 1$ in $\Omega \cup A$, hence in $\bar{\Omega}$. Therefore (4.19) is strict.

(4.47) can be shown in the same manner, by taking $b = 0$ instead, using the actual boundary condition $L = \sqrt{1 - \epsilon}$ on P_L and P_R and considering $\epsilon < \delta_{LS}, \delta_{L\Omega}$. □

4.7 Second Derivatives on Arcs

Consider a fixed point $\chi \in \bar{\mathcal{F}}$ of \mathcal{K} so that $L^2 = 1 - \epsilon$ is satisfied on P_L and P_R . This can be restated as

$$(4.48) \quad \frac{1}{2} |\nabla \chi|^2 + \frac{(1 - \epsilon)((\gamma - 1)\chi - c_0^2)}{\gamma + 1 - \epsilon(\gamma - 1)} = 0.$$

In what follows, \vec{n} will be an outer normal to P_L or P_R and $\vec{t} = \vec{n}^\perp$; ∂_n and ∂_t are defined accordingly. When considering a particular point on an arc, we also use τ and ν , which are Cartesian coordinates such that $\partial_\tau = \partial_t$ and $\partial_\nu = \partial_n$ in that point (only).

Take ∂_t of (4.48) (using $\partial_t \partial_\tau = \partial_\tau^2$ and $\partial_t \partial_\nu = \partial_\tau \partial_\nu$):

$$(4.49) \quad \begin{aligned} 0 &= \chi_\tau \chi_{\tau\tau} + \chi_\nu \chi_{\nu\tau} + \frac{(\gamma - 1)(1 - \epsilon)}{\gamma + 1 - \epsilon(\gamma - 1)} \chi_\tau \\ &= \chi_\tau \psi_{\tau\tau} + \chi_\nu \psi_{\nu\tau} - \frac{2}{\gamma + 1 - \epsilon(\gamma - 1)} \chi_\tau. \end{aligned}$$

We have two equations ((4.49) and the interior equation (2.14)) for the three components of $D^2 \chi$. That is the case for other equation and boundary conditions as well; it is not really sufficient for any control. But here there is an additional tool: (4.47) implies that L attains its maximum in $\bar{\Omega}$ in every point on $P_L \cup P_R$. Therefore $(L^2)_\nu \geq 0$ on $P_L \cup P_R$, or equivalently

$$(4.50) \quad \chi_\tau \psi_{\tau\nu} + \chi_\nu \psi_{\nu\nu} - \frac{2}{\gamma + 1 - \epsilon(\gamma - 1)} \chi_\nu \geq 0.$$

From here on—and in the next few sections—we focus on P_R ; all results have an analogous extension to P_L .

Combine (4.50), (4.49), and (2.15):

$$(4.51) \quad \begin{bmatrix} \chi_\tau \chi_\nu & 0 \\ 0 & \chi_\tau & \chi_\nu \\ c^2 - \chi_\tau^2 & -2\chi_\tau \chi_\nu & c^2 - \chi_\nu^2 \end{bmatrix} \begin{bmatrix} \psi_{\tau\tau} \\ \psi_{\nu\tau} \\ \psi_{\nu\nu} \end{bmatrix} = \frac{2}{\gamma + 1 - \epsilon(\gamma - 1)} \begin{bmatrix} \chi_\tau \\ \chi_\nu \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$$

where we use a “slack variable” $a \geq 0$. We obtain in particular that

$$(4.52) \quad \psi_{\tau\tau} = \frac{2}{\gamma + 1 - \epsilon(\gamma - 1)} \left(\frac{1 + \epsilon}{1 - \epsilon} \cdot \frac{\chi_\tau^2}{c^2} - \epsilon \right) - \underbrace{\frac{(c^2 - \chi_\nu^2)\chi_\nu}{(1 - \epsilon)c^4}}_{\geq 0} \cdot a,$$

(where we used $\chi_\nu^2 = (1 - \epsilon)c^2 - \chi_\tau^2$). The a term is positive due to $\chi_n < 0$ (by (4.32)).

Now consider radial coordinates, centered in the origin, with $\phi = 0$ corresponding to the positive ξ -axis. P_R is at a fixed radius $r = (1 - \epsilon)^{1/2}c_R$ and covers $\phi \in [0, \bar{\phi}]$. In these coordinates,

$$\begin{aligned} \chi_{\phi\phi} &= r^2(\chi_{\tau\tau} - r^{-1}\chi_\nu) = r^2(\psi_{\tau\tau} - 1 - r^{-1}\chi_\nu) \geq f(c^2, \chi_\phi), \\ f(h, p) &:= \frac{2}{\gamma + 1 - \epsilon(\gamma - 1)} \left(\frac{1 + \epsilon}{1 - \epsilon} \cdot \frac{p^2}{h} - \epsilon r^2 \right) - r^2 \\ &\quad + \sqrt{r^2 h(1 - \epsilon) - p^2}. \end{aligned}$$

On the other hand, $c^2 = c_0^2 + (1 - \gamma)(\chi + |\nabla\chi|^2/2)$ and (4.48) yield

$$(4.53) \quad (c^2)_\phi = g(c^2, \chi_\phi), \quad g(h, p) := \underbrace{\frac{-2(\gamma - 1)}{\gamma + 1 - \epsilon(\gamma - 1)}}_{=: \sigma_g} p.$$

We seek stationary points of the ODE system $(p_\phi, h_\phi) = (f, g)$. $g(h_0, p_0) = 0$ obviously requires $p_0 = 0$.

$$(4.54) \quad \begin{aligned} 0 &= f(h_0, 0) = -r^2 \left(1 + \frac{2\epsilon}{\gamma + 1 - \epsilon(\gamma - 1)} \right) + r\sqrt{h_0}\sqrt{1 - \epsilon} \\ \Rightarrow \quad h_0 &= \left(1 + \frac{2\epsilon}{\gamma + 1 - \epsilon(\gamma - 1)} \right)^2 \frac{r^2}{1 - \epsilon}. \end{aligned}$$

g is already linear; we linearize f around the stationary point:

$$\partial_h f(h_0, 0) = r \frac{\sqrt{1 - \epsilon}}{2\sqrt{h_0}} = \frac{1 - \epsilon}{2(1 + \frac{2\epsilon}{\gamma + 1 - \epsilon(\gamma - 1)})} =: \sigma_f.$$

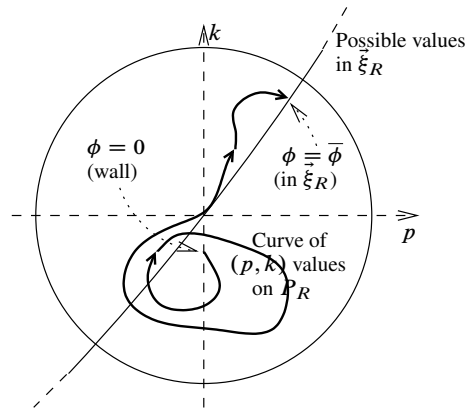


FIGURE 4.5. $\chi, \nabla\chi$ on P_R define a path $\phi \mapsto (p(\phi), k(\phi))$ (bold curve). The path is constrained by (4.57) and (4.56). The possible k, p in $\vec{\xi}_R$ are on the thin solid curve which is parametrized by η_R ; one endpoint of the path is on this curve. The other endpoint is on the k axis.

For easier treatment of the isentropic case, we change coordinates again (see Figure 4.5): take

$$(4.55) \quad k := \left(-\frac{\sigma_f}{\sigma_g} \right)^{1/2} (h - h_0).$$

From linearization we have

$$(4.56) \quad \begin{aligned} p_\phi &\geq \sigma_f (h - h_0) + O((h - h_0)^2 + p^2) \\ &= \underbrace{(-\sigma_f \sigma_g)^{1/2}}_{=: \sigma_\theta} \cdot k + O(k^2 + p^2), \end{aligned}$$

$$(4.57) \quad k_\phi = \left(-\frac{\sigma_f}{\sigma_g} \right)^{1/2} \sigma_g p = -(-\sigma_f \sigma_g)^{1/2} p = -\sigma_\theta p.$$

Here and later, $O(x^\alpha)$ is a term with absolute value $\leq Cx^\alpha$ for $|x| \leq R$, where C and R may depend only on C_L , but none of the other constants. While the elliptic equation degenerates as $\epsilon \downarrow 0$, all ODE we discuss here are well-behaved for $\epsilon \downarrow 0$. However, some $\gamma \downarrow 1$ are delicate, because $\sigma_g = \gamma - 1 + O(\epsilon)$, so (4.55) has a singularity; we make detailed comments in each case.

Represent (p, k) by radial coordinates (q, θ) , with $\theta = 0$ corresponding to the positive p -axis and $\theta = \frac{\pi}{2}$ to the positive k -axis. Then

$$(4.58) \quad q_\phi = \partial_\phi \sqrt{p^2 + k^2} = \frac{pp_\phi + kk_\phi}{q} = \begin{cases} \geq O(q^2), & p \geq 0, \\ \leq O(q^2), & p \leq 0. \end{cases}$$

$$(4.59) \quad \theta_\phi = \frac{pk_\phi - kp_\phi}{q^2} = \begin{cases} \leq O(q) - \sigma_\theta, & k \geq 0, \\ \geq O(q) - \sigma_\theta, & k \leq 0. \end{cases}$$

4.8 Arc Control

PROPOSITION 4.16 *If $C_{P_t} < \infty$ is sufficiently large, if $\delta_{P_n} > 0$ is sufficiently small, if ϵ is sufficiently small, and if C_{P_v} and C_{P_ρ} are sufficiently large, with bounds depending on C_{P_t} , then for any fixed point χ of \mathcal{K} , (4.31) and (4.32) are strict, and*

$$(4.60) \quad |\rho - \rho_C| \leq C_{P_\rho} \epsilon^{1/2},$$

$$(4.61) \quad |\vec{v} - \vec{v}_C| \leq C_{P_v} \epsilon^{1/2} \quad \text{on } P_C \quad (C = L, R).$$

In the case where $\gamma > 1$,

$$(4.62) \quad \begin{aligned} \theta(\phi) &\notin \left(\frac{\pi}{2}, \frac{3\pi}{2} - \sigma_\theta \bar{\phi} \right) + 2\pi\mathbb{Z} \quad \text{on } P_R, \\ \theta(\phi) &\notin \left(-\frac{\pi}{2} + \sigma_\theta \bar{\phi}, \frac{\pi}{2} \right) + 2\pi\mathbb{Z} \quad \text{on } P_L \end{aligned}$$

(if $q(\phi) \neq 0$).

PROOF: We focus on P_R first.

For $\vec{\xi}_R = \vec{\xi}_R^*$ we have $p(\bar{\phi}) = 0$ and $c(\bar{\phi}) = c_R$, by construction of the R shock in Section 4.3. $p(\bar{\phi})$ and $c(\bar{\phi})$ depend smoothly on $\vec{\xi}_R$, so (4.14) yields

$$(4.63) \quad |p(\bar{\phi})| \leq C_{pk} \epsilon^{1/2},$$

$$(4.64) \quad |c(\bar{\phi}) - c_R| \leq C_{pk} \epsilon^{1/2},$$

in $\vec{\xi}_R$ for some constant C_{pk} . $L^2 = 1 - \epsilon$, combined with (4.31), as well as (4.31) integrated in the tangential direction, then implies

$$(4.65) \quad q(\phi) \leq C_q \epsilon^{1/2} \quad \forall \phi \in [0, \bar{\phi}]$$

for $C_q = C_q(C_{P_t})$. (C^q may be $\geq C_{P_t}$, so this does not imply the sharp form of (4.31) yet.)

Isothermal Case

For $\gamma = 1$ we need to consider only a single differential inequality, since c is constant. (4.53) takes the form

$$(4.66) \quad p_\phi \geq -(1 - \epsilon)c^2 \cdot \epsilon + O(p^2).$$

$\phi = 0$ is the corner between A and P_R where $\chi_\phi = r\chi_\eta = 0$ by the boundary condition on A , so $p(\phi = 0) = 0$. Integrating (4.66) from 0 to ϕ yields

$$(4.67) \quad p(\phi) \geq O(\epsilon).$$

On the other hand $p(\bar{\phi})$ is controlled by (4.63). Integrating (4.66) from $\bar{\phi}$ to ϕ yields

$$(4.68) \quad p(\phi) \leq O(\epsilon^{1/2}).$$

(4.67) and (4.68) combine to

$$c^{-1}r^{-1} \max_{\phi \in [0, \bar{\phi}]} p(\phi) < C_{P_t} \epsilon^{1/2}$$

for ϵ sufficiently small and C_{P_t} sufficiently large. This is the strict form of (4.31).

Isentropic Case

For $\gamma > 1$, we first show (4.62). We fix $\theta \in [0, 2\pi)$ here.

Assume that $q(\phi') > 0$ and $\theta(\phi') \in (\pi, \frac{3\pi}{2} - \sigma_\theta \bar{\phi})$ for some $\phi' \in [0, \bar{\phi}]$. Necessarily $\phi' > 0$ because $p(\phi = 0) = 0$, so either $q(\phi = 0) = 0$ or $\theta(\phi = 0) = \pi \pm \frac{\pi}{2}$.

Let $\phi_0 \in [0, \phi')$ be maximal so that

$$\theta(\phi_0) \notin (\pi, \frac{3\pi}{2}) \quad \text{or} \quad q(\phi_0) = 0.$$

Such a ϕ_0 must exist because $p(\phi = 0) = 0$. For $\phi \in (\phi_0, \phi')$, $k(\phi) < 0$ and $p(\phi) < 0$.

By the $p \leq 0$ part of (4.58) (in the reverse direction), $q(\phi) > 0$ for $\phi \in [\phi_0, \phi']$. On the other hand, (4.65) applies. Therefore $q(\phi_0) = 0$ is not possible, so either $\theta(\phi_0) = \pi$ or $\theta(\phi_0) = \frac{3\pi}{2}$. (4.57) with $p < 0$ (in the reverse direction) shows that $k(\phi_0) \leq k(\phi') < 0$, so $\theta(\phi_0) = \pi$ is not possible.

For ϵ sufficiently small, with bound depending on C_q , the $k \leq 0$ part of (4.59) yields

$$\theta(\phi_0) \leq \theta(\phi') + \sigma_\theta \underbrace{(\phi' - \phi_0)}_{\leq \bar{\phi}} + O(\epsilon^{1/2}),$$

so since $\bar{\phi} < \frac{\pi}{2}$ and $\sigma_\theta < 1$ (for ϵ small),

$$\theta(\phi_0) \in (\pi, \frac{3\pi}{2}).$$

Contradiction! So ϕ' cannot exist; $\theta(\phi) \notin (\pi, \frac{3\pi}{2} - \sigma_\theta \bar{\phi})$ for any $\phi \in [0, \bar{\phi}]$.

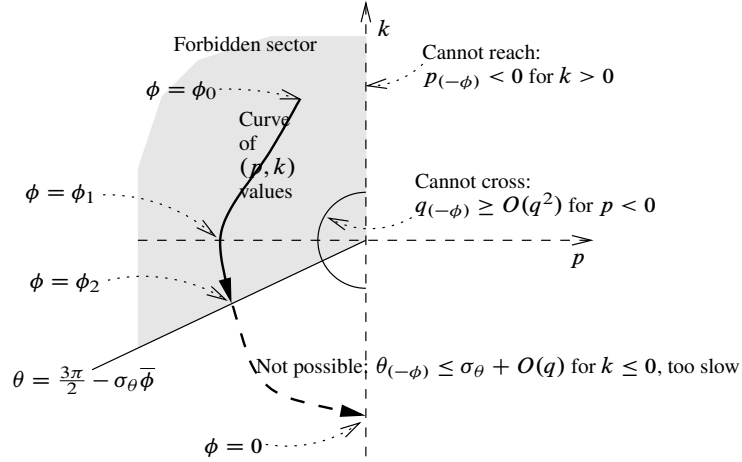


FIGURE 4.6. The second-derivative inequality (4.59), combined with the wall condition $p = 0$, rules out a sector of χ_ϕ, c^2 values.

Now assume $q(\phi_0) \neq 0$ and $\theta(\phi_0) \in (\frac{\pi}{2}, \pi]$ for some ϕ_0 (see Figure 4.6). Let $\phi_1 \in [0, \phi_0)$ be maximal so that $q(\phi_1) = 0$ or $\theta(\phi_1) \notin (\frac{\pi}{2}, \pi]$. Again, $p(\phi = 0) = 0$, so such a ϕ_1 must exist. For $\phi \in (\phi_1, \phi_0]$, $k(\phi) \geq 0 > p(\phi)$.

The $p < 0$ part of (4.58) (in the reverse direction) shows that $q(\phi) > 0$ for all $\phi \in [\phi_1, \phi_0]$, so $q(\phi_1) = 0$ is not possible.

For sufficiently small ϵ , by using (4.65), (4.56) yields $p_\phi(\phi) > 0$ for those ϕ . Thus $p(\phi_1) < p(\phi_0)$, so $\theta(\phi_1) = \frac{\pi}{2}$ is not possible either.

Only $\theta(\phi_1) = \pi$ remains, but then $p(\phi_1) < 0 = k(\phi_1)$, so (4.57) yields $k_\phi(\phi_1) > 0$. Therefore $\theta(\phi_1 - \delta_1) = \pi + \delta_2$ for some small $\delta_1 \in (0, \phi_1)$ and $\delta_2 > 0$. This is in the sector we have already ruled out—contradiction. The proof of (4.62) is complete.

Define

$$Q := \begin{cases} q, & p \geq 0, \\ |k|, & p \leq 0. \end{cases}$$

By (4.62), for $p < 0$ necessarily $k \leq p \cot(\sigma_\theta \bar{\phi}) < 0$, so

$$Q_{(-\phi)} = -k_{(-\phi)} \stackrel{(4.57)}{=} -\sigma_\theta p \leq -\sigma_\theta k \tan(\sigma_\theta \bar{\phi}) = \sigma_\theta \tan(\sigma_\theta \bar{\phi}) Q.$$

For $p \geq 0$, (4.58) yields $Q_{(-\phi)} \leq O(Q^2)$. Altogether $Q_{(-\phi)} \leq O(Q)$ (where O is with respect to $|Q| \downarrow 0$), so integrating from $\bar{\phi}$ backwards yields

$$Q(\phi) \leq O(Q(\bar{\phi})) \quad (\phi \in [0, \bar{\phi}]).$$

Using $|p| \leq |k| \tan(\sigma_\theta \bar{\phi})$ for $p \leq 0$ again, we get

$$c^{-1} r^{-1} q(\phi) \leq c^{-1} r^{-1} O(q(\bar{\phi})) < C_{P_t} \epsilon^{1/2} \quad (\phi \in [0, \bar{\phi}]),$$

for sufficiently large C_{P_t} and for ϵ sufficiently small with bound depending on C_{P_t} only. This implies the strict form of (4.31).

The strict form of (4.32) is immediate from $L^2 = 1 - \epsilon$, in the form

$$\chi_n^2 = -\sqrt{(1 - \epsilon)c^2 - \chi_t^2}$$

(note that (4.32) (nonstrict) fixes the sign). Here δ_{P_n} is sufficiently small.

Density and Velocity

Finally, we obtain (4.60). (4.14) combined with $\chi = \chi^I$ on the shock yields

$$|\chi(\vec{\xi}_R) - \chi^I(\vec{\xi}_R^*)| \leq C\epsilon^{1/2}$$

for some constant C . Integrating (4.31) along P_R , we obtain

$$\max_{P_R} |\chi - \chi^I(\vec{\xi}_R^*)| \leq C'\epsilon^{1/2}$$

for some other constant C' . Combined with (4.31) and $L^2 = 1 - \epsilon$, this implies (4.60) for C , C' , and C_{P_ρ} depending only on C_{P_t} .

(4.61) is shown in the same manner.

Left Arc

P_L can be discussed in the same fashion by noticing that in the L picture (Figure 4.1) it has the same properties as P_R , except for the wall not being horizontal, which does not matter. Note that the mirror reflection in going back to the R picture reverses the direction of ϕ and therefore changes θ to $-\theta$ (see (4.62)). □

4.9 Corners Moving along Arcs

Since our shock is a free boundary, we cannot be sure where the shock-arc corner is located. We study the behavior of shock normal and downstream data when keeping the upstream data ρ_u, \vec{v}_u fixed, imposing $L_d = \sqrt{1 - \epsilon}$ and restricting the shock location to be on P_R . (This is different from Proposition 2.12, where we imposed $v_d^y = 0$ instead of the location.)

All the calculations in this section are done for the corner between S and P_R , but each has an analogous result for the left corner, using L -coordinates (Figure 4.1).

We abbreviate $\omega := \eta_R$. In this section ∂_ω refers to derivatives of upstream and downstream quantities as ω is varied, while keeping $\vec{\xi}_R \in P_R$ and maintaining the shock conditions and the parabolic boundary condition. We use dot notation, such as $\dot{\cdot}$, on relations that hold only for $\omega = \eta_R^*$. No ∂_ω may be taken of such relations. In addition, we use the notation from Section 2.4; $\vec{v}_u = \vec{v}_I$, $\vec{z} = \nabla\chi$, $\vec{v} = \nabla\psi$, etc.

We required $\vec{\xi}_R \in \hat{P}_R^{(0)}$, so

$$\xi = \sqrt{(1 - \epsilon)c_R^2 - \eta^2}.$$

Moreover,

$$\begin{aligned}\vec{n} &\stackrel{\bullet}{=} (0, -1), & \vec{t} &\stackrel{\bullet}{=} (1, 0), \\ z_u^x &= -\xi, & z_u^y &= v_u^y - \eta, & z_u^n &\stackrel{\bullet}{=} \eta - v_u^y, & z_u^t &\stackrel{\bullet}{=} -\xi, \\ z_d^x &\stackrel{\bullet}{=} -\xi, & z_d^y &\stackrel{\bullet}{=} -\eta, & z_d^n &\stackrel{\bullet}{=} \eta, & z_d^t &\stackrel{\bullet}{=} -\xi.\end{aligned}$$

Thus,

$$\begin{aligned}\partial_\omega \xi &= \partial_\omega \left(\sqrt{(1-\epsilon)c_R^2 - \eta^2} \right) = -\frac{\eta}{\xi}, \\ \partial_\omega n^y &= -\partial_\omega \sqrt{1 - (n^x)^2} = -\frac{n^x}{n^y} \partial_\omega n^x \stackrel{\bullet}{=} 0, \\ \partial_\omega z_u^x &= -\partial_\omega \xi = \frac{\eta}{\xi}, & \partial_\omega z_u^y &= -1, \\ \partial_\omega (|\vec{z}_u|^2) &= \partial_\omega (\xi^2 + (\eta - v_u^y)^2) = 2\xi \left(-\frac{\eta}{\xi} \right) + 2(\eta - v_u^y) = -2v_u^y.\end{aligned}$$

We use

$$\begin{aligned}L_d^2 &= 1 - \epsilon \\ \Leftrightarrow |\vec{z}_d|^2 &= (1 - \epsilon)c_d^2 = (1 - \epsilon)c_u^2 + \frac{(\gamma - 1)(1 - \epsilon)}{2} (|\vec{z}_u|^2 - |\vec{z}_d|^2) \\ (4.69) \quad \Leftrightarrow |\vec{z}_d|^2 &= \frac{2(1 - \epsilon)}{\gamma + 1 + \epsilon(1 - \gamma)} c_u^2 + \frac{(\gamma - 1)(1 - \epsilon)}{\gamma + 1 + \epsilon(1 - \gamma)} |\vec{z}_u|^2\end{aligned}$$

(4.69) yields

$$(4.70) \quad \partial_\omega (|\vec{z}_d|^2) = \frac{\gamma - 1}{\gamma + 1} \partial_\omega (|\vec{z}_u|^2) + O(\epsilon) = \frac{-2(\gamma - 1)}{\gamma + 1} v_u^y + O(\epsilon).$$

On the other hand,

$$\partial_\omega (|\vec{z}_d|^2) = 2z_d^x \partial_\omega (z_d^x) + 2z_d^y \partial_\omega (z_d^y) \stackrel{\bullet}{=} -2\xi \partial_\omega (z_d^x) - 2\eta \partial_\omega (z_d^y),$$

so

$$(4.71) \quad \xi \partial_\omega (z_d^x) + \eta \partial_\omega (z_d^y) \stackrel{\bullet}{=} \frac{\gamma - 1}{\gamma + 1} v_u^y + O(\epsilon).$$

(2.19) can be restated:

$$\begin{aligned}0 &= \rho_u z_u^n - \rho_d z_d^n \\ &= \rho_u (z_u^x n^x + z_u^y n^y) - \pi^{-1} \left(\pi(\rho_u) + \frac{|\vec{z}_u|^2 - |\vec{z}_d|^2}{2} \right) (z_d^x n^x + z_d^y n^y).\end{aligned}$$

Take ∂_ω :

$$\begin{aligned}
 \Rightarrow \quad 0 &\equiv \rho_u(n^x \partial_\omega z_u^x + z_u^x \partial_\omega n^x + n^y \partial_\omega z_u^y + z_u^y \partial_\omega n^y) \\
 &\quad - \frac{\rho_d}{2c_d^2} (\partial_\omega (|\vec{z}_u|^2) - \partial_\omega (|\vec{z}_d|^2)) z_d^n \\
 &\quad - \rho_d (n^x \partial_\omega z_d^x + z_d^x \partial_\omega n^x + n^y \partial_\omega z_d^y + z_d^y \partial_\omega n^y) \\
 &\stackrel{(4.70)}{\equiv} \rho_u (-\xi \partial_\omega n^x + 1) + \frac{2\rho_d v_u^y z_d^n}{c_d^2 (\gamma + 1)} - \rho_d (-\xi \partial_\omega n^x - \partial_\omega z_d^y) \\
 \Rightarrow \quad &\xi \left(1 - \frac{\rho_u}{\rho_d}\right) \partial_\omega n^x + \partial_\omega z_d^y \stackrel{\bullet}{=} -\frac{\rho_u}{\rho_d} - \frac{2v_u^y \eta}{c_d^2 (\gamma + 1)} \\
 (4.72) \quad \Rightarrow \quad &\frac{\xi v_u^y}{v_u^y - \eta} \partial_\omega n^x + \partial_\omega z_d^y \stackrel{\bullet}{=} \frac{-\eta}{\eta - v_u^y} - \frac{2v_u^y \eta}{c_d^2 (\gamma + 1)}
 \end{aligned}$$

using $\rho_u/\rho_d = z_d^n/z_u^n \stackrel{\bullet}{=} \eta/(\eta - v_u^y)$. Finally, $z_u^t = z_d^t$ yields

$$0 = n^x (z_d^y - z_u^y) - n^y (z_d^x - z_u^x),$$

so take ∂_ω :

$$\begin{aligned}
 0 &= \partial_\omega n^x (z_d^y - z_u^y) + n^x (\partial_\omega z_d^y - \partial_\omega z_u^y) - \partial_\omega n^y (z_d^x - z_u^x) \\
 &\quad - n^y (\partial_\omega z_d^x - \partial_\omega z_u^x) \\
 &\stackrel{\bullet}{=} -v_u^y \partial_\omega n^x + \partial_\omega z_d^x - \frac{\eta}{\xi} \\
 (4.73) \quad \Rightarrow \quad &-v_u^y \partial_\omega n^x + \partial_\omega z_d^x \stackrel{\bullet}{=} \frac{\eta}{\xi}.
 \end{aligned}$$

(4.71), (4.72), and (4.73) form a linear (nondegenerate) system for the three derivatives. The solution is

$$(4.74) \quad \partial_\omega z_d^y = \frac{(\gamma - 1)v_u^y - 2(\gamma + 1)\eta - 2c_d^{-2}v_u^y \eta(\eta - v_u^y)}{(\gamma + 1)(2\eta - v_u^y)} + O(\epsilon).$$

Using $v_d^y = z_d^y + \eta$, we have

$$(4.75) \quad \partial_\omega v_d^y \stackrel{\bullet}{=} 2v_u^y \frac{\eta(v_u^y - \eta)c_d^{-2} - 1}{(\gamma + 1)(2\eta - v_u^y)} + O(\epsilon).$$

Note that

$$(4.76) \quad \partial_\omega v_d^y \stackrel{\bullet}{>} 0$$

(for sufficiently small ϵ) because $v_u^y = v_I^y < 0$ and $\eta - v_u^y \stackrel{\bullet}{=} z_u^n > 0$.

Transform (4.71) to

$$(4.77) \quad \partial_\omega (z_d^x) \stackrel{\bullet}{=} -\frac{\eta}{\xi} \partial_\omega (z_d^y) + \frac{\gamma - 1}{\gamma + 1} \cdot \frac{v_u^y}{\xi} + O(\epsilon)$$

(no need to evaluate further; $\partial_\omega (n^x)$ is not needed).

Finally, the counterclockwise unit tangent for P_R is

$$\vec{t}_P = \frac{\vec{\xi}^\perp}{|\vec{\xi}|} = r^{-1} \vec{\xi}^\perp,$$

so

$$p = \chi\phi = r\chi_t = r\vec{z} \cdot \vec{t}_P = \vec{\xi}^\perp \cdot \vec{z} = \xi z_d^y - \eta z_d^x.$$

Thus

$$\begin{aligned} \partial_\omega p &= \partial_\omega(\xi z_d^y - \eta z_d^x) = -\frac{\eta z_d^y}{\xi} + \xi \partial_\omega z_d^y - z_d^x - \eta \partial_\omega z_d^x \\ &\stackrel{\bullet}{=} \frac{\eta^2}{\xi} + \xi + \xi \partial_\omega z_d^y - \eta \partial_\omega z_d^x = \underbrace{\frac{\eta^2 + \xi^2}{\xi}}_{=(1-\epsilon)c_R^2/\xi} + \xi \partial_\omega z_d^y - \eta \partial_\omega z_d^x \\ &\stackrel{(4.77)}{=} \frac{\eta((\gamma+1)v_u^y - 2\gamma\eta) - 2c_d^2}{(\gamma+1)(2\eta - v_u^y)} \cdot \frac{v_u^y}{\xi} + O(\epsilon). \end{aligned}$$

Using $\eta \stackrel{\bullet}{=} z_d^n =: \sigma c_u$, $v_u^y = z_u^y + \eta \stackrel{\bullet}{=} \eta - z_u^n = (\sigma - L_u^n)c_u$, as well as

$$c_d^2 = c_u^2 + \frac{\gamma-1}{2}((z_u^n)^2 - (z_d^n)^2) = \left(1 + \frac{\gamma-1}{2}((L_u^n)^2 - \sigma^2)\right) c_u^2,$$

we obtain a more convenient formula:

$$(4.78) \quad p_\omega \stackrel{\bullet}{=} \frac{2 + L_u^n((\gamma+1)\sigma + (\gamma-1)L_u^n)}{L_u^n + \sigma} \cdot \frac{-v_u^y c_u}{(\gamma+1)\xi} + O(\epsilon).$$

Since $\sigma = z_d^n/c_u < z_u^n/c_u = L_u^n$ for any admissible shock, we can argue that

$$\begin{aligned} p_\omega &\stackrel{\bullet}{\geq} \frac{2 + (\gamma-1)L_u^n(\sigma + L_u^n)}{L_u^n + \sigma} \cdot \frac{-v_u^y c_u}{(\gamma+1)\xi} + O(\epsilon) \\ &\geq \left(\frac{2}{L_u^n + \sigma} + (\gamma-1)L_u^n\right) \cdot \frac{-v_u^y c_u}{(\gamma+1)\xi} + O(\epsilon) \\ (4.79) \quad &\geq (\delta_{p\eta} + (\gamma-1)\sigma) \cdot \frac{-v_u^y c_u}{(\gamma+1)\xi} + O(\epsilon) \end{aligned}$$

for some $\delta_{p\eta} > 0$. (Note: $L_u^n + \sigma$ is uniformly bounded because the set of possible shock locations is bounded.) Also,

$$(4.80) \quad \begin{aligned} (c_d^2 - h_0)_\omega &= (c_d^2)_\omega \stackrel{L_d^2=1-\epsilon}{=} (1-\epsilon)\partial_\omega(|\vec{z}_d|^2) \\ &\stackrel{(4.70)}{=} \frac{-2(\gamma-1)}{\gamma+1} v_u^y + O(\epsilon) > 0 \end{aligned}$$

for ϵ sufficiently small.

$$(4.81) \quad k_\omega \stackrel{(4.55)}{\stackrel{(4.80)}{=}} \left(-\frac{\sigma_f}{\sigma_g} \right)^{1/2} \cdot \frac{-2(\gamma-1)}{\gamma+1} v_u^y = \sqrt{\frac{\gamma-1}{\gamma+1}} (-v_u^y) + O(\epsilon).$$

(The O term is uniform in $\gamma \downarrow 1$ because $(-\sigma_f/\sigma_g)^{1/2}$ has a $(\gamma-1)^{-1/2}$ singularity that is cancelled by the $\gamma-1$ numerator.)

Extreme Corner Locations

Now we study the behavior of $p, k, q,$ and θ (as introduced in Section 4.7) for $\eta = \eta_R^\pm$. We use a superscript \pm to indicate quantities evaluated for $\eta = \eta_R^\pm$; a superscript $*$ indicates η_R^* . We omit superscripts if the choice is unimportant (e.g., if the difference incurs an $O(\epsilon^{1/2})$ term that is dominated by something else). This is the case for $p_\omega, k_\omega,$ and other derivatives; we may conveniently evaluate them at η_R^* (note that $\eta_R^+ - \eta_R^* = O(\epsilon^{1/2})$).

$p^* = 0$ and $k^* = O(\epsilon)$ by (4.54) (with $r = \sqrt{1-\epsilon}c_R$ and $c_* = c_R$). Then

$$(4.82) \quad p^+ = p_\omega(\eta_R^+ - \eta_R^*) + O(\epsilon),$$

$$(4.83) \quad k^+ = k^* + k_\omega(\eta_R^+ - \eta_R^*) + O(\epsilon) \stackrel{(4.54)}{=} k_\omega(\eta_R^+ - \eta_R^*) + O(\epsilon),$$

where p_ω and k_ω are the values at $\eta = \eta_R^*$. Therefore

$$(4.84) \quad q^+ = \sqrt{p_\omega^2 + k_\omega^2}(\eta_R^+ - \eta_R^*) + O(\epsilon).$$

We estimate q^+ :

$$\begin{aligned} & 1 - \left(\frac{\xi \sqrt{p_\omega^2 + k_\omega^2}}{c_d v_u^y} \right)^2 \\ & \stackrel{(4.81)}{\stackrel{(4.78)}{=}} 1 - c_d^{-2} (v_u^y)^{-2} \left(\frac{\gamma-1}{\gamma+1} (v_u^y)^2 + \left(\frac{\frac{2}{\gamma+1} + L_u^n(\sigma + \frac{\gamma-1}{\gamma+1} L_u^n)}{(L_u^n + \sigma)\xi/c_u} v_u^y \right)^2 \right) \xi^2 + O(\epsilon) \\ & = \frac{c_u^2}{c_d^2} \left(\frac{c_d^2}{c_u^2} - \frac{\gamma-1}{\gamma+1} \frac{\xi^2}{c_u^2} - \left(\frac{\frac{2}{\gamma+1} + L_u^n(\sigma + \frac{\gamma-1}{\gamma+1} L_u^n)}{(L_u^n + \sigma)} \right)^2 \right) + O(\epsilon) \\ & = \frac{c_u^2}{c_d^2} \left(\frac{c_d^2}{c_u^2} - \frac{\gamma-1}{\gamma+1} \frac{c_d^2 - \eta^2}{c_u^2} - \left(\frac{\frac{2}{\gamma+1} + L_u^n(\sigma + \frac{\gamma-1}{\gamma+1} L_u^n)}{(L_u^n + \sigma)} \right)^2 \right) + O(\epsilon) \\ & = \frac{c_u^2}{c_d^2} \left(\frac{2}{\gamma+1} \frac{c_d^2}{c_u^2} + \frac{\gamma-1}{\gamma+1} \sigma^2 - \left(\frac{\frac{2}{\gamma+1} + L_u^n(\sigma + \frac{\gamma-1}{\gamma+1} L_u^n)}{(L_u^n + \sigma)} \right)^2 \right) + O(\epsilon) \\ & = \frac{c_u^2}{c_d^2} \left(\frac{2}{\gamma+1} \frac{c_u^2 + \frac{\gamma-1}{2} (|\vec{z}_u|^2 - |\vec{z}_d|^2)}{c_u^2} + \frac{\gamma-1}{\gamma+1} \sigma^2 \right. \\ & \quad \left. - \left(\frac{\frac{2}{\gamma+1} + L_u^n(\sigma + \frac{\gamma-1}{\gamma+1} L_u^n)}{(L_u^n + \sigma)} \right)^2 \right) + O(\epsilon) = \end{aligned}$$

$$\begin{aligned}
 &= \frac{c_u^2}{c_d^2} \left(\frac{2}{\gamma+1} \left(1 + \frac{\gamma-1}{2} ((L_u^n)^2 - \sigma^2) \right) + \frac{\gamma-1}{\gamma+1} \sigma^2 \right. \\
 &\quad \left. - \left(\frac{\frac{2}{\gamma+1} + L_u^n(\sigma + \frac{\gamma-1}{\gamma+1} L_u^n)}{(L_u^n + \sigma)} \right)^2 \right) + O(\epsilon) \\
 (4.85) \quad &= \frac{2c_u^2((L_u^n)^2 - 1)}{(\gamma+1)^2 c_d^2 (L_u^n + \sigma)^2} (2 - (\gamma+1)\sigma^2 + (\gamma-1)(L_u^n)^2) + O(\epsilon).
 \end{aligned}$$

The last factor is positive: (2.41) yields

$$\sigma \leq \frac{\gamma-1}{\gamma+1} L_u^n + \frac{2}{\gamma+1},$$

so

$$2 - (\gamma+1)\sigma^2 + (\gamma-1)(L_u^n)^2 \geq \frac{2(\gamma-1)}{\gamma+1} (L_u^n - 1)^2.$$

$L_u^n - 1$ is uniformly positive since the corner shocks allowed by (4.14) are uniformly not vanishing for ϵ sufficiently small. All other factors are trivially positive (note $L_u^n > 1, \sigma$). The right-hand side of (4.85) is positive, so

$$(4.86) \quad \sqrt{p_\omega^2 + k_\omega^2} < \frac{-c_R v_I^y}{\xi}$$

and therefore

$$(4.87) \quad q^+ \stackrel{(4.84)}{\leq} \frac{-c_R v_I^y}{\xi_R} (\eta_R^+ - \eta_R^*) + O(\epsilon).$$

Extreme θ -Value

$$\begin{aligned}
 \tan\left(\frac{\pi}{2} - \theta^+\right) &= \frac{p^+}{k^+} \stackrel{(4.83)}{\stackrel{(4.82)}}{=} \frac{p_\omega}{k_\omega} + O(\epsilon) \\
 &\stackrel{(4.81)}{\stackrel{(4.78)}}{=} \sqrt{\frac{\gamma+1}{\gamma-1}} \cdot \frac{\frac{2}{\gamma+1} + L_u^n(\sigma + \frac{\gamma-1}{\gamma+1} L_u^n)}{(L_u^n + \sigma)\xi/c_u} + O(\epsilon).
 \end{aligned}$$

On the other hand, by the convexity of \tan on $[0, \frac{\pi}{2})$, with $\sigma_\theta \bar{\phi} < \bar{\phi} < \frac{\pi}{2}$,

$$\tan(\sigma_\theta \bar{\phi}) \leq \sigma_\theta \tan \bar{\phi} = \sigma_\theta \frac{\eta_R^+}{\xi_R^+} + O(\epsilon) = \sqrt{\frac{\gamma-1}{\gamma+1}} \cdot \frac{\sigma}{\xi/c_u} + O(\epsilon).$$

Then

$$\frac{\tan(\sigma_\theta \bar{\phi})}{\tan(\frac{\pi}{2} - \theta^+)} \leq \frac{\gamma-1}{\gamma+1} \cdot \frac{\sigma(L_u^n + \sigma)}{\frac{2}{\gamma+1} + L_u^n(\sigma + \frac{\gamma-1}{\gamma+1} L_u^n)} + O(\epsilon)$$

The right-hand side is < 1 for sufficiently small ϵ if and only if

$$(\gamma-1)\sigma(L_u^n + \sigma) < 2 + L_u^n((\gamma+1)\sigma + (\gamma-1)L_u^n).$$

Since $\sigma = z_d^n/c_u < z_u^n/c_u = L_u^n$ and $L_u^n > 1$, this is always true. Therefore

$$(4.88) \quad \theta^+ \in (0, \frac{\pi}{2} - \sigma_\theta \cdot \bar{\phi}) + 2\pi\mathbb{Z}.$$

The result for $\bar{\xi}_R^-$ follows from symmetry:

$$(4.89) \quad \theta^- \in (\pi, \frac{3\pi}{2} - \sigma_\theta \cdot \bar{\phi}) + 2\pi\mathbb{Z}.$$

4.10 Corner Bounds

PROPOSITION 4.17 *For ϵ sufficiently small, for any fixed point $\psi \in \bar{\mathcal{F}}$ of \mathcal{K} , the lower bounds in (4.14) are strict:*

$$\eta_L > \eta_L^-, \quad \eta_R > \eta_R^-.$$

PROOF: For $\gamma = 1$, we may borrow (4.67) which contradicts

$$p^- = \underbrace{p^*}_{=0} + p_\omega(\eta_R^- - \eta_R^*) + O(\epsilon) = -p_\omega\epsilon^{1/2} + O(\epsilon)$$

if ϵ is sufficiently small, because $p_\omega > 0$ by (4.78).

For $\gamma > 1$, (4.62) contradicts (4.89) for $\epsilon > 0$ sufficiently small.

Again the proof for the left corner is analogous: in L -coordinates (Figure 4.1) $P_L, \bar{\xi}_L$, and $\bar{\xi}_L^*$ have the same properties as $P_R, \bar{\xi}_R$, and $\bar{\xi}_R^*$ except that $\bar{\xi}_{BL}$ is not on the horizontal axis, which is irrelevant. □

To prove that $\eta_R = \eta_R^+$ is impossible, a more global argument is needed.

PROPOSITION 4.18 *Consider $\eta_R = \eta_R^+$. There is a \bar{c} so that for ϵ sufficiently small,*

$$c(\phi) < \bar{c} \quad \text{for } \phi \in [0, \bar{\phi}].$$

For $\gamma > 1$, \bar{c} satisfies

$$(4.90) \quad \bar{c} = \sqrt{h_0 + C_{\bar{c}}\epsilon + (-\sigma_g/\sigma_f)^{1/2}q^+ \sin(\theta^+ + \sigma_\theta\bar{\phi})}$$

where $C_{\bar{c}}$ is some constant, as in the O terms; for $\gamma = 1$ we may take any $\bar{c} > c$.

PROOF: Consider $\gamma > 1$. For $k \leq 0$ the result is trivial. Assume that there is a $\phi_0 \in [0, \bar{\phi})$ so that $\phi \mapsto k(\phi)$ has a positive local maximum in $\phi = \phi_0$ (see Figure 4.7). For $\phi_0 > 0$ we need $k_\phi(\phi_0) = 0$, which implies $p(\phi_0) = 0$ by (4.57); for $\phi_0 = 0$ we have $p = 0$ anyway. By (4.56), $k > 0$ means $p_\phi > 0$ (if ϵ is small enough, so that the $O(k^2) = O(\epsilon^{1/2}k)$ term is dominated by $\sigma_\theta k$). Therefore $p > 0$ on $(\phi_0, \phi_2]$ for some $\phi_2 > \phi_0$. Take $\phi_2 \in (\phi_0, \bar{\phi}]$ maximal with this property.

Assume that $\phi_2 < \bar{\phi}$. Then $p(\phi_2) = 0$ necessarily.

(4.58) for $p > 0$ implies $q > 0$ on $[\phi_0, \phi_2]$, so θ is well-defined. Moreover, k is strictly decreasing on $[\phi_0, \phi_2]$ by (4.57).

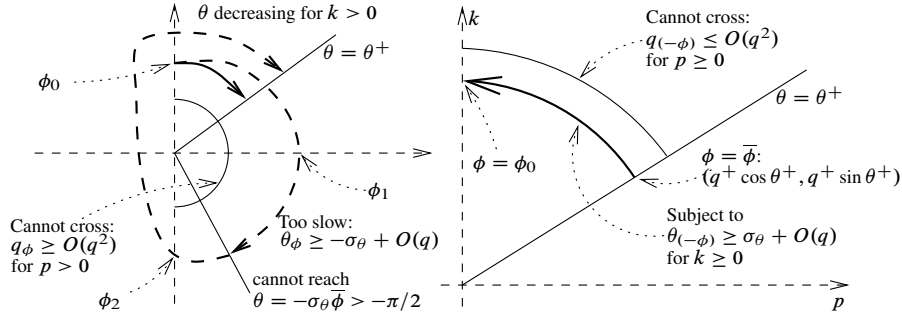


FIGURE 4.7. Left: (p, k) value curve (bold) from a k maximum on P_R to the upper endpoint for $\eta_R = \eta_R^+$. The curve cannot go past θ^+ because (4.59) does not allow it to pass through the lower right quadrant “fast” enough. Right: (p, k) curve (bold) in the opposite direction; the constraints limit the value of k maxima.

Assume there is a $\phi_1 \in (\phi_0, \phi_2]$ with $k(\phi_1) = 0$; hence $k \leq 0$ on $[\phi_1, \phi_2]$. (4.59) (for $k \leq 0$) integrated from ϕ_1 to $\phi \in [\phi_1, \phi_2]$ implies

$$\theta(\phi) \geq \underbrace{\theta(\phi_1)}_{=0} - \underbrace{\sigma_\theta}_{<1} \underbrace{\bar{\phi}}_{<\pi/2} + O(\epsilon^{1/2}) > -\frac{\pi}{2}.$$

In particular, $0 \geq \theta(\phi_2) > -\frac{\pi}{2}$ —which contradicts $p(\phi_2) = 0$. The assumption is wrong; necessarily $k > 0$ on $[\phi_0, \phi_2]$.

(4.59) (for $k \geq 0$) implies that θ is strictly decreasing on $[\phi_0, \phi_2]$ (for $O(q) = O(\epsilon^{1/2})$, i.e., ϵ is sufficiently small). Then $\theta(\phi_2) \in (0, \frac{\pi}{2})$, which contradicts $p(\phi_2) = 0$. The assumption is wrong; therefore $\phi_2 = \bar{\phi}$.

By integrating (4.59) (for $k \geq 0$) from $\bar{\phi}$ back to ϕ ,

$$\begin{aligned} \theta(\phi) &\geq \theta(\bar{\phi}) + \sigma_\theta(\bar{\phi} - \phi) + O(\epsilon^{1/2}) \\ &= \theta^+ + \sigma_\theta(\bar{\phi} - \phi) + O(\epsilon^{1/2}) \quad (\phi \in [\phi_0, \bar{\phi}]). \end{aligned}$$

Integrate (4.58) (for $p \geq 0$) backwards:

$$q(\phi) \leq q(\bar{\phi}) + O(\epsilon) = q^+ + O(\epsilon) \quad (\phi \in [\phi_0, \bar{\phi}]).$$

Then

$$\begin{aligned} k_\phi(\phi) &= -\sigma_\theta p(\phi) = -\sigma_\theta q(\phi) \cos \theta(\phi) \\ &\geq -\sigma_\theta q^+ \cos(\theta^+ + \sigma_\theta(\bar{\phi} - \phi)) + O(\epsilon) \quad (\phi \in [\phi_0, \bar{\phi}]) \end{aligned}$$

so

$$\begin{aligned}
 k(\phi_0) &\leq k(\bar{\phi}) + \sigma_\theta q^+ \int_{\phi_0}^{\bar{\phi}} \cos(\theta^+ + \sigma_\theta(\bar{\phi} - \phi)) d\phi + O(\epsilon) \\
 &= q^+ \sin \theta^+ - q^+ \left(\sin \theta^+ - \sin(\theta^+ + \sigma_\theta \underbrace{(\bar{\phi} - \phi_0)}_{\leq \bar{\phi}}) \right) + O(\epsilon) \\
 &\leq q^+ \sin(\theta^+ + \sigma_\theta \bar{\phi}) + O(\epsilon).
 \end{aligned}$$

Finally, we obtain a bound for c : let ϕ_0 be the *global* maximum point of k on $[0, \bar{\phi}]$. Then

$$\begin{aligned}
 \sup_{[0, \bar{\phi}]} c(\phi)^2 &= h_0 + \sqrt{\frac{-\sigma_g}{\sigma_f}} k(\phi) \leq h_0 + \sqrt{\frac{-\sigma_g}{\sigma_f}} k(\phi_0) \\
 &\leq h_0 + \sqrt{\frac{-\sigma_g}{\sigma_f}} q^+ \sin(\theta^+ + \sigma_\theta \bar{\phi}) + C_{\bar{c}} \epsilon
 \end{aligned}$$

for $\epsilon > 0$ small enough. This is exactly the statement. □

PROPOSITION 4.19 *Again consider the case $\eta_R = \eta_R^+$. For $\gamma > 1$, define*

$$(4.91) \quad \tilde{a}(\phi) := \sqrt{1 - \epsilon} \cdot c_R \left(\left[\left(\frac{\bar{c}}{c_R} \right)^2 - \sin^2 \phi \right]^{1/2} - \cos \phi \right),$$

$$(4.92) \quad a := \tilde{a}(0) = \sqrt{1 - \epsilon} (\bar{c} - c_R),$$

where \bar{c} is as in Proposition 4.18; for $\gamma = 1$ take $a = \tilde{a} = 0$. For ϵ sufficiently small,

$$a = \max_{\phi \in [0, \bar{\phi}]} \tilde{a}(\phi),$$

and $\psi + a\xi$ cannot have a local minimum (with respect to $\bar{\Omega}$) on $P_R \cup \{\bar{\xi}_{BR}\}$.

PROOF: For $\gamma = 1$, $L^2 = 1 - \epsilon$ implies $\chi_r \geq -\sqrt{1 - \epsilon} \cdot c = -r$, so $\psi_r \geq 0$. By the Hopf lemma, this does not allow a local minimum of ψ at P_R . In $\bar{\xi}_{BR}$ we argue that by Remark 4.7, we may consider the even reflection of ψ across \bar{A} , which still satisfies the same equation, so the Hopf lemma also rules out a local minimum of ψ in $\bar{\xi}_{BR}$.

For $\gamma > 1$,

$$\begin{aligned}
 \frac{\tilde{a}}{\sqrt{1 - \epsilon} \cdot c_R} &\stackrel{(4.90)}{=} \left(\frac{h_0}{c_R^2} - 1 + \frac{C_{\bar{c}}}{c_R^2} \epsilon + \cos^2 \phi \right. \\
 &\quad \left. + \sqrt{\frac{-\sigma_g}{\sigma_f}} \frac{q^+}{c_R^2} \sin(\theta^+ + \sigma_\theta(\bar{\phi} - \phi)) \right)^{1/2} - \cos \phi.
 \end{aligned}$$

Call the right-hand side f and take ∂_ϕ of it:

$$\frac{2(f - \cos(\phi)) \sin \phi - \sqrt{\frac{-\sigma_g}{\sigma_f} \frac{q^+}{c_R^2}} \sigma_\theta \cos(\theta^+ + \sigma_\theta(\bar{\phi} - \phi))}{2f}.$$

$f > 0$ by (4.54) and $r = \sqrt{1 - \epsilon} \cdot c_R$, so the denominator is positive. By (4.88), $\cos(\theta^+ + \sigma_\theta(\bar{\phi} - \phi)) > 0$, so the numerator is negative unless $f \geq \cos \phi$. In that case $\phi = \frac{\pi}{2} + O(\epsilon^{1/2})$, because f is $O(\epsilon^{1/2})$ due to $q^+ = O(\epsilon^{1/2})$ and $h_0 - c_R^2 = O(\epsilon)$ (by (4.54)). But $0 \leq \phi \leq \bar{\phi} = \arctan(\eta_R^+ / \xi_R^+) \leq \frac{\pi}{2} - \delta$, with $\delta > 0$ uniformly in $\epsilon \downarrow 0$, so for sufficiently small ϵ there is a contradiction. Thus f is decreasing in ϕ , so it attains its maximum in $\phi = 0$.

For a local minimum (with respect to $\bar{\Omega}$) of $\psi + a\xi$ on P_R , we need

$$0 = (\psi + a\xi)_\phi = \psi_\phi + a\xi_\phi = \psi_\phi - ar \sin \phi,$$

so $\psi_t = r^{-1}\psi_\phi = a \sin \phi$ there. In $\bar{\xi}_{BR}$ this still holds (minimum or not) because we have $\psi_\phi = 0 = a \sin \phi$. Another minimum condition is

$$\begin{aligned} 0 &\geq (\psi + a\xi)_r = \psi_r + a\xi_r \stackrel{L^2=1-\epsilon}{=} r - \sqrt{c^2(1 - \epsilon) - \psi_t^2} + a \cos \phi \\ &= \sqrt{1 - \epsilon} \cdot c_R - \sqrt{c^2(1 - \epsilon) - a^2 \sin^2 \phi} + a \cos \phi. \end{aligned}$$

This is equivalent (by squaring to eliminate the root and solving a quadratic inequality for a) to

$$a \leq \sqrt{1 - \epsilon} \cdot c_R \left(\sqrt{\left(\frac{c}{c_R}\right)^2 - \sin^2 \phi} - \cos \phi \right).$$

But $a \geq \tilde{a}(\phi)$, which, by Proposition 4.18, is greater than the right-hand side. Contradiction! \square

LEMMA 4.20

$$(4.93) \quad a \leq \left(\frac{-v_I^y}{\xi_R} - \delta_a \right) (\eta_R^+ - \eta_R^*)$$

for some $\delta_a > 0$ (depending continuously on λ), and $\epsilon > 0$ sufficiently small.

PROOF: For $\gamma = 1$, this is trivial since $a = 0$ and $v_I^y < 0 < \xi_R$.

For $\gamma > 1$, (4.54) (with $r^2 = c_R^2(1 - \epsilon)$) means $h_0 = c_R^2(1 + O(\epsilon))$. Taylor-expand the square root in (4.90) around h_0 :

$$\begin{aligned} \bar{c} &= \sqrt{h_0} + O(\epsilon) + \frac{1}{2\sqrt{h_0}} q^+ \left(-\frac{\sigma_g}{\sigma_f} \right)^{1/2} \sin(\theta^+ + \sigma_\theta \bar{\phi}) + O(\epsilon) \\ &\stackrel{(4.54)}{=} c_R + \frac{q^+}{2c_R} \left(-\frac{\sigma_g}{\sigma_f} \right)^{1/2} \sin(\theta^+ + \sigma_\theta \bar{\phi}) + O(\epsilon) \stackrel{(4.87)}{\leq} \end{aligned}$$

$$\begin{aligned}
 &\stackrel{(4.87)}{\leq} c_R + (\eta_R^+ - \eta_R^*) \frac{-v_I^y}{2\xi_R} \left(-\frac{\sigma_g}{\sigma_f} \right)^{1/2} \sin(\theta^+ + \sigma_\theta \bar{\phi}) + O(\epsilon) \\
 &\leq c_R + (\eta_R^+ - \eta_R^*) \left(\frac{-v_I^y}{\xi_R} - \delta_a \right)
 \end{aligned}$$

for some $\delta_a > 0$ because

$$\left(-\frac{\sigma_g}{\sigma_f} \right)^{1/2} = 2\sqrt{\frac{\gamma - 1}{\gamma + 1}} + O(\epsilon) < 2$$

for $\epsilon > 0$ small enough and because $\sin(\theta^+ + \sigma_\theta \bar{\phi}) < 1$ by (4.88). Use (4.92) to get (4.93). \square

PROPOSITION 4.21 Consider a as in (4.92). For ϵ sufficiently small, the shock through $\vec{\xi}_R^+$ with upstream data \vec{v}_I and ρ_I and tangent $(1, a/-v_I^y)$ has $v_d^y > 0$.

PROOF: The shock through $\vec{\xi}_R^*$ with tangent $(1, 0)$ is the R shock where $v_d^y = 0$ by construction. v_d^y for the new slope and location differs from 0 by

- (1) moving the shock up to $\vec{\xi}_R^+$ from $\vec{\xi}_R^*$, while keeping it horizontal, and
- (2) rotating it while holding it in $\vec{\xi}_R^+$ to make its slope the above.

Both of these changes are $O(\epsilon^{1/2})$, so it is sufficient to consider a first-order expansion using known derivatives. As before, we use uniformity in $\epsilon \downarrow 0$.

For (1), we use (2.42) (note $v_d^n = \vec{v} \cdot \vec{n} = -v_d^y$, $\sigma = \vec{\xi} \cdot \vec{n} = -\eta$): v_d^y changes by

$$\left(1 - \frac{\partial z_d^n}{\partial z_u^n} \right) (\eta_R^+ - \eta_R^*) + O(\epsilon).$$

For (2), we use (2.44) with $\vec{n} = (0, -1)$:

$$\partial_\beta v_d^y = (\partial_\beta \vec{v}_d) \cdot \vec{n} \underbrace{n^y}_{=-1} + (\partial_\beta \vec{v}_d) \cdot \vec{t} \underbrace{t^y}_{=0} \stackrel{(2.44)}{=} z^t \left(1 - \frac{\partial z_d^n}{\partial z_u^n} \right).$$

Here $\beta = \arctan(a/-v_I^y) = O(\epsilon^{1/2})$. So v_d^y changes by

$$\begin{aligned}
 &z^t \left(1 - \frac{\partial z_d^n}{\partial z_u^n} \right) \cdot \arctan \frac{a}{-v_I^y} + O(\epsilon) \\
 &= \left(1 - \frac{\partial z_d^n}{\partial z_u^n} \right) \frac{z^t}{-v_I^y} a + O(\epsilon) \\
 &\stackrel{(4.93)}{\geq}_{z^t = -\xi_R} \left(1 - \frac{\partial z_d^n}{\partial z_u^n} \right) \left(\frac{\xi_R}{-v_I^y} \delta_a - 1 \right) (\eta_R^+ - \eta_R^*) + O(\epsilon).
 \end{aligned}$$

With (1) and (2) combined, the change is

$$\geq \left(1 - \frac{\partial z_d^n}{\partial z_u^n}\right) \frac{\xi_R}{-v_I^y} \delta_a \underbrace{(\eta_R^+ - \eta_R^*)}_{=\epsilon^{1/2}} + O(\epsilon).$$

By (2.40) the first factor is $\geq 2/(\gamma + 1)$. Therefore $v_d^y > 0$ for $\eta_R = \eta_R^+$ for ϵ sufficiently small. \square

PROPOSITION 4.22 *Let $\chi \in \overline{\mathcal{F}}$ be a fixed point of \mathcal{K} . For C_η sufficiently large and for $\epsilon > 0$ sufficiently small, the upper part of (4.14) is strict:*

$$\eta_C < \eta_C^+ \quad (C = L, R).$$

PROOF: Let a be defined as in (4.92). As shown in Proposition 4.19, $\psi + a\xi$ cannot have a local minimum at $P_L \cup \{\vec{\xi}_{BR}\}$. For $\eta_R = \eta_R^*$, we have $(\psi + a\xi)_2 = \psi_2 > 0$ in $\vec{\xi}_R$ by (4.76) (for sufficiently small ϵ), so the minimum cannot be in $\vec{\xi}_R$ either (note that the domain locally contains the ray downward from the corner).

On the shock, $\psi + a\xi = \psi^I + a\xi$, so

$$\partial_t(\psi + a\xi) = \partial_t(\psi^I + a\xi) = \vec{v}_I \cdot \vec{t} + \frac{a}{(1 + s_1^2)^{1/2}} = \frac{v_I^y s_1 + a}{(1 + s_1^2)^{1/2}}.$$

For a local minimum at the shock, we need $\partial_t(\psi + a\xi) = 0$, so

$$s_1 = \frac{a}{-v_I^y}.$$

A global minimum, in particular $\leq \psi(\vec{\xi}_R) + a\xi_R$, additionally requires that $\vec{\xi}_R$ (as well as the rest of the shock) be on or below the tangent through the minimum point, because ψ^I and thus $\psi^I + a\xi$ are decreasing in η . Proposition 4.21 shows that the shock through $\vec{\xi}_R$ with that tangent has $v_d^y > 0$ for $\eta_R = \eta_R^+$. In the minimum point the tangent has the same slope but is at least as high, so the shock speed is at least as high, so $v_d^y = \psi_2 > 0$ is at least as high, in particular > 0 , too. But that contradicts a minimum (the ray vertically downwards from any shock point is locally contained in $\overline{\Omega}$). Hence $\psi + a\xi$ cannot have a global minimum on the shock.

$\psi_n = 0$ on A contradicts a minimum on A (by the Hopf lemma).

The equation (2.15) yields

$$(c^2 I - \nabla \chi^2) : \nabla^2(\psi + a\xi) = 0$$

($a\xi$ is linear), so the classical strong maximum principle rules out a minimum in the interior (unless $\psi + a\xi$ is constant, which means we are looking at the unperturbed solution that has $\eta_R = \eta_R^* < \eta_R^+$).

On $\overline{P_L}$, for $\gamma = 1$ we argue that $\psi_r \leq 0$ on $\overline{P_L}$ (in fact, $\psi_r < 0$ except in the unperturbed case $\eta_L^* = \eta_R^*$), as in the first paragraph of the proof of Proposition 4.19, so the Hopf lemma rules out a maximum at $P_L \cup \{\vec{\xi}_{BL}\}$. In $\vec{\xi}_L$, either

$\eta_L \leq \eta_L^* \leq \eta_R^* < \eta_R^+$ so that $\psi(\vec{\xi}_L) > \psi(\vec{\xi}_R) = \psi(\vec{\xi}_R^+)$ due to $\psi = \psi^I$ on \bar{S} and $\psi_{\vec{\xi}}^I = 0 > \psi_{\eta}^I$, or $\eta_L > \eta_L^*$ so that $\psi_{\eta} > 0$ in $\vec{\xi}_L$, by the same analysis as for $\vec{\xi}_R$, if we rule out a minimum.

For $\gamma > 1$,

$$(\psi + a\xi)_{\xi} = \psi_{\xi} + O(\epsilon^{1/2}) \stackrel{(4.61)}{=} v_L^x + O(\epsilon^{1/2}).$$

For C_{η} sufficiently large and ϵ sufficiently small,¹⁷ the right-hand side is negative: $v_L^x = 0$ for $\eta_L^* = \eta_R^*$, it is strictly decreasing in η_L^* (Proposition 2.12), and C_{η} bounds η_L^* away from η_R^* (see (4.7)). Again, no minimum is possible.

Conclusion: there is no $\psi + a\xi$ minimum anywhere—a contradiction!

The argument for η_L is similar, using the L picture (Figure 4.1). The wall still passes through the origin, so the wall boundary condition $\chi_n = 0$ implies $\psi_n = 0$. For $\gamma = 1$ this implies that ψ cannot have minima at the wall (Hopf lemma). For $\gamma > 1$ we have to modify the argument at the wall since it is no longer horizontal. A minimum of $\psi + a\xi$ requires $(\psi + a\xi)_n < 0$ (where \vec{n} points out of Ω), so $a\xi_n < 0$, so $\xi_n < 0$ (by $a > 0$). But the wall slope is positive, so $\xi_n > 0$ (for outward \vec{n})—a contradiction.

The remaining arguments are as in R -coordinates. □

4.11 Density Bounds and Shock Strength

PROPOSITION 4.23 *If ϵ and δ_{ρ} are sufficiently small (bounds depending only on C_{Pt}), then for any fixed point $\psi \in \bar{\mathcal{F}}$ of \mathcal{K} , the inequality (4.17) is strict.*

PROOF: By Proposition 4.13, ψ and hence s are analytic. Thus we may use Proposition 3.5, which rules out minima of ρ in Ω and (using Remark 4.7) at A , unless ψ and s are linear so that $\rho = \rho_R$ is constant and (4.17) is trivially strict. However, $\rho \in C(\bar{\Omega})$, so it must attain a global minimum somewhere.

Case 1. ρ attains its global minimum on S (excluding $\vec{\xi}_{L,R}$). Then by Proposition 3.13, the rest of the shock, including the corners $\vec{\xi}_L$ and $\vec{\xi}_R$, must be below the tangent in that point (see Figure 4.8). We assumed in Definition 4.1 that the line through $\vec{\xi}_L^*$ and $\vec{\xi}_R^*$ does not touch or intersect the circle with radius c_I centered in \vec{v}_I . Hence if ϵ is sufficiently small (depending on η_L^*), then by (4.14) the shock tangent must have positive distance from the circle as well. Thus the global minimum of ρ must be bounded below away from ρ_I .

Case 2. ρ attains its global minimum on \bar{P}_L or \bar{P}_R . By (4.60) we know ρ up to $O(\epsilon^{1/2})$.

Combining both (nonexclusive) cases, we see that for sufficiently small δ_{ρ} and ϵ , depending continuously on C_{Pt} , λ , and η_L^* , (4.17) is *strict*. □

¹⁷We need $C_{\eta} > 0$; otherwise the upper bound on ϵ would become 0 as $\eta_L^* \uparrow \eta_R^*$.

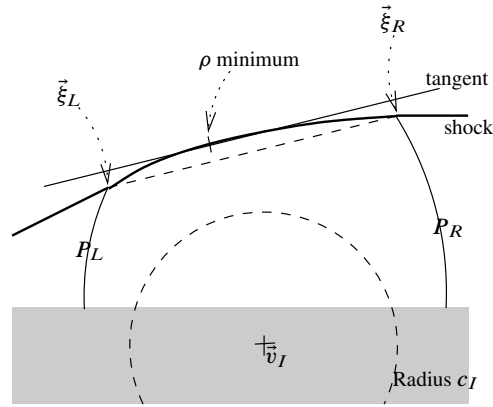


FIGURE 4.8. In a $\rho = \rho_-$ minimum at S , the shock tangent must be above $\vec{\xi}_L$ and $\vec{\xi}_R$, so by (1.1) the minimum is separated from $B_{c_I}(\vec{v}_I)$. Therefore $\rho_- \gg \rho_I$.

4.12 Velocity and Shock Normal Control

PROPOSITION 4.24 *If δ_{SA} is sufficiently small, if C_{vx} and C_{vL} are sufficiently large (bounds depending only on C_{Pt}), if C_{Sn} is sufficiently large (bound depending only on C_{vx}, C_{vL}), if δ_{vy} and ϵ are sufficiently small (δ_{vy} bound depending only on δ_ρ and C_{Sn}), and ϵ bound depending only on C_{Sn}), and if δ_{Cc} is sufficiently small, then for any fixed point $\psi \in \overline{\mathcal{F}}$ of \mathcal{K} , the inequalities (4.12), (4.33), (4.18), (4.34), (4.35), and (4.15) are strict. Moreover,*

$$(4.94) \quad |\chi_t| \geq \delta_{\chi t} \quad (\text{on } S \cap B_{\delta_d}(\vec{\xi}_C), C = L, R)$$

for some constants $\delta_{\chi t}, \delta_d > 0$.

PROOF:

(1) For horizontal velocity (4.33), assume that v^x attains a positive global maximum (with respect to $\overline{\Omega}$) in a point $\vec{\xi}_0$ on S . Since $\vec{v}_I = (0, v_I^y)$ with $v_I^y < 0$, this means $n^x \dot{<} 0$ (because $n^y \dot{<} 0$), i.e., $s_1(\xi_0) < 0$.

$s_1(\xi_0)$ can be expressed as a continuous function of $v^x(\xi_0)$ and $\vec{\xi}_0$. The set of possible $\vec{\xi}_0$ is contained in the set of possible shock locations, which is precompact. Therefore if v^x has a maximum $= C_{vx}\epsilon^{1/2}$ in some $\vec{\xi}_0 \in S$, then

$$(4.95) \quad s_1(\xi_0) \leq -C_{s1}\epsilon^{1/2}$$

where $C_{s1} = C_{s1}(C_{vx}) > 0$ is uniformly increasing in C_{vx} .

Since S and ψ are analytic (Proposition 4.13), we can apply Proposition 3.10 with $\vec{w} = (1, 0)$. For a constant-state solution, (4.33) is immediate. Otherwise (3.8) and (3.9) are satisfied. $n^x \dot{<} 0$ means $w^n \dot{<} 0$, so by (3.9) $\kappa \dot{<} 0$, i.e., $s_{11} \dot{>} 0$.

Now we can use a similar argument to that for the density in Proposition 4.23 and Proposition 3.13 (see Figures 3.2 and 4.8): $s_{11} > 0$ implies $s_1(\xi) < s_1(\xi_0)$ for $\xi < \xi_0$ near ξ_0 . On the other hand, for $\vec{\xi}_L = \vec{\xi}_L^*$ we have $s_1(\xi_L) \geq 0$ by construction of the L shock in Section 4.3, so for arbitrary $\vec{\xi}_L$ (satisfying (4.14)) the continuous dependence of $s_1(\xi_L)$ on η_L and by (4.14) shows $s_1(\xi_L) \geq -C_2\epsilon^{1/2}$ for some constant C_2 independent of ϵ . If we pick C_{vx} so large that $C_{s1} > C_2$, then $s_1(\xi_L) > s_1(\xi_0)$ by (4.95) for any possible location of the left corner.

Therefore we can pick $\xi_a \in (\xi_L, \xi_0)$ maximal so that $s_1(\xi_a) = s_1(\xi_0)$. Then $s_1(\xi) < s_1(\xi_0)$ for $\xi \in (\xi_a, \xi_0)$, so by integration

$$s(\xi_a) > s(\xi_0) + s_1(\xi_0) \cdot (\xi_a - \xi_0).$$

But that means the shock tangent in ξ_a is parallel to the one in ξ_0 but *higher*, so $\sigma := \vec{\xi} \cdot \vec{n}$ is smaller in ξ_a . By (2.42), that means v_d^n is smaller in ξ_a , whereas v^t is the same (parallel tangents). $n^x < 0$, so v_d^x is *bigger* in ξ_a . Contradiction—we assumed that we have a *global* maximum of v^x in ξ_0 .

Propositions 3.8 and 3.9 rule out local maxima of v^x in Ω and on A , where we use that χ is analytic and that \vec{w} is not vertical, i.e., not normal to the wall.

On $\overline{P_L} \cup \overline{P_R}$ we can use (4.61), increasing C_{vx} to $> C_{Pv}$ if necessary (this makes C_{vx} depend on C_{Pt} as well). Now (4.33) is strict.

(2) In any point $\vec{\xi} \in S$, $s_1(\xi)$ is a function of v_d^x , with $\text{sgn } s_1 = -\text{sgn } v^x$. $s_1(\xi)$ is continuous in $\vec{\xi}$ and v_d^x , and the set of possible shock locations $\vec{\xi}$ is pre-compact, so (4.33) implies

$$\sup \angle(\vec{n}, \vec{n}_R) < C_{Sn}\epsilon^{1/2}$$

where $C_{Sn} = C_{Sn}(C_{vx})$.

(3) The arguments for (4.34) are analogous to those for (4.33): the transformation from R - to L -coordinates (Figure 4.1) turns \vec{n}_L^\perp into $\vec{n}_R^\perp = (1, 0)$ (and vice versa). The wall is never vertical in L -coordinates, so Proposition 3.9 still applies to v^x . The other arguments are as before.

Moreover, (4.34) implies

$$\sup \angle(\vec{n}_L, \vec{n}) < C_{Sn}\epsilon^{1/2},$$

where $C_{Sn} = C_{Sn}(C_{vx}, C_{vL})$ now. (4.35) is strict with these choices.

(4) $v^y = 0$ on A ; $v^y = O(\epsilon^{1/2})$ on $P_L \cup P_R$ by (4.61). v^y has no extrema in Ω (Proposition 3.10); to show (4.18) it remains to discuss S .

(4.17) is in particular a lower bound for the shock strength, so $|v_d^n - v_u^n|$ is bounded away from 0. (4.35) bounds the shock normals away from horizontal. Both combined imply (4.18) is strict if $\delta_{vy} > 0$ and $\epsilon > 0$ are chosen small enough, with upper bound on δ_{vy} depending on δ_ρ and C_{Sn} , and upper bound on ϵ depending on C_{Sn} only.

(5) The shock normal bounds also imply (4.15) is strict, for $\delta_{C_c} > 0$ and $\epsilon > 0$ sufficiently small(er), with the ϵ bound depending only on C_{S_n} . Here we use (4.5): (4.35) shows shock tangents are between \vec{n}_L^\perp and \vec{n}_R^\perp (up to $O(\epsilon^{1/2})$), and these are bounded away from arc tangents.

(6) Since the left corner is above the wall, the shock normal bounds imply (4.12) is strict for ϵ sufficiently small(er), with bound depending only on C_{S_n} and for δ_{S_A} sufficiently small.

(7) Near each corner the shock normal bound bounds \vec{n} away from the $\vec{\xi}$ -direction, so $|\chi_t^I| \geq \delta_{\chi t}$ and therefore (4.94) for some $\delta_{\chi t}$.

□

4.13 Fixed Points

PROPOSITION 4.25 *For δ_o sufficiently small, with bounds depending only on δ_ρ and C_L , for C_d sufficiently large and δ_d sufficiently small, with bounds depending only on δ_ρ and C_L , and for ϵ sufficiently small, with bounds depending only on C_{P_t} , C_L , and δ_ρ :*

If $\chi \in \overline{\mathcal{F}}$ is a fixed point of \mathcal{K} , then (4.36) and (4.37) are strict.

PROOF: First, we check (4.36).

(1) For the wall boundary operator (4.28), the strict inequality (4.36) is obvious.

(2) The parabolic boundary operator (4.26) has \vec{p} derivative $g_{\vec{p}}^i = \nabla \hat{\chi}$. We use (4.31) with sufficiently small ϵ , depending on C_{P_t} , to obtain (4.36) strictly (with $\delta_o = \frac{1}{2}$, for example).

(3) For the shock boundary operator (4.27),

$$(4.96) \quad g_{\vec{p}}^i = \hat{\rho}(1 - \hat{c}^{-2} \nabla \hat{\chi}^2) \frac{\vec{v}_I - \nabla \hat{\psi}}{|\vec{v}_I - \nabla \hat{\psi}|} - |\vec{v}_I - \nabla \hat{\psi}|^{-1} \left(1 - \left(\frac{\vec{v}_I - \nabla \hat{\psi}}{|\vec{v}_I - \nabla \hat{\psi}|} \right)^2 \right) \cdot (\hat{\rho} \nabla \hat{\chi} - \rho_I \nabla \chi^I).$$

We exploit that for a *fixed point* we have $\hat{\psi} = \psi = \psi^I$ at the shock, so

$$\vec{n} = \frac{\vec{v}_I - \nabla \hat{\psi}}{|\vec{v}_I - \nabla \hat{\psi}|} = \frac{\nabla \chi^I - \nabla \hat{\chi}}{|\nabla \chi^I - \nabla \hat{\chi}|},$$

where \vec{n} is the downstream shock normal. Note that

$$|\nabla \chi^I - \nabla \hat{\chi}| = \chi_n^I - \hat{\chi}_n$$

because $\chi_t^I = \hat{\chi}_t$ and $\chi_n^I > \hat{\chi}_n$.

$$\begin{aligned}
 g_{\vec{p}}^i &= \hat{\rho}(1 - \hat{c}^{-2} \nabla \hat{\chi}^2) \vec{n} - |\vec{v}_I - \nabla \hat{\psi}|^{-1} \vec{t} \cdot (\hat{\rho} \nabla \hat{\chi} - \rho_I \nabla \chi^I) \\
 &\stackrel{\hat{\chi}_t = \chi_t^I}{=} \hat{\rho}(1 - \hat{c}^{-2} \nabla \hat{\chi}^2) \vec{n} - (\chi_n^I - \hat{\chi}_n)^{-1} (\hat{\rho} - \rho_I) \hat{\chi}_t \vec{t} \\
 &\stackrel{(2.19)}{=} \hat{\rho}((1 - \hat{c}^{-2} \nabla \hat{\chi}^2) \vec{n} - (\hat{\chi}_n)^{-1} \hat{\chi}_t \vec{t}) \\
 (4.97) \quad &= \hat{\rho}((1 - \hat{c}^{-2} \hat{\chi}_n^2) \vec{n} - \hat{\chi}_t ((\hat{\chi}_n)^{-1} + \hat{c}^{-2} \hat{\chi}_n) \vec{t}),
 \end{aligned}$$

$$\begin{aligned}
 \vec{n} \cdot g_{\vec{p}}^i &= \hat{\rho} \vec{n}^\top (1 - \hat{c}^{-2} \hat{\chi}_n^2) \vec{n} - \underbrace{\vec{n} \cdot \vec{t}}_{=0} \dots \\
 &= \hat{\rho}(1 - \hat{c}^{-2} \hat{\chi}_n^2) \geq \delta'_o
 \end{aligned}$$

for some δ'_o depending only on δ_ρ , because (4.17) bounds downstream ρ away from ρ_I , and that means $1 - c^{-2} \hat{\chi}_n^2 = 1 - (L_d^n)^2$ is lower-bounded away from 0 by the shock analysis in Section 2.4. $|g_{\vec{p}}^i|$ is easily bounded from above, from (4.30), so (4.36) is strict for a sufficiently small δ_o , depending only on δ_ρ and C_L .

Now we check (4.37). For the parabolic-wall corners it is trivial: on P_L or P_R , $g_{\vec{p}}^i = \nabla \hat{\chi}$ is almost normal, by (4.31), by using a sufficiently small ϵ (bound depending only on C_{Pt}); on A the vector $g_{\vec{p}}^i$ is normal; the corners enclose an angle exactly equal to $\frac{\pi}{2}$, so the $g_{\vec{p}}^i$ directions are independent (almost orthogonal).

For shock-parabolic corners, on P_L and P_R , respectively, again $g_{\vec{p}}^i = \nabla \hat{\chi}$. On S , use (4.97). We normalize both derivative vectors and consider their cross product:

$$\begin{aligned}
 (4.98) \quad &\frac{\nabla \chi}{|\nabla \chi|} \times \frac{(1 - c^{-2} \chi_n^2) \vec{n} - \chi_t ((\chi_n^I)^{-1} + c^{-2} \chi_n) \vec{t}}{\sqrt{(1 - c^{-2} \chi_n^2)^2 + \chi_t^2 ((\chi_n^I)^{-1} + c^{-2} \chi_n)^2}} \\
 &= - \frac{\chi_t (1 + \chi_n / \chi_n^I)}{|\nabla \chi| \sqrt{(1 - c^{-2} \chi_n^2)^2 + \chi_t^2 ((\chi_n^I)^{-1} + c^{-2} \chi_n)^2}}.
 \end{aligned}$$

In the denominator, $|\nabla \chi| \geq \sqrt{1 - \epsilon} \cdot c \geq c/2$ for $\epsilon \leq \frac{1}{2}$. The square root is lower-bounded by $1 - c^{-2} \chi_n^2 \geq \delta > 0$ for some δ depending only on δ_ρ (see above). (4.94) bounds the numerator away from 0.

Hence (A.9) is strict, for C_d sufficiently large, δ_d and then ϵ sufficiently small, depending only and continuously on δ_ρ . □

PROPOSITION 4.26 *If the constants in (4.6) in Definition 4.4 are chosen sufficiently small (or large, depending on the conventions explained after (4.6)), then for any $\lambda \in \Lambda$, \mathcal{K}_λ cannot have fixed points on $\overline{\mathcal{F}}_\lambda - \mathcal{F}_\lambda$.*

PROOF: Let $\chi \in \overline{\mathcal{F}}$ be a fixed point of \mathcal{K} . We show that every inequality in the definition of $\overline{\mathcal{F}}$ is strict, so $\chi \in \mathcal{F}$.

- (4.30) and (4.10) are strict by Proposition 4.13.
- (4.12) is strict by Proposition 4.24.
- (4.17) is strict by Proposition 4.23.
- A fixed point satisfies $\psi = \hat{\psi}$, so $\|\psi - \hat{\psi}\| = r_I(\psi) > 0$ cannot be true. (4.21) is strict.
- (4.19) strict is provided by Proposition 4.15.
- Due to Proposition 4.15, $L^2 = 1 - \epsilon$ on each point of $\overline{P}_L \cup \overline{P}_R$, so we are in the situation of Section 4.7. Proposition 4.16 shows that (4.31) and (4.32) are strict.
- (4.33) is strict by Proposition 4.24.
- (4.18) is strict by Proposition 4.24.
- (4.34) is strict by Proposition 4.24.
- Propositions 4.17 and 4.22 rule out $\eta_L = \eta_L^* \pm \delta^{-1}\epsilon$ and $\eta_R = \eta_R^* \pm \delta^{-1}\epsilon$ if δ is small enough, so (4.14) is strict.
- (4.15) is strict by Proposition 4.24.
- (4.35) is strict by Proposition 4.24.
- Proposition 4.25 shows that (4.36) and (4.37) are strict.
- All inequalities are strict. □

4.14 Leray-Schauder Degree

We determine the Leray-Schauder degree of \mathcal{K} on \mathcal{F} for a particular choice of parameters λ : a straight horizontal shock ($\eta_L^* = \eta_R^*$), with $\gamma = 1$ (see Figure 4.9). This problem is simple enough to compute the degree precisely, although the discussion is still difficult, especially due to the free boundary.

Another option is to introduce a homotopy to an even simpler problem (linear, fixed boundary, etc.). But it would be necessary to prove a new set of a priori estimates for a family of arbitrary, unphysical problems, hence a lot of work without useful side effects.

PROPOSITION 4.27 *For sufficiently small ϵ , for $\gamma = 1$ and $\eta_L^* = \eta_R^*$, there are no fixed points of \mathcal{K} in \mathcal{F} other than the unperturbed solution.*

Remark 4.28. This result makes essential use of $\gamma = 1$ as well as the considerable simplifications from using the constant-state solution. At the time of writing we do not know a way of proving uniqueness for $\gamma > 1$ or $M_I < \infty$; this is why we choose to argue by homotopy to $\gamma = 1$, $\eta_L^* = \eta_R^*$.

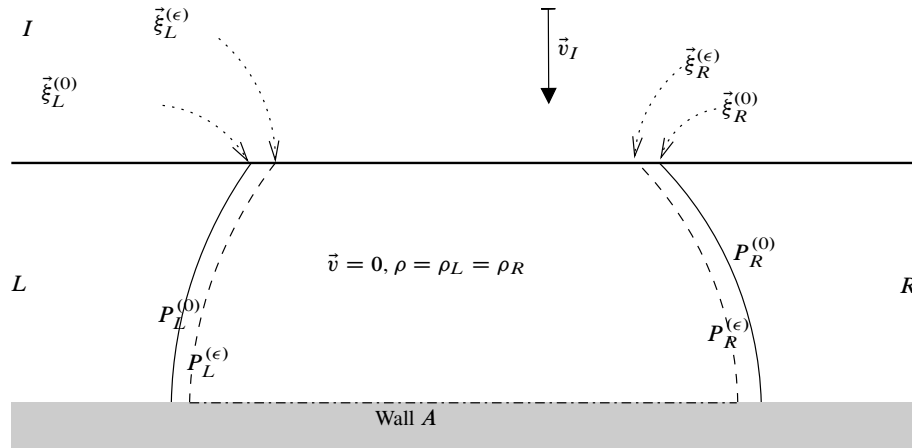


FIGURE 4.9. The unperturbed solution: a straight shock. This is the asymptotic limit for $M_I \uparrow \infty$ while holding $M_I \sin \tau$ constant. In this limit, the wall corner moves to $\vec{\xi} = (-\infty, 0)$.

PROOF OF PROPOSITION 4.27: Let $\bar{\psi}$ be the unperturbed solution and \bar{s} the corresponding shock ($\bar{s}(\xi) = \eta_R^*$ for all ξ). Consider another fixed point ψ for the same case $\gamma = 1$ and $\eta_L^* = \eta_R^*$. Let s be its shock.

Consider $\vec{\xi}$ -coordinates as defined by $\psi = \psi(\sigma, \zeta)$ (see Definition 4.4). Then

$$0 = (c^2 I - \nabla \chi^2) : \nabla^2 \psi = (c^2 I - \nabla \chi^2) : \nabla^2 (\psi - \bar{\psi}) \quad (\text{in } \Omega)$$

because $\bar{\psi}$ is constant. The classical weak maximum principle [17, theorem 10.1] (the strong version applies but is not needed) implies that $\bar{\psi} - \psi$ must attain its global minimum and maximum on the boundary. This excludes the bottom boundary because we can make it interior by reflection (Remark 4.7).

Assume ψ has an extremum (with respect to $\bar{\Omega}$) in $\vec{\xi} \in P_L \cup P_R$. Then $\psi_t = 0$ there, so $\chi_t = 0$ because $\vec{\xi} \cdot \vec{t} = 0$ on P_R and P_L for $\eta_L^* = \eta_R^*$. So by $|\nabla \chi|^2 = (1 - \epsilon)c^2$ (this argument breaks down for $\gamma > 1$), we get $\chi_n = -(1 - \epsilon)^{1/2}c$ (the sign is fixed by (4.32)) and finally

$$\psi_n = \chi_n + \vec{\xi} \cdot \vec{n} = -c\sqrt{1 - \epsilon} + c\sqrt{1 - \epsilon} = 0.$$

Again, there is a contradiction to the Hopf lemma. A reflection argument (Remark 4.7) also rules out extrema (with respect to $\bar{\Omega}$) in $\vec{\xi}_{BL}$ and $\vec{\xi}_{BR}$.

Assume ψ has a maximum $> \bar{\psi}$ on \bar{S} . By $\psi = \psi^I$ on \bar{S} , with ψ^I strictly decreasing in η , this corresponds to a minimum of s smaller than $\bar{s} = \eta_R^*$. Then $s_\xi = 0$ in that point, so graph s is horizontal there, like graph \bar{s} . Since it is lower than graph \bar{s} , the shock S is weaker there, so by (2.42) we have $\psi_2 < 0$. But this is incompatible with a maximum.

Analogously, a minimum $< \bar{\psi}$ is ruled out.

Assume ψ has a maximum $> \bar{\psi}$ in $\vec{\xi}_R$. Then $\psi = \psi^I$ on S , and ψ^I decreasing in η means $s(\xi_R) < \eta_R^*$, so $\psi_2(\vec{\xi}_R) < 0$ by (4.76) for sufficiently small ϵ . This contradicts a maximum of ψ , because the negative vertical direction from $\vec{\xi}_R$ is contained in the domain. After reversing some signs and inequalities we rule out a minimum $< \bar{\psi}$ in the same manner. Analogous arguments apply to $\vec{\xi}_L$.

Altogether we must have $\psi = \bar{\psi}$. □

PROPOSITION 4.29 *The unperturbed solution with $\gamma = 1$ has index $\iota \neq 0$ as a fixed point of \mathcal{K} from Definition 4.11.*

PROOF: We use [30, prop. 14.5]. Since \mathcal{K} is compact (Proposition 4.14), we have to show that $I - \partial\mathcal{K}/\partial\psi(\bar{\psi})$ has trivial kernel, where $\partial\mathcal{K}/\partial\psi(\bar{\psi})$ is the Fréchet derivative \mathcal{K} at $\psi = \bar{\psi}$. If this is true, then the index of $\bar{\psi}$ as a fixed point of \mathcal{K} is ± 1 .

We consider $\psi = \hat{\psi} = \bar{\psi}$ in $\vec{\sigma} = (\sigma, \zeta) \in [0, 1]$ coordinates (from Definition 4.4). We consider first variations ψ' of ψ in these coordinates: i.e., consider $\psi + t\psi'$ for $t \in \mathbb{R}$ and evaluate ∂_t at $t = 0$. Let $\mathcal{K}(\psi)'$, ρ' , $(\nabla_{\vec{\xi}}\psi)'$, etc., denote the resulting first variations of other objects.

Assume that $\psi' = \partial\mathcal{K}/\partial\psi(\bar{\psi})\psi'$ (which is $\mathcal{K}(\psi)'$). We have to show $\psi' = 0$. $\psi' = \mathcal{K}(\psi)'$ implies that the variations of the $\vec{\sigma} \mapsto \vec{\xi}$ transforms defined by ψ and $\mathcal{K}(\psi)$ (see Definition 4.4) are identical as well. We may write $(\vec{\xi})'$ without distinction. Moreover, $\psi' = \hat{\psi}'$.

The following relations are all meant to hold for $\psi = \hat{\psi} = \bar{\psi}$ and $\psi' = \hat{\psi}'$ only.

We emphasize here that the variation ψ' is taken in $\vec{\sigma}$ -coordinates. The variation of $\chi = \psi - \frac{1}{2}|\vec{\xi}|^2$ is *not* (necessarily) $\psi' - \frac{1}{2}|\vec{\xi}'|^2$ because $\vec{\xi}$ varies as well. We deliberately vary ψ , not χ , because $\nabla\bar{\psi} = 0$ in *all* coordinates, allowing considerable simplification.

In particular,

$$(\nabla_{\vec{\xi}}\psi)' = (\nabla_{\vec{\xi}}^T\vec{\sigma}\nabla_{\vec{\sigma}}\psi)' = (\nabla_{\vec{\xi}}^T\vec{\sigma})' \underbrace{\nabla_{\vec{\sigma}}\psi}_{=0} + \nabla_{\vec{\xi}}^T\vec{\sigma}(\nabla_{\vec{\sigma}}\psi)' = \nabla_{\vec{\xi}}^T\vec{\sigma}\nabla_{\vec{\sigma}}\psi' = \nabla_{\vec{\xi}}\psi'$$

and

$$\begin{aligned} (\nabla_{\vec{\xi}}^2\psi)' &= \left(\sum_k \frac{\partial\psi}{\partial\sigma^k} \nabla_{\vec{\xi}}^2\sigma^k + \nabla_{\vec{\xi}}^T\vec{\sigma}\nabla_{\vec{\sigma}}^2\psi\nabla_{\vec{\xi}}\vec{\sigma} \right)' \\ &= \left(\sum_k \frac{\partial\psi}{\partial\sigma^k} \right)' \nabla_{\vec{\xi}}^2\sigma^k + \sum_k \underbrace{\frac{\partial\psi}{\partial\sigma^k}}_{=0} (\nabla_{\vec{\xi}}^2\sigma^k)' \\ &\quad + (\nabla_{\vec{\xi}}^T\vec{\sigma})' \underbrace{\nabla_{\vec{\sigma}}^2\psi}_{=0} \nabla_{\vec{\xi}}\vec{\sigma} + \nabla_{\vec{\xi}}^T\vec{\sigma}(\nabla_{\vec{\sigma}}^2\psi)' \nabla_{\vec{\xi}}\vec{\sigma} + \nabla_{\vec{\xi}}^T\vec{\sigma} \underbrace{\nabla_{\vec{\sigma}}^2\psi}_{=0} (\nabla_{\vec{\xi}}\vec{\sigma})' = \end{aligned}$$

$$\begin{aligned}
 (4.99) \quad &= \sum_k \frac{\partial \psi'}{\partial \sigma^k} \nabla_{\vec{\xi}}^2 \sigma^k + \nabla_{\vec{\xi}}^T \vec{\sigma} \nabla_{\vec{\sigma}}^2 \psi' \nabla_{\vec{\xi}} \vec{\sigma} \\
 &= \nabla_{\vec{\xi}}^2 \psi'.
 \end{aligned}$$

The same relations hold for $\hat{\psi}$ instead of ψ .

The Fréchet derivative of the interior equation is

$$\begin{aligned}
 0 &= (c^2 I - (\nabla_{\vec{\xi}} \hat{\chi})^2)' : \underbrace{\nabla_{\vec{\xi}}^2 \hat{\psi}}_{=0} + (c^2 I - (\nabla_{\vec{\xi}} \hat{\chi})^2) : (\nabla_{\vec{\xi}}^2 \hat{\psi})' \\
 &= (c^2 I - (\nabla_{\vec{\xi}} \hat{\chi})^2) : \nabla_{\vec{\xi}}^2 \psi'.
 \end{aligned}$$

The resulting right-hand side is a linear elliptic operator without a zeroth-order term, applied to ψ' . The classical maximum principle shows that ψ' cannot have a minimum in the interior.

On the parabolic arcs,

$$c^2 = |\nabla_{\vec{\xi}} \hat{\chi}|^2$$

linearizes to

$$\begin{aligned}
 0 &= \nabla_{\vec{\xi}} \chi \cdot (\nabla_{\vec{\xi}} \hat{\chi})' = \underbrace{(\nabla_{\vec{\xi}} \psi - \vec{\xi})}_{=0} \cdot ((\nabla_{\vec{\xi}} \hat{\psi})' - (\vec{\xi})') \\
 &= \vec{\xi} \cdot (\vec{\xi})' - \underbrace{\vec{\xi}}_{=|\vec{\xi}|\vec{n}} \cdot \nabla_{\vec{\xi}} \psi' = \underbrace{\frac{1}{2}(|\vec{\xi}|^2)'}_{=(r^2)'=0} + |\vec{\xi}| \psi'_n
 \end{aligned}$$

$$(4.100) \quad \Rightarrow \quad \psi'_n = 0.$$

Here we use that variation of ψ may move $\vec{\xi}$ but keep it on P_R . (4.100) does not admit any extrema of $\hat{\psi}'$ by the Hopf lemma.

By reflection across A , we can also rule out extrema on \bar{A} by applying the arguments for P_L , P_R , and Ω from above a second time.

For shock, $\psi = \psi^I = \psi^I(0, 0) + v_I^y \eta$, so

$$(4.101) \quad \eta' = (v_I^y)^{-1} \psi' \quad \text{on shock.}$$

Moreover,

$$\begin{aligned}
 (4.102) \quad &\left(\frac{\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}}{|\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}|} \right)' = \frac{-1}{|\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}|} \underbrace{\left(1 - \left(\frac{\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}}{|\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}|} \right)^2 \right)}_{\substack{= \vec{n} \\ = (\vec{t})^2}} \nabla_{\vec{\xi}} \hat{\psi}' \\
 &= \frac{-(\hat{\psi}')_t}{|\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}|} \vec{t}.
 \end{aligned}$$

Take the Fréchet derivative of (4.27):

$$\begin{aligned}
0 &= \left((\hat{\rho} \nabla_{\vec{\xi}} \hat{\chi} - \rho_I \nabla_{\vec{\xi}} \chi^I) \cdot \frac{\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}}{|\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}|} \right)' \\
&= ((\hat{\rho})' \nabla_{\vec{\xi}} \hat{\chi} + \rho (\nabla_{\vec{\xi}} \hat{\chi})' - \rho_I (\nabla_{\vec{\xi}} \chi^I)') \cdot \vec{n} \\
&\quad + (\rho \nabla_{\vec{\xi}} \hat{\chi} - \rho_I \nabla_{\vec{\xi}} \chi^I) \cdot \left(\frac{\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}}{|\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}|} \right)' \\
&= \rho c^{-2} (-\hat{\chi}' - \nabla_{\vec{\xi}} \hat{\chi} \cdot (\nabla_{\vec{\xi}} \hat{\chi})') \hat{\chi}_n + \rho (\nabla_{\vec{\xi}} \hat{\psi} - \vec{\xi})' \cdot \vec{n} + \rho_I (\vec{\xi})' \cdot \vec{n} \\
&\quad + (\rho \nabla_{\vec{\xi}} \hat{\chi} - \rho_I \nabla_{\vec{\xi}} \chi^I) \cdot \left(\frac{\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}}{|\vec{v}_I - \nabla_{\vec{\xi}} \hat{\psi}|} \right)' \\
&\stackrel{(4.102)}{=} \rho c^{-2} (-\hat{\psi}' + \vec{\xi} \cdot (\vec{\xi})' - \nabla_{\vec{\xi}} \hat{\chi} \cdot (\nabla_{\vec{\xi}} \hat{\psi})' + \underbrace{\nabla_{\vec{\xi}} \hat{\psi} \cdot (\vec{\xi})' - \vec{\xi} \cdot (\vec{\xi})'}_{=0}) \hat{\chi}_n \\
&\quad + \rho (\nabla_{\vec{\xi}} \hat{\psi})' \cdot \vec{n} - \rho (\vec{\xi})' \cdot \vec{n} + \rho_I (\vec{\xi})' \cdot \vec{n} - \frac{(\rho \nabla_{\vec{\xi}} \chi - \rho_I \nabla_{\vec{\xi}} \chi^I) \cdot \vec{t}}{|\vec{v}_I - \nabla_{\vec{\xi}} \psi|} (\psi')_t \\
&= -\rho c^{-2} \psi' + \rho (1 - c^{-2} \chi_n^2) (\psi')_n - \rho c^{-2} \chi_n \chi_t (\psi')_t + (\rho - \rho_I) \eta' \\
&\quad - \frac{\rho - \rho_I}{|\vec{v}_I - \nabla_{\vec{\xi}} \psi|} \chi_t (\psi')_t \\
&= \rho (1 - c^{-2} \chi_n^2) (\psi')_n + \left(\frac{\rho_I - \rho}{|\vec{v}_I - \nabla_{\vec{\xi}} \psi|} - \rho c^{-2} \chi_n \right) \chi_t (\psi')_t \\
&\quad + (\rho - \rho_I) \eta' - \rho c^{-2} (\psi') \\
&\stackrel{(4.101)}{=} \rho (1 - c^{-2} \chi_n^2) (\hat{\psi}')_n + \left(\frac{\rho_I - \rho}{|\vec{v}_I - \nabla_{\vec{\xi}} \psi|} - \rho c^{-2} \chi_n \right) \chi_t (\psi')_t \\
&\quad + \left(\frac{\rho - \rho_I}{v_I^y} - \rho c^{-2} \right) \psi' \\
&^* = \underbrace{-\rho (1 - c^{-2} \chi_2^2)}_{<0} (\hat{\psi}')_2 + \left(\underbrace{\frac{\rho_I - \rho}{|\vec{v}_I - \nabla_{\vec{\xi}} \psi|}}_{<0} + \underbrace{\rho c^{-2} \chi_2}_{<0} \right) \chi_1 (\psi')_1 \\
(4.103) \quad &+ \left(\underbrace{\frac{\rho - \rho_I}{v_I^y}}_{<0} - \underbrace{\rho c^{-2}}_{>0} \right) \psi'.
\end{aligned}$$

Now consider the case of a positive maximum of $\hat{\psi}$ on the shock. Then $(\psi')_t = (\psi')_1 = 0$, which also implies $(\vec{n})' = 0$ by (4.102). (This is natural because then we perturb the shock in a way that keeps it horizontal in that point, so the normal

does not change in first order.) The two remaining terms are

$$0 = \underbrace{-\rho(1 - c^{-2}\chi_2^2)}_{<0}(\psi')_2 + \underbrace{\left(\frac{\rho - \rho I}{v_I^y} - \rho c^{-2}\right)}_{<0}\psi'.$$

For a positive local maximum of ψ' , we need $(\psi')_2 \geq 0$, but this is incompatible with the previous equation. In the same way, a negative local minimum is ruled out.

Finally, consider a positive maximum of ψ' in $\vec{\xi}_R$. The parabolic condition (4.100) means that $\nabla_{\vec{\xi}}\psi'$ must be tangential to P_R in the corner; for a maximum we need that it points counterclockwise, hence into the upper left quadrant, unless it is 0. On the other hand, the vector consisting of the $(\psi')_2$ and $(\psi')_1$ coefficients in (4.103) points into the lower right quadrant (because $\chi_1 < 0$ in the right corner), so the scalar product is ≤ 0 . But the coefficient of ψ' is < 0 , so the right-hand side of (4.103) is < 0 —contradiction!

The same argument, with certain signs and inequalities reversed, rules out a positive maximum in the left corner as well as a negative minimum in either corner.

We have ruled out that ψ' has a positive maximum or negative minimum anywhere in the domain. Therefore $\psi' = 0$, which is precisely what we had to show. \square

4.15 Existence of Fixed Points

PROPOSITION 4.30 *For sufficiently small (or large, as explained after (4.6)) constants in (4.6), \mathcal{K} has a fixed point for all $\lambda \in \Lambda$.*

PROOF: Let $t \in [0, 1] \mapsto \lambda(t)$ be any path in Λ . We have shown in Proposition 4.10 that $U := \bigcup_{t \in [0, 1]} (\{t\} \times \mathcal{F}_{\lambda(t)})$ is open in $[0, 1] \times C_{\beta}^{2,\alpha}([0, 1]^2)$, and that its closure is contained in $W := \bigcup_{t \in [0, 1]} (\{t\} \times \overline{\mathcal{F}}_{\lambda(t)})$, so

$$\partial U \subset V := W - U = \bigcup_{t \in [0, 1]} (\{t\} \times (\overline{\mathcal{F}}_{\lambda(t)} - \mathcal{F}_{\lambda(t)})).$$

By Proposition 4.14, \mathcal{K} is a continuous and compact map on W . Finally, by Proposition 4.26 we know \mathcal{K} cannot have fixed points on V , hence on ∂U . U is bounded. Thus we may apply property (D4*) in [30, sec. 13.6] to argue that the Leray-Schauder index of $\mathcal{K}_{\lambda(t)}$ on $\mathcal{F}_{\lambda(t)}$ is constant in $t \in [0, 1]$.

For $\lambda(0) = (\gamma, \eta_L^*) = (1, \eta_R^*)$, Proposition 4.29 shows that the Leray-Schauder degree of \mathcal{K}_{λ} is $\iota \neq 0$. Since Λ is path-connected (Lemma 4.3), there is a path to any other $\lambda = \lambda(1)$. So \mathcal{K}_{λ} has degree $\iota \neq 0$ for all $\lambda \in \Lambda$. Nonzero degree requires at least one fixed point.¹⁸ \square

¹⁸We expect that there is only one, but we do not need this fact. For all we prove, there could be more, e.g., two with index ι and one with index $-\iota$ (which would still yield a total Leray-Schauder degree ι).

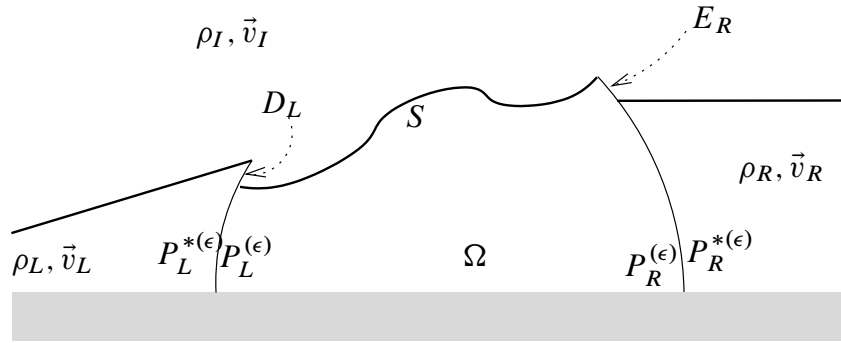


FIGURE 4.10. Extension from Ω to the entire domain.

4.16 Construction of the Entire Flow

PROOF OF THEOREM 1.1: For all $\rho_I, c_I \in (0, \infty)$ and $M_I^y \in (-\infty, 0)$, for each choice of $\bar{\gamma}$ and $\underline{\eta}_L^*$, and for all sufficiently small ϵ , a separate Λ is defined. For sufficiently small constants in (4.6), Proposition 4.30 yields fixed points ψ for all $\lambda \in \Lambda$. Note that there is no lower bound on ϵ except that α, β , and some other constants may deteriorate as $\epsilon \downarrow 0$. By Definition 4.4, Remark 4.6, Proposition 4.16, and (4.44), the fixed points satisfy

$$(4.104) \quad (c^2 I - \nabla \chi^2) : \nabla^2 \psi = 0 \quad \text{in } \Omega^{(\epsilon)},$$

$$(4.105) \quad |\psi - \psi^I(\vec{\xi}_C^*)| = O(\epsilon^{1/2}),$$

$$(4.106) \quad |\rho - \rho_C| = O(\epsilon^{1/2}),$$

$$(4.107) \quad |\nabla \psi - \vec{v}_C| = O(\epsilon^{1/2}) \quad \text{on } P_C^{(\epsilon)} \quad (C = L, R),$$

$$(4.108) \quad \chi = \chi^I,$$

$$(4.109) \quad (\rho \nabla \chi - \rho_I \nabla \chi^I) \cdot \vec{n} = 0 \quad \text{on } S,$$

$$(4.110) \quad \chi_n = 0 \quad \text{on } A,$$

$$(4.111) \quad |\vec{\xi}_C - \vec{\xi}_C^{*(\epsilon)}| = O(\epsilon^{1/2}) \quad (C = L, R),$$

where the O constants are independent of ϵ . For regularity we have

$$(4.112) \quad \|\psi\|_{C^{0,1}(\bar{\Omega}^{(\epsilon)})} \leq C_1,$$

$$(4.113) \quad \|\psi\|_{C^{k,\alpha}(K \cap \bar{\Omega}^{(\epsilon)})}, |S|_{C^{k,\alpha}(K \cap \bar{S}^{(\epsilon)})} \leq C_2(d),$$

where $d := d(K, \hat{P}_L^{(\epsilon)} \cup \hat{P}_R^{(\epsilon)}) > 0$.

for constants C_1 and $C_2(d)$ independent of ϵ .

We extend each ψ to a function $\psi^{(\epsilon)}$ defined on $\mathbb{C}W$ as shown in Figure 4.10: we use ρ_L and \vec{v}_L in the region enclosed by the wall, L shock, and $P_L^{*(\epsilon)}$; we use ρ_R and \vec{v}_R in the region enclosed by R shock and $P_R^{*(\epsilon)}$, and ρ_I and \vec{v}_I elsewhere.

In each of the four regions, $\psi^{(\epsilon)}$ is a strong solution of self-similar potential flow, so we may multiply the divergence form (2.13) with any test function $\vartheta \in C_c^\infty(\mathbb{C}W)$ and integrate over each of the four regions to obtain a sum of boundary integrals. Set

$$D_C := P_C^{*(\epsilon)} - P_C^{(\epsilon)}, \quad E_C := P_C^{(\epsilon)} - P_C^{*(\epsilon)}, \quad C = L, R.$$

These curves have length $O(\epsilon^{1/2})$, so the integrals over them can be neglected because $\psi^{(\epsilon)}$ has ϵ uniformly bounded $C^{0,1}$ norm in each region. The two integrals over $S^{(\epsilon)}$ cancel because of (4.109). The two integrals over $P_L^{(\epsilon)} \cap P_L^{*(\epsilon)}$ are $O(\epsilon^{1/2})$ due to (4.106) and (4.107), and the same for P_R . The integrals over each piece of the wall vanish because $\chi_n = 0$ in all cases. Therefore,

$$(4.114) \quad \int_{\mathbb{C}W} \rho^{(\epsilon)} \nabla \chi^{(\epsilon)} \cdot \nabla \vartheta - 2\rho^{(\epsilon)} \vartheta d\vec{\xi} = O(\epsilon^{1/2}).$$

By (4.113) with a diagonalization argument, for every compact $K \subset \mathbb{C}W - \overline{P_L^{*(0)}} - \overline{P_R^{*(0)}}$ we can find a sequence $(\epsilon_k) \downarrow 0$ so that $\psi^{(\epsilon_k)}$ converges to $\psi^{(0)}$ in $C^{0,1}(K)$. Moreover, the convergence is bounded in $C^{0,1}(\mathbb{C}W)$, so we can take $\epsilon \downarrow 0$ in (4.114) to obtain

$$(4.115) \quad \int_{\mathbb{C}W} \rho^{(0)} \nabla \chi^{(0)} \cdot \nabla \vartheta - 2\rho^{(0)} \vartheta d\vec{\xi} = 0.$$

In addition, (4.108), (4.105), and (4.112) show that

$$(4.116) \quad \psi^{(0)} \in C(\mathbb{C}W).$$

Finally, by construction of $\psi^{(\epsilon)}$,

$$(4.117) \quad \rho^{(0)}(s\vec{\xi}), \vec{v}^{(0)}(s\vec{\xi}) \rightarrow \rho_I, \vec{v}_I \quad (s \rightarrow \infty)$$

for any $\vec{\xi} \in \mathbb{C}\overline{W} - \{0\}$ (note that any ray from the origin is either in \overline{W} or enters and stays in the I region).

(4.115), (4.116), and (4.117) show that $\phi(t, \vec{x}) := \psi^{(0)}(t^{-1}\vec{x})$ defines a solution of (1.2), (1.3), and (1.4).

Taking $\bar{\gamma} \uparrow \infty$ and $\eta_L^*(\gamma) \downarrow \eta_L^x(\gamma)$, we obtain a solution for every $\gamma \in [1, \infty)$ and $\eta_L^* \in (\eta_L^x(\gamma), \bar{\eta}_R^*]$. Note that $\bar{\eta}_L^* \uparrow \bar{\eta}_R^*$ for $\epsilon \downarrow 0$ by Definition 4.1.

It remains to show that we have covered every tip shock that satisfies the conditions of Theorem 1.1. Consider *standard* coordinates (Figure 4.1). By Proposition 2.12, the shocks with some upstream velocity $\vec{v}_I = (0, v_I^y)$, $v_I^y < 0$, and density ρ_I and downstream velocity parallel to the wall are *uniquely* determined by η_L^* . The choices with $\eta_L^* \geq \eta_R^*$ cannot intersect the horizontal axis left of $\vec{\xi}_L^*$, as their shock slope is negative by Proposition 2.12. The choices with $\eta_L^* \leq 0$ are possible, but they are supersonic-subsonic at the intersection point (note that

their other $L_d = 1$ point must be above the axis, as their slope is positive and $\partial B_{c_d}(\vec{v}_d)$ is centered on the horizontal axis). Therefore, all supersonic-supersonic tip shocks have $\eta_L^* \in (0, \eta_R^*]$ after passing from original to standard coordinates; there is only one such shock for each η_L^* , and we have constructed a solution for each that satisfies (1.1). \square

Remark 4.31. It remains to discuss whether there *are* tip shocks that satisfy the conditions of Theorem 1.1. Proposition 2.12, as discussed in the last paragraph of the proof of Theorem 1.1, already settles that question. By Lemma 4.2, for any $M_I^y < 0$, there is an interval $(\eta_L^x(\gamma), \eta_R^*]$ of η_L^* so that (1.1) is satisfied. $\eta_L^x < \eta_R^*$, so there are some nontrivial solutions. For small $|M_I^y|$ we have $\eta_L^x > 0$, so there are cases where (1.1) is violated. But if $|M_I^y|$ is sufficiently large, then $\eta_L^x = 0$. In particular, if $M_I^y \leq -1$, then $\overline{B}_{c_I}(\vec{v}_I)$ is below the wall (in R -coordinates), so (1.1) is always satisfied.

Remark 4.32. In addition to mere existence, we obtain some structural information in the proof:

- The solution has the structure shown in Figure 1.4, with $L > 1$ in the I , L , and R regions, $L < 1$ in the elliptic region.
- The solution has constant density and velocity in each of the I , L , and R regions.
- The solution is analytic everywhere except perhaps at the L , Ω and Ω , R interfaces; it is analytic on each side of S .
- The shock is Lipschitz; it is straight between the I , L and I , R regions; it is analytic between I and the elliptic region (away from the corners).
- Density and velocity are bounded.

It is expected that density and velocity are at least continuous. However, the methods developed in this article yield boundedness everywhere, but continuity only away from $\overline{P}_L^* \cup \overline{P}_R^*$. Note that $P_L^{(0)}$ and $P_R^{(0)}$ *cannot* be classical shocks with smooth data on each side, because the one-sided limit of L on the hyperbolic side of the parabolic arcs is $= 1$ everywhere (> 1 is needed for positive shock strength).

In some points, continuity can be obtained a posteriori. For example $v^x \leq 0$, as implied by (4.33), yields continuity in ξ_{BR} .

Some other structural information:

- The possible (downstream) normals of the curved shock portion are between the R and L shock normals (counterclockwise).
- The shock is admissible and does not vanish anywhere.
- Therefore, the shock is above the line connecting the arc-shock corners.
- In the elliptic region, the velocity v^x tangential to the wall is between v_L^x and v_R^x .
- In the elliptic region, the density ρ is greater than ρ_I .

Additional information can be obtained from the inequalities in Definition 4.4.

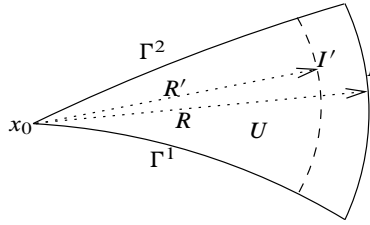


FIGURE A.1. Corner

Appendix A: Regularity in Two-Dimensional Corners

We obtain $C_\beta^{2,\alpha}$ bounds in a corner neighborhood from merely C^1 (with a uniform $C^{0,1}$ bound). Although our argument follows [23, theorem 2.1] without major modification, we prefer to repeat it explicitly, because we need a uniformly large neighborhood.

We keep the proof simple by restricting ourselves to smooth coefficient functions and a priori C^1 rather than $C^{0,1}$ regularity.

PROPOSITION A.1

(1) Consider a point $x_0 \in \mathbb{R}^2$ and polar coordinates (r, ϕ) with origin in that point. Let $R > 0$. Consider two functions $\phi^1, \phi^2 : [0, R] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ so that $\phi^1 < \phi^2$. For $j = 1, 2$ let Γ^j be the curve $\{(r, \phi^j(r)) : r \in (0, R)\}$. Assume that Γ^j are $C^{1,\sigma}$ curves ($\sigma \in (0, 1]$), including the endpoints:

$$(A.1) \quad |\bar{\Gamma}^j|_{C^{1,\sigma}} \leq C_\Gamma.$$

Let $I = \{(R, \phi) : \phi \in (\phi^1(R), \phi^2(R))\}$. Let $U = \{(r, \phi) : r \in (0, R), \phi \in (\phi^1(r), \phi^2(r))\}$. Assume there is a $\theta < \pi$ so that

$$(A.2) \quad \sup_{r \in (0, R)} \phi^2(r) \leq \frac{\theta}{2}, \quad \inf_{r \in (0, R)} \phi^1(r) \geq -\frac{\theta}{2}.$$

Consider a function $u \in C^3(U) \cap C^2(U \cup \Gamma^1 \cup \Gamma^2) \cap C^1(\bar{U})$ with

$$(A.3) \quad \|u\|_{C^1(\bar{U})} \leq C_u.$$

Assume that u satisfies

$$(A.4) \quad a^{ij}(x, u(x), Du(x))u_{ij}(x) = 0 \quad \forall x \in U$$

(we use the Einstein convention in this section) as well as boundary conditions

$$(A.5) \quad g^k(x, u(x), Du(x)) = 0 \quad \forall x \in \Gamma^k, \quad k \in \{1, 2\}.$$

Here a^{ij} and g^k ,¹⁹ as functions of x, u , and ∇u , are assumed to satisfy

$$(A.6) \quad \|a^{ij}\|_{C^1(\bar{U} \times \mathbb{R} \times \mathbb{R}^2)}, \|g^k\|_{C^1(\bar{U} \times \mathbb{R} \times \mathbb{R}^2)} \leq C_c.$$

¹⁹Note that g^k is defined for all arguments. See [23] for more general circumstances.

Assume uniform ellipticity; there is a $\delta_e > 0$, independent of $x \in \bar{U}$, $z \in \mathbb{R}$, $p \in \mathbb{R}^2$, and $y \in \mathbb{R}^2$, so that

$$(A.7) \quad a^{ij}(\bar{x}, z, p)y^i y^j \geq \delta_e |y|^2.$$

Moreover, assume uniform nondegenerate obliqueness. There is a $\delta_o > 0$, independent of $x \in \Gamma^k$, $z \in \mathbb{R}$, and $p \in \mathbb{R}^2$ so that

$$(A.8) \quad g_p^k(x, z, p) \cdot n(x) \geq \delta_o,$$

where n is the outer unit normal to Γ^k ($k = 1, 2$).

Finally, let C_d be independent of $x \in \bar{U}$, $z \in \mathbb{R}$, and $p \in \mathbb{R}^2$, so that

$$(A.9) \quad \|G^{-1}\| \leq C_d,$$

where G is the 2×2 matrix with k^{th} column $g_p^k(x, z, p)$.

Then there are $R' \in (0, R)$, $\lambda > 0$, and $C_r < \infty$ so that

$$(A.10) \quad |\nabla u(x) - \nabla u(x_0)| \leq C_r |x - x_0|^\lambda \quad \text{for } x \in U.$$

R' , λ , and C_r depend only on C_d , δ_e , C_c , δ_o , C_u , C_Γ , and R except R' may also depend on σ .

(2) Assume in addition that

$$(A.11) \quad \|G\|_{C^{1,\sigma}(\bar{U} \times \mathbb{R} \times \mathbb{R}^2)} \leq C_\Gamma.$$

Then there are $\kappa > 0$, $C_r < \infty$, and $R' > 0$ so that

$$(A.12) \quad \|u\|_{C_{1+\lambda}^{2,\kappa}(B_{R'}(x_0) \cap \bar{U}, \{x_0\})} \leq C_r.$$

λ is as before. R' may have changed but has the same dependencies. C_r may have changed and may depend on σ now. κ depends continuously and only on C_d , δ_e , C_c , δ_o , C_u , C_Γ , σ , and R .

PROOF: In this proof let C , C_i , δ , and δ_i (for numbers i) represent constants that may depend continuously and only on C_d , δ_e , C_c , δ_o , C_u , C_Γ , and R (but not σ). C and δ may change from occurrence to occurrence. $\delta > 0$ is meant to be small, $|C| < \infty$ large.

Constructing the Interior of w . Define

$$v(x) := g^1(x, u(x), Du(x)) + Mw(x).$$

Set

$$w(r, \phi) := r^\lambda f(\phi), \quad f(\phi) := 1 - \mu e^{-B\phi},$$

with $\lambda \in (0, \frac{1}{2}]$ and $\mu, B > 0$ to be determined. We take μ so small that $f \geq \frac{1}{2}$. In a given point of U , rotate coordinates to be angular and radial and let a^{rr} , $a^{r\phi}$, and $a^{\phi\phi}$ be the corresponding components of a^{ij} .

$$\begin{aligned} a^{ij} w_{ij} &= a^{rr} w_{rr} + 2a^{r\phi}(r^{-1}w_{r\phi} - r^{-2}w_\phi) + a^{\phi\phi}(r^{-2}w_{\phi\phi} + r^{-1}w_r) \\ &= (a^{rr}\lambda(\lambda-1)f + 2a^{r\phi}(\lambda-1)f' + a^{\phi\phi}(f'' + \lambda f))r^{\lambda-2} = \end{aligned}$$

$$\begin{aligned}
 &= \left(\underbrace{a^{rr} \lambda (\lambda - 1) f}_{\leq 0} + \underbrace{2a^{r\phi} (\lambda - 1) \cdot \mu B e^{-B\phi}}_{\leq 2|a^{r\phi}|} \right. \\
 &\quad \left. + \underbrace{a^{\phi\phi}}_{\geq 0} (\lambda \underbrace{f}_{\leq 1} - \mu B^2 e^{-B\phi}) \right) r^{\lambda-2} \\
 &\leq (e^{-B\phi} \mu B (2|a^{r\phi}| - Ba^{\phi\phi}) + \lambda a^{\phi\phi}) r^{\lambda-2}.
 \end{aligned}$$

We pick²⁰

$$B = \frac{1 + 2 \sup |a^{r\phi}|}{\inf a^{\phi\phi}} > 0$$

so that

$$1a^{ij} w_{ij} \leq (-\mu B e^{-B\phi} + \lambda a^{\phi\phi}) r^{\lambda-2}.$$

For any $\mu > 0$ we can pick a $\lambda = \lambda(\mu) > 0$ so small that

$$(A.13) \quad a^{ij} w_{ij} \leq -\frac{\mu B}{2} e^{-B\phi} r^{\lambda-2}.$$

Boundary Condition for w. Let ν and τ be Cartesian coordinates that are normal and tangential, respectively, to a given point on Γ^2 (then $\partial_\nu = \partial_n$ and $\partial_\tau = \partial_t$, but higher derivatives differ). Let ν be the outer normal and τ point away from the corner (clockwise from the ν direction).

The angle between the radial and tangential direction is $\leq C(R')^\sigma$ because of (A.1). We can make R' small to control it. Note that C can change from line to line.

$$\begin{aligned}
 w_\nu &\geq (1 - C(R')^\sigma) r^{-1} w_\phi - C(R')^\sigma |w_r| \\
 &= (1 - C(R')^\sigma) r^{\lambda-1} \mu B e^{-B\phi} - C(R')^\sigma \lambda r^{\lambda-1} (1 - \mu e^{-B\phi}) \\
 &= (B\mu e^{-B\phi} - C(R')^\sigma) r^{\lambda-1} \\
 |w_\tau| &\leq (1 - C(R')^\sigma) |w_r| + C(R')^\sigma |w_\phi| \leq (\lambda + C(R')^\sigma B\mu) r^{\lambda-1}.
 \end{aligned}$$

For any constant T ,

$$(A.14) \quad w_\nu + T w_\tau \geq (B\mu e^{-B\phi} - C(R')^\sigma - T(\lambda + C(R')^\sigma B\mu)) r^{\lambda-1}.$$

D^2u and $D(g^1)$.

$$(g^1)_i := \partial_{x^i} (g^1(x, u, Du(x))) = g_{pk}^1 u_{ik} + C \quad (i = 1, 2).$$

(C is a remnant of terms containing only $u(x)$ and $Du(x)$, which can be estimated by C_u via (A.3).) Combined with (A.4) we have a system

$$\begin{bmatrix} a^{11} & 2a^{12} & a^{22} \\ g_{p1}^1 & g_{p2}^1 & 0 \\ 0 & g_{p1}^1 & g_{p2}^1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ (g^1)_1 \\ (g^1)_2 \end{bmatrix} + C.$$

²⁰We need full ellipticity later, but here only a^{rr} and $a^{\phi\phi} > 0$ matter.

The system matrix has determinant

$$\begin{aligned} a^{11}(g_{p_2}^1)^2 - 2a^{12}g_{p_1}^1g_{p_2}^1 + a^{22}(g_{p_1}^1)^2 \\ = ((g_p^1)^\perp)^\top A(g_p^1)^\perp \stackrel{(A.7)}{\geq} \delta_e |g_p^1|^2 \stackrel{(A.6)}{\geq} \delta_e C_c > 0. \end{aligned}$$

Combined with an upper bound on the matrix (from (A.6)) we obtain that

$$(A.15) \quad \begin{bmatrix} u_{11} \\ u_{12} \\ u_{22} \end{bmatrix} = D(x)\nabla(g^1) + C$$

where $D(x) \in \mathbb{R}^{3,2}$ with $|D(x)| \leq C$.

Interior PD Inequality.

$$(A.16) \quad a^{ij}v_{ij} = g_{p^k}^1 a^{ij}u_{kij} + (g_{p^k p^\ell}^1 u_{j\ell} + C)u_{ik} + Ma^{ij}w_{ij} + C.$$

Again we use C to abbreviate terms depending on u and Du only, which are bounded by (A.3).

Take ∂_k of (A.4):

$$a^{ij}u_{ijk} + (a_{p^\ell}^{ij}u_{k\ell} + C)u_{ij} = 0.$$

Multiply by $g_{p^k}^1$ and substitute into (A.16):

$$a^{ij}v_{ij} = -(C + g_{p^k}^1 a_{p^\ell}^{ij}u_{k\ell})u_{ij} + (g_{p^k p^\ell}^1 u_{j\ell} + C)u_{ik} + Ma^{ij}w_{ij} + C.$$

Apply (A.15):

$$a^{ij}v_{ij} = Ma^{ij}w_{ij} + e^{ij}(g^1)_i(g^1)_j + e^i(g^1)_i + C$$

for some coefficients $e^{ij}(x), e^i(x) = C$. Use $v_i = (g^1)_i + Mw_i$ to obtain

$$\begin{aligned} a^{ij}v_{ij} + \underbrace{(-e^{ij}v_j + M(e^{ij} + e^{ji})w_j - e^i)}_{=:q^i} \cdot v_i \\ = Ma^{ij}w_{ij} + M^2e^{ij}w_iw_j - Me^iw_i + C \\ \stackrel{(A.13)}{\leq} -\frac{M\mu B}{2}e^{-B\phi}r^{\lambda-2} + CM^2|\nabla w|^2 + C \\ \leq -\frac{M\mu B}{2}e^{-B\phi}r^{\lambda-2} + CM^2(\lambda^2 \underbrace{f^2}_{\leq 1} + \mu^2 B^2 e^{-2B\phi})r^{2\lambda-2} + C. \end{aligned}$$

We take $M = M'(R')^{-\lambda}$ with $M' \geq 1$ to be determined:

$$a^{ij} v_{ij} + q^i v_i \leq \underbrace{\left(-\frac{M' \mu B}{2} e^{-B\phi} \left(\frac{r}{R'} \right)^\lambda \right)}_{(I)} + C(M')^2 \left(\frac{r}{R'} \right)^{2\lambda} \left(\underbrace{\lambda^2}_{(II)} + \underbrace{\mu^2 B^2 e^{-2B\phi}}_{(III)} \right) r^{-2} + C.$$

Given any M' and $R' \in (0, 1]$, we take μ (and λ , which depends on it) so small that the (III) term is dominated by the (I) term. Then we pick λ even smaller so that the (II) term is also dominated by the (I) term. In both cases we use that $(r/R')^\lambda \geq (r/R')^{2\lambda}$ because $0 < r \leq R'$. Now

$$a^{ij} v_{ij} + q^i v_i \leq -\frac{M' \mu B}{4} e^{-B\phi} \underbrace{\left(\frac{r}{R'} \right)^\lambda r^{-2}}_{\geq (R')^{-2}} + C.$$

Finally, we take R' so small that the C term is dominated. We obtain

$$a^{ij} v_{ij} + q^i v_i < 0 \quad \text{in } U \cap B_{R'}(x_0).$$

Therefore v cannot have minima in that set.

Boundary PD Inequality. Take ∂_τ of the boundary condition (A.5) on Γ^2 to obtain

$$g_{p^\tau}^2 u_{\tau\tau} + g_{p^v}^2 u_{v\tau} + C = 0.$$

We plan to take a linear combination of this with the interior equation (A.4) to obtain a new boundary condition of the form

$$(A.17) \quad \begin{aligned} C &= (g^1)_v + T(g^1)_\tau \\ &= g_{p^v}^1 u_{vv} + (Tg_{p^v}^1 + g_{p^\tau}^1) u_{\tau v} + Tg_{p^\tau}^1 u_{\tau\tau}. \end{aligned}$$

This problem can be written

$$[y_1 \quad y_2 \quad -1] \begin{bmatrix} a^{\tau\tau} & 2a^{\tau v} & a^{\nu\nu} \\ g_{p^\tau}^2 & g_{p^v}^2 & 0 \\ Tg_{p^\tau}^1 & Tg_{p^v}^1 + g_{p^\tau}^1 & g_{p^v}^1 \end{bmatrix} \stackrel{!}{=} 0.$$

It is solvable if and only if the determinant is 0, which is a linear equation for T , with solution

$$T = \frac{C}{a^{\nu\nu} \cdot (g_{p^\tau}^2 g_{p^v}^1 - g_{p^v}^2 g_{p^\tau}^1)} = \frac{C}{a^{\nu\nu} \det G}.$$

The denominator is bounded away from 0 by (A.7) and (A.9), so $|T| \leq C$. The solutions are

$$y_1 = \frac{g_{p^v}^1}{a^{\nu\nu}}, \quad y_2 = \frac{a^{\tau\tau} (g_{p^v}^1)^2 - 2a^{\tau v} g_{p^\tau}^1 g_{p^v}^1 + a^{\nu\nu} (g_{p^\tau}^1)^2}{a^{\nu\nu} \det G};$$

clearly $|y_1|, |y_2| \leq C$. Therefore we are justified in writing C on the left-hand side of (A.17).

Now:

$$\begin{aligned} v_\nu + Tv_\tau &= M(w_\nu + Tw_\tau) + (g^1)_\nu + T(g^1)_\tau \\ &\stackrel{(A.14)}{\geq} M(\mu B e^{-B\phi} - C(R')^\sigma - T(\lambda + C(R')^\sigma B\mu))r^{\lambda-1} + C \\ &\stackrel{(A.17)}{=} M'(R')^{-\lambda}(\mu B e^{-B\phi} - C(R')^\sigma - T(\lambda + C(R')^\sigma B\mu))r^{\lambda-1} + C. \end{aligned}$$

We can choose λ and R' so small that

$$v_\nu + Tv_\tau \geq M'(R')^{-\lambda} \frac{\mu B}{2} r^{\lambda-1} e^{-B\phi} + C \quad \text{for all } r \in (0, R').$$

The C term is bounded, so we can pick R' so small that

$$v_\nu + Tv_\tau > 0 \quad \text{for all } r \in (0, R').$$

This inequality implies that v cannot have minima on $\Gamma^2 \cap B_{R'}(\bar{x})$ because otherwise $v_\nu \leq 0$ and $v_\tau = 0$ in the minimum point would cause a contradiction.

Conclusion of Corner $C^{1,\lambda}$. Let

$$\begin{aligned} I' &:= \{(R', \phi) : \phi \in (\phi^1(R'), \phi^2(R'))\}, \\ U' &:= \{(r, \phi) : r \in (0, R'), \phi \in (\phi^1(r), \phi^2(r))\}. \end{aligned}$$

We are still free to choose $M' (\geq 1)$. Since $Mw \geq M'/2$ on \bar{I}' , whereas

$$g^1(x, u(x), Du(x)) = C,$$

we can take M' so large that $v \geq 0$ on \bar{I}' .

v must attain its minimum over \bar{U}' on $\bar{I}' \cup \{x_0\} \cup \Gamma^1$. On $\bar{\Gamma}^1$ the boundary condition $g^1 = 0$ yields $v = w \geq 0$. Therefore v attains its minimum $v = 0$ in x_0 . Hence $v = g^1 + Mw \geq 0$ in \bar{U} , so

$$g^1 \geq -Mw \geq -\frac{M'}{2} \left(\frac{r}{R'}\right)^\lambda.$$

Moreover, we can apply the same arguments to $v := g^1 - Mw$ by switching inequalities and signs and replacing “minimum” with “maximum” wherever appropriate. Then

$$g^1 \leq \frac{M'}{2} \left(\frac{r}{R'}\right)^\lambda.$$

The same arguments, but with the roles of Γ^1 and Γ^2 reversed, yield a bound on g^2 . Now we argue that g^1 and g^2 , as functions of ∇u with x and $u(x)$ held fixed, are uniformly functionally independent, by (A.9). Therefore

$$|\nabla u(x) - \nabla u(x_0)| \leq Cr^\lambda.$$

Full $C_\beta^{2,\alpha}$ Result. This concludes the proof of (A.10). The $C_{1+\lambda}^{2,\kappa}$ result follows from [23, cor. 1.4], where R is the distance from the corner, and

$$\text{osc}_{\Omega_R} \frac{Du}{R} = R^{\lambda-1}$$

by (A.10). □

Appendix B: Free Boundary Transformation

For a column vector x and a column-vector-valued function u , let u_x be the transpose of the gradient and u_{xx} the Hessian. For “row” instead of “column” let u_x be the gradient. Boundary normals are row vectors.

PROPOSITION B.1 Consider a coordinate transformation $y = y(x, u(x))$.

(1) It is nondegenerate if and only if

$$y_x + y_u u_x$$

is regular. Let C_d be an upper bound for the norms of the matrix and its inverse (for some fixed matrix norm).

(2) A quasi-linear equation

$$A(x, u(x), u_x(x)) : u_{xx}(x) + b(x, u(x), u_x(x)) = 0$$

transforms into another quasi-linear equation. The first equation is elliptic if and only if the second one is, and the ellipticity constants are comparable, up to constant factors depending only and continuously on C_d .

(3) Consider a boundary condition

$$g(x, u(x), u_x(x)) = 0.$$

As before we say it is oblique, with obliqueness constant $\delta_o > 0$, if

$$|ng_p| \geq \delta_o |g_p|,$$

where p represents u_x . If

$$(B.1) \quad D := |1 - u_x(y_x + y_u u_x)^{-1} y_u| \geq \delta_D > 0,$$

then the new boundary condition is oblique if and only if the old one is, and the obliqueness constants are comparable, up to constant factors depending only and continuously on δ_D and C_d .

(4) Under the same assumptions, consider two functions g^1 and g^2 of u_x . The smaller angle between g_p^1 and g_p^2 is nonzero if and only if the smaller angle between g_q^1 and g_q^2 is nonzero. They are comparable, up to constant factors depending only and continuously on C_d and δ_D . Here q represents u_y .

PROOF:

- (1) Obvious.
- (2) The new top-order coefficient matrix is

$$(y_x + y_u u_x)A(y_x + y_u u_x)^\top.$$

It is still symmetric positive definite, and the ellipticity constant bound is obvious.

- (3) We omit the x and $u(x)$ arguments from the notation of the boundary conditions.

$$u(x) = u(y(x, u(x))).$$

Take the x derivative:

$$u_x = u_y(y_x + y_u u_x) \quad \Rightarrow \quad u_y = u_x(y_x + y_u u_x)^{-1}.$$

If $h(u_y) = 0$ is the boundary condition in y -coordinates, then

$$g(u_x) = h(u_x(y_x + y_u u_x)^{-1}).$$

Expressed with p and q :

$$g(p) = h(p(y_x + y_u p)^{-1}),$$

so

$$\begin{aligned} g_p &= ((y_x + y_u p)^{-1} - p(y_x + y_u p)^{-1} y_u (y_x + y_u p)^{-1}) h_q \\ &= (1 - p(y_x + y_u p)^{-1} y_u) (y_x + y_u p)^{-1} h_q. \end{aligned}$$

With $p = u_x$ we have

$$g_p = (1 - u_x(y_x + y_u u_x)^{-1} y_u) (y_x + y_u u_x)^{-1} h_q.$$

Let n be the normal in x -coordinates, N the normal in y -coordinates. Then

$$n = \frac{N(y_x + y_u u_x)}{|N(y_x + y_u u_x)|}.$$

So

$$n g_p = \frac{1 - u_x(y_x + y_u u_x)^{-1} y_u}{|N(y_x + y_u u_x)|} N h_q.$$

Clearly $n g_p \neq 0$ if and only if $N h_q \neq 0$ as long as $D > 0$. The constant comparison is obvious.

- (4) The two functions transform like the boundary condition above:

$$g_p^k = (1 - u_x(y_x + y_u u_x)^{-1} y_u) (y_x + y_u u_x)^{-1} h_q^k$$

where $g^k(u_x) = h^k(u_y)$. Then clearly g_p^1 and g_p^2 are collinear if and only if h_q^1 and h_q^2 are, and the constants are obvious.

□

Remark B.2. We use the Sherman-Morrison formula: for an invertible matrix M , the rank 1 perturbation $M + ab^T$ is invertible if and only if $b^T M^{-1} a \neq -1$. Taking $M = y_x + y_u u_x$, $a = -y_u$, and $b = u_x^T$, we see that (B.1) is precisely the condition for y_x to be invertible as well.

Acknowledgment. The research of Tai-Ping Liu is supported in part by National Science Council (NSC) Grant 96-2628-M-001-011 and National Science Foundation (NSF) Grant DMS 0709248.

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Received February 2007.