HIERARCHICAL DECOMPOSITION SYNTHESIS IN OPTIMAL SYSTEMS DESIGN

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ABSTRACT

Optimal design of large systems is easier if the optimization model can be decomposed and solved as a set of smaller, coordinated subproblems. Casting a given design problem into a particular optimization model by selecting objectives and constraints is generally a subjective task. This article describes how such a subjective selection can be made so that the resulting optimal design model can be directly partitioned into an appropriate decomposed form. This process is termed *decomposition synthesis*. A particular methodology for synthesizing hierarchically decomposed optimal design models is presented together with examples.

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INTRODUCTION

Solving an optimal design problem by decomposition methods involves partitioning a given optimal design problem (ODP) into several smaller problems and coordinating their solutions to obtain the solution to the original problem. The process of identifying and executing appropriate partitioning of a given ODP is referred to as *Decomposition Analysis* (Wagner 1993, Wagner and Papalambros 1993a, 1993b). Decomposition of a design problem that has been cast in an optimization model form is linked to the mathematical structure of the already selected objective and constraint functions. For example, if the objective is expressed as a sum of terms, then the solution to the problem using a decomposed form is enhanced. In general, casting a given design problem as an optimization model is subjective. Therefore, one may seek to synthesize an ODP by defining the appropriate model functions so that the resulting model can be directly partitioned and solved in a decomposed form. *Decomposition Synthesis* is defined as the process of synthesizing a decomposable ODP from a general design problem. Successful decomposition synthesis will allow an ODP of identified decomposition to be composed and solved by a desired decomposition method. This is especially useful in the optimal design of large systems (Papalambros 1995).

A general design problem (GDP) is modeled as

$$\begin{aligned} &\text{Find } \mathbf{x} \in F \\ &\text{subject to } \mathbf{h}(\mathbf{x}, \mathbf{p}) = \mathbf{0} \\ &\qquad\qquad\quad \mathbf{g}(\mathbf{x}, \mathbf{p}) \leq \mathbf{0} \end{aligned} \tag{1}$$

where $\mathbf{h}$, $\mathbf{g}$ are vectors of design criteria represented by equalities and inequalities that are functions of the design variables $\mathbf{x}$ and parameters $\mathbf{p}$, and F is the set constraint on the design variables. The GDP is transformed to an ODP by selecting one or more design criteria from above and composing a scalar objective, namely,

$$\begin{aligned} &\text{Minimize } f(\mathbf{x}, \mathbf{p}') \\ &\text{subject to } \mathbf{h}(\mathbf{x}, \mathbf{p}) = \mathbf{0} \\ &\qquad\qquad\quad \mathbf{g}(\mathbf{x}, \mathbf{p}) \leq \mathbf{0} \\ &\text{and } \mathbf{x} \in F \end{aligned} \tag{2}$$

where **p**' is a vector of parameters that includes any weights used in composing the *scalar substitute* objective f. In general, the objective is a vector function which must be scalarized into a scalar substitute form so that the methods of mathematical programming can be used. For example, if there are q objectives $f_i(\mathbf{x}_i, \mathbf{p}')$, $i = 1, \ldots, q$, then $f(\mathbf{x}, \mathbf{p}') = \sum_{i=1}^{q} f_i(\mathbf{x}_i, \mathbf{p}')$ with $\mathbf{x}_i$ being a subvector (*any* vector defined from the *components* of a given vector) of the design vector **x**. See Athan (1994) for further details on scalar substitute functions.

The functional representations in Eq. (1) and (2) can be converted to equivalent matrix and graph representations: the functional dependence table (FDT) and a bipartite graph or a hypergraph (Wagner and Papalambros 1993a, Michelena and Papalambros 1995a). Functional, matrix, and bipartite graph representations for an example GDP are shown in Fig. 1. Note that a graph G is bipartite if its vertex set V can be divided into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2 (Deo 1990). Representation of an ODP is similar, with each term f_i in a scalar objective represented by a vertex in the graph or a row in the FDT. All graphs are assumed to be connected. (A graph G is connected if a path exists between every pair of vertices.)

$g_1(x_1, x_2) \leq 0$

$g_2(x_1, x_2) \leq 0$

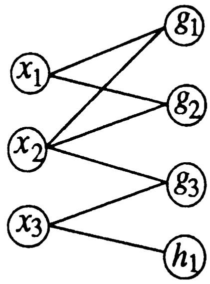
$g_3(x_2, x_3) \leq 0$

$h_1(x_3) = 0$

(a)

$g;h$	x_1	x_2	x_3
g_1	1	1	0
g_2	1	1	0
g_3	0	1	1
h_1	0	0	1

(b)



(c)

Fig. 1 Alternate Representations of a General Design Problem

Decomposition methods for solving ODPs can be classified into hierarchical and nonhierarchical, and further into primal and dual methods. A given problem may be decomposed and/or solved in more than one way. A recent review and analysis of the decomposability of optimal design problems is given in Wagner (op. cit.). Rigorous partitioning techniques for decomposition analysis have been presented by Michelena and Papalambros (1994, 1995a, b).

The decomposition synthesis methodology proposed below makes use of the partitioning tools of decomposition analysis. The methodology consists of two main steps. First, a "block-angular" or a "dual-angular" structure in the GDP is found using a graph partitioning method. Next, an ODP is composed with an identified hierarchical decomposition based on the structure found for the GDP.

The next section formalizes some basic concepts for hierarchical and nonhierarchical decomposition. A methodology for hierarchical decomposition synthesis is then presented followed by examples illustrating its application.

HIERARCHICAL AND NONHIERARCHICAL DECOMPOSITION

A decomposition method is characterized by the mathematical structure that defines the subproblems and by the coordination strategy that connects them in the solution process. Before describing a synthesis methodology it is necessary to define rigorously what we mean by hierarchical or nonhierarchical decomposition of an ODP. In this section we present definitions that can be applied to both primal and dual formulations.

Primal Decomposition

The primal formulation of the ODP is the functional representation in Eq.(2). Assuming a *summation objective* (expressed as sum of terms) as in Eq. (2) we have

$$\underset{\mathbf{x} \in F}{\textit{minimize}} \quad f(\mathbf{x}, \mathbf{p}) = \sum_{i=1}^{q} f_i(\mathbf{x}_i, \mathbf{p}') \tag{3}$$

subject to: $g_j(\mathbf{x}, \mathbf{p}) \leq 0 \qquad j = 1,\dots, J,$

$h_m(\mathbf{x}, \mathbf{p}) = 0 \qquad m = 1,\dots, M.$

Consider the two sets C, X defined as

$$C = \{c_1, c_2,\dots, c_t,\dots, c_T\}, \; X = \{x_1, x_2,\dots, x_i,\dots, x_N\} \tag{4}$$

where each g_j, h_m, and each f_i term in the objective is represented by a member c_t in the set C. The set C is composed of T functions where $T = (q + J + M)$. Each design variable is represented by an x_i in the set X.

Definition 1: A *decomposition* of an optimal design problem into K subproblems is the ordered triple $P_i = (Z_i, V_i, W_i)$, $\forall\ i = 1,\dots, K$, where Z_i is the set of design variables to be optimized in problem i, V_i is the set of functions in problem i, W_i is the set of variables that the functions in the set V_i depend on, and Z_i, V_i, W_i satisfy the following:

$$Z_i\ (\neq \emptyset) \subseteq X, \quad V_i \subseteq C, \quad W_i \subseteq X \tag{5}$$

$$\bigcup_i Z_i = X, \qquad \bigcup_i V_i = C$$

$$Z_i \cap Z_j = \emptyset, \quad V_i \cap V_j = \emptyset, \quad \forall\ i \neq j, \text{ and } i, j = 1,\dots, K$$

with each function in the set V_i dependent on one or more of the design variables that are elements of the set Z_i.

Definition 2: The *input* to problem i from problem j is defined by the set I_{ij}, where

$$I_{ij} = W_i \cap Z_j, \qquad j \neq i. \tag{6}$$

Definition 3: The *linking variables of two subproblems i* and j are defined by the set $L_{ij} = L_{ji}$, where

$$L_{ij} = I_{ij} \cup I_{ji}, \qquad j \neq i. \tag{7}$$

Definition 4: The *linking variables in a decomposition* are defined by the set L_v, where

$$L_v = \bigcup_{i\ j\ ;\ i \neq j} L_{ij}. \tag{8}$$

In a directed graph representation of an ODP (Deo 1990, Wagner 1993) the decomposed problems are the nodes and the inputs are the directed edges.

Definition 5: A directed graph G is said to be an *out-tree* or an *arborescence* (Deo op.cit.) if

(a) G contains no directed circuit or semicircuit,

(b) there is precisely one vertex of zero in-degree, the *root* of the out-tree.

Definition 6: A decomposition of an optimal design problem is *hierarchical* (*nonhierarchical*) if its directed graph representation is (not) an out-tree (Figs. 2, 3).

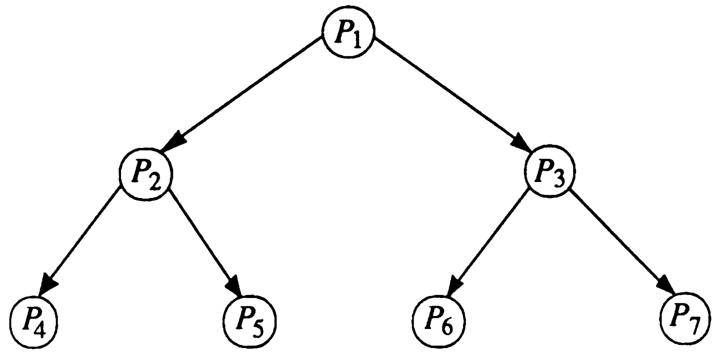


Fig. 2 Hierarchical decomposition graph

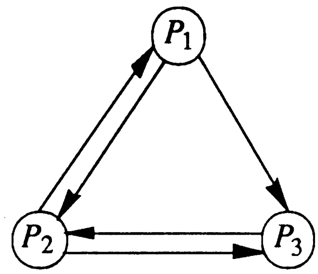


Fig. 3 Nonhierarchical decomposition graph

As an example, consider the ODP

$$\underset{(x_1, x_2, x_3)}{\text{minimize}} \quad x_1 + x_1x_2 + x_2x_3 + x_3$$
$$f_1 = x_1, f_2 = x_1x_2, f_3 = x_2x_3, f_4 = x_3 \tag{9}$$

subject to:

$g_1 = x_1x_2x_3 - 8 \le 0$ $\qquad g_2 = -\ (x_1 + x_2 + x_3 - 2) \le 0$

$g_3 = -\ (x_1) \le 0$ $\qquad g_4 = x_1 - 4 \le 0$ $\qquad g_5 = -(x_2) \le 0$

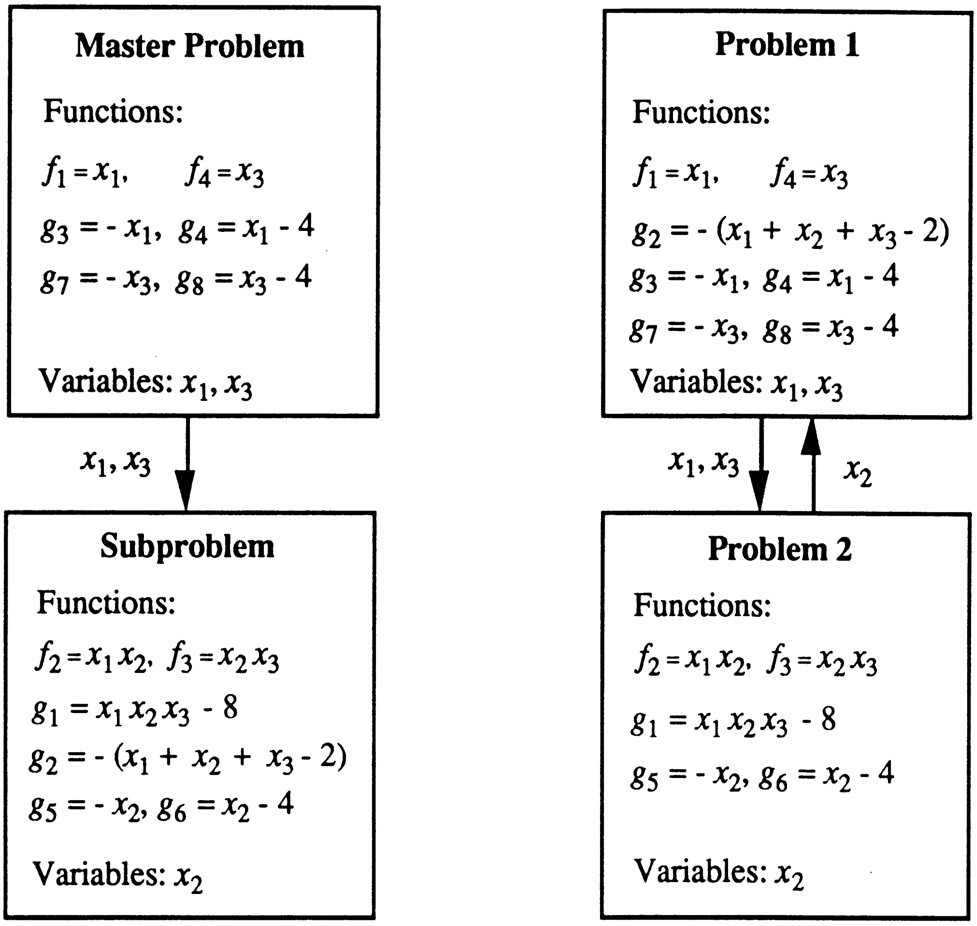
$g_6 = x_2 - 4 \le 0$ $\qquad g_7 = -(x_3) \le 0$ $\qquad g_8 = x_3 - 4 \le 0.$

In a hierarchical decomposition, Fig. 4(a), input flow is unidirectional (from the master problem to the subproblems) and linking variables are (x_1, x_3); in a nonhierarchical one, Fig. 4(b) inputs are bi-directional and (x_1, x_2, x_3) link the decomposed problems.

Dual Decomposition

The Lagrangian dual problem of the primal ODP in Eq.(2) is defined as (e.g., Bazaraa et al. 1993)

$$\underset{(\mu;\, \lambda \ge \mathbf{0})}{\text{maximize}} \quad \{\min_{\mathbf{x}} L = f(\mathbf{x}, \mathbf{p}') + \mu^T \mathbf{h}(\mathbf{x}, \mathbf{p}) + \lambda^T \mathbf{g}(\mathbf{x}, \mathbf{p})\} \tag{10}$$



(a) (b)

Fig. 4 (a) Hierarchical, and (b) nonhierarchical decomposition of the example formulation

where μ, λ (vectors) are the Lagrangian multipliers or dual variables. Under convexity assumptions and suitable constraint qualifications, the primal and dual problems have the same optimal objective values. In such cases it may be advantageous to apply decomposition methods on the dual problem instead of the primal. Definitions of hierarchical and nonhierarchical decomposition remain the same in the dual space, except that the set X will include both primal and dual variables, and c_t in general will be a function of both primal design and dual variables. In the definitions for the dual, the primal objective f is replaced by the Lagrangian L, and f_i *by* L_i.

A simple two-level hierarchical decomposition of the dual is shown in Fig. 5. Problem A is the master problem defined in the dual space, linking variables are the dual ones, and Problem B is the subproblem defined in the primal space. If Problem A is not "pure," i.e., primal variables must be included in it, then the dual decomposition will become nonhierarchical.

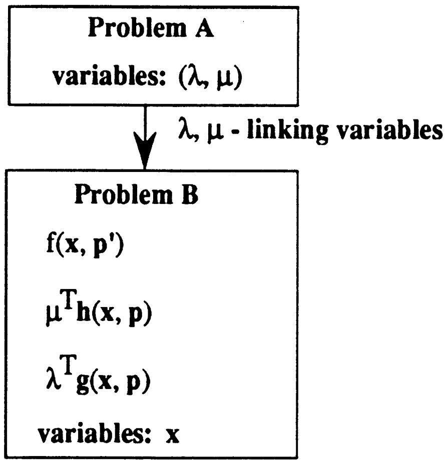


Fig. 5 Hierarchy in dual decomposition

Note that instead of including all constraints in the Lagrangian, some of them may be left out and treated separately in the primal. Then the dual variables will correspond to linking constraints that are relaxed when solving the problem (Lootsma 1990). Also, a Lagrangian separable in the primal will allow partition into several subproblems. These cases are illustrated further in the example section.

METHODOLOGY FOR HIERARCHICAL DECOMPOSITION SYNTHESIS

A method for hierarchical decomposition, primal or dual, can be applied when the ODP model has the exact form required by the relevant coordination strategy. A methodology for decomposition synthesis aims at transforming a GDP into the desired form of a decomposable ODP.

Primal Decomposition Synthesis

Here we consider ODPs that can be solved by primal hierarchical decomposition methods. Assume the decomposed model has a master problem and *q* subproblems. Then the original GDP must be cast into the *block-angular* structure shown in Fig. 6 and Eq. (11).

$$\mathbf{g}_0(\mathbf{x}_0) \le \mathbf{0} \qquad \mathbf{h}_0(\mathbf{x}_0) = \mathbf{0} \qquad (11a)$$

$$\mathbf{g}_i(\mathbf{x}_0, \mathbf{x}_i) \le \mathbf{0} \qquad \mathbf{h}_i(\mathbf{x}_0, \mathbf{x}_i) = \mathbf{0} \qquad i = 1, \dots, q \qquad (11b)$$

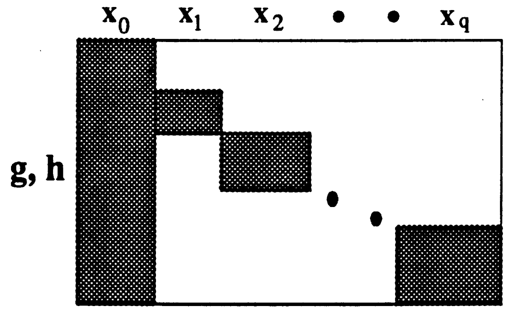


Fig. 6 Block-angular structure in the GDP

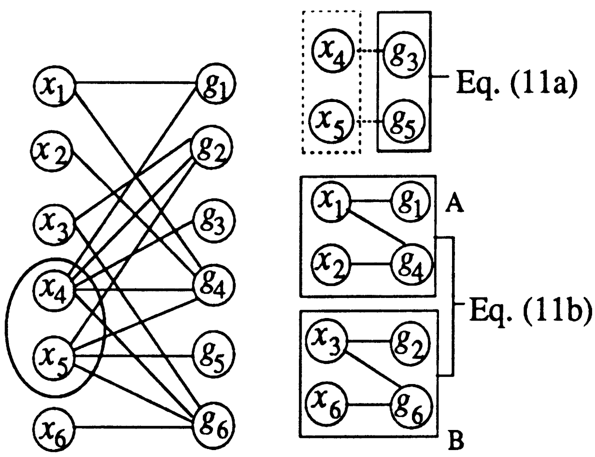
Identifying Structure

The required structural form is sought using graph partitioning methods applied to the bipartite graph representation of the GDP. As linking variables (vertices) $\mathbf{x}_0$ are removed, the functions that depend exclusively on these variables correspond to $\mathbf{g}_0$, $\mathbf{h}_0$ in Eq. (11a). The corresponding function vertices will be isolated after removal of linking variable vertices from the graph. (A vertex is isolated if it is not connected to any other vertex by an edge.) In Fig. 7(a) the FDT of a general design problem is shown. In the bipartite graph, Fig. 7(b), the set $\{x_4, x_5\}$ is selected as linking variables. Removing the corresponding vertices leaves g_3 and g_5 isolated.

Each connected component in the remaining portion of the graph that excludes the vertices already identified with Eq. (11a) must have the structure of Eq. (11b). (A graph is a connected component if a path exists between every pair of vertices in that graph. An isolated vertex is also a connected component but with just one vertex.) Fig. 7(c) shows the resulting partitions $A = \{g_1, g_4, x_1, x_2\}$, $B = \{g_2, g_6, x_3, x_6\}$. The new FDT shown in Fig. 7(d) is block-angular.

	x_1	x_2	x_3	x_4	x_5	x_6
g_1	1	0	0	1	0	0
g_2	0	0	1	1	1	0
g_3	0	0	0	1	0	0
g_4	1	1	0	1	1	0
g_5	0	0	0	0	1	0
g_6	0	0	1	1	1	1

(a)



(b) (c)

	x_4	x_5	x_1	x_2	x_3	x_6
g_3	1	0	0	0	0	0
g_5	0	1	0	0	0	0
g_1	1	0	1	0	0	0
g_4	1	1	1	1	0	0
g_2	1	1	0	0	1	0
g_6	1	1	0	0	1	1

(d)

Fig. 7 Steps in identifying block-angular structure in the GDP

Algorithms for identifying connected components in a graph can be used to find partitions as in Fig. 7(c) for a particular choice of linking variables. Correct choice of linking variables is a key decision as studied in Decomposition Analysis (Wagner 1993, Michelena and Papalambros 1995a, b). Also, partitions such as *A* and *B* in Fig. 7(c) need not be the only way the graph can be partitioned to fit the particular structural form. For example, one could combine A and B into a larger single cluster and still produce the required structure. The resulting clusters, however, may not be internally connected and hence not always acceptable (Wagner op.cit.). Identifying a suitable structure for synthesis must include acceptability criteria determined by the user.

Composing the ODP

After identifying the block-angular structure in the GDP, an ODP can be synthesized by creating a summation objective function as follows: (i) select a function from the block $\{\mathbf{g}_0, \mathbf{x}_0\}$, Eq.(11a); (ii) select a function from each block i, i = 1,..., q, Eq.(11b); (iii) add all selected functions with appropriate weights w_i , $i = 0, ..., q$; this is the objective function; (iv) append all constraints to the objective to complete the model. The resulting structure will have the form of Eq. (12) and Fig. 8.

$$\underset{\mathbf{x}_0,\, \mathbf{x}_i}{minimize} \quad f = f_0(\mathbf{x}_0, \mathrm{w}_0) + \sum_{i=1}^{q} f_i(\mathbf{x}_0, \mathbf{x}_i, \mathrm{w}_i) \tag{12}$$

subject to:

$$\mathbf{g}_0(\mathbf{x}_0) \leq \mathbf{0}, \qquad \mathbf{h}_0(\mathbf{x}_0) = \mathbf{0}$$

$$\mathbf{g}_i(\mathbf{x}_0, \mathbf{x}_i) \leq \mathbf{0}, \qquad \mathbf{h}_i(\mathbf{x}_0, \mathbf{x}_i) = \mathbf{0} \qquad i = 1, \ldots, q.$$

This ODP can be decomposed into a master problem in the linking variables $\mathbf{x}_0$ and q subproblems in the local variables $\mathbf{x}_i$. A hierarchical coordination method such as those proposed by Kirsch (1981) or Azarm (1988) can be used to solve it. The functions selected need not be weighted linearly; exponential or other nonlinear weighting forms can be used (Athan 1994).

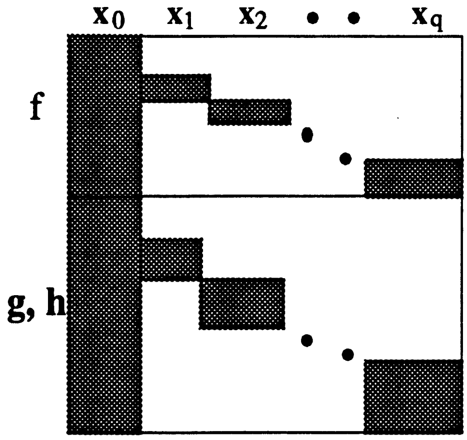


Fig. 8 Hierarchically Decomposed ODP

Not all terms need be present in Eq. (12). For a given choice of linking variables, no functions may exist in the set corresponding to Eq. 11(a). Also, objective function terms for composing the ODP may be selectable only from the set corresponding to Eq. 11(a) as required in the formulations used by Sobieski (1982), and Haftka (1984). The objective composed in the above two cases will not include all terms in *f* of Eq. (12).

Dual Decomposition Synthesis

To apply dual hierarchical decomposition methods we must identify a *dual-angular* structure in the GDP, as in Eq. (13) and Fig. 9.

$$\sum_{i=1}^{q} g_{ji}(\mathbf{x}_i) \le 0 \qquad j = (J - C_g),\ldots, J \qquad \text{(a)}$$

$$\sum_{i=1}^{q} h_{mi}(\mathbf{x}_i) = 0 \qquad m = (M - C_h),\ldots, M \qquad \text{(b)}$$

$$\mathbf{g}_j(\mathbf{x}_j) \le \mathbf{0} \qquad j = 1,\ldots, q \qquad \text{(c)} \qquad (13)$$

$$\mathbf{h}_m(\mathbf{x}_m) = \mathbf{0} \qquad m = 1,\ldots, q \qquad \text{(d)}$$

Here C_g and C_h are the number of *coupling* inequality and equality functions identified in the GDP, respectively.

Identification of structure in the GDP is similar to the primal case. The graph is partitioned by removing vertices that correspond to *functions*.

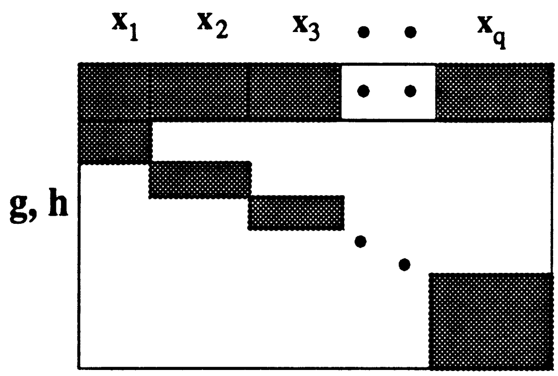


Fig. 9 Dual-angular structure in the GDP

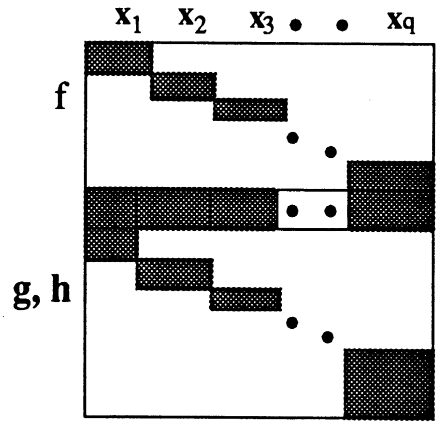


Fig. 10 Hierarchically decomposed ODP

However, it is necessary that these linking functions be separable in the variables $\mathbf{x}_i$ of the independent connected components to ensure separability in the dual objective. Thus, a particular partitioning of the GDP is *acceptable* only if the above separability criterion is met. Given a dual-angular partitioning, a hierarchically decomposed ODP solvable by dual methods can be created by constructing an appropriate objective, as in the primal case, namely,

$$\textit{minimize } f = \sum_{i=1}^{q} f_i\,(\mathbf{x}_i\,,\mathbf{w}_i) \tag{14}$$

subject to the constraints in Eq. (13). The resulting FDT will have a structure as in Fig. 10. The Lagrangian of this formulation is additively separable in $\mathbf{x}_i$ and the problem can be solved with dual decomposition methods, e.g., Lasdon (1968). For a linear model the Dantzig-Wolfe (1960) method can be used.

EXAMPLES

The methodology for primal or dual decomposition synthesis is demonstrated below with some simple examples.

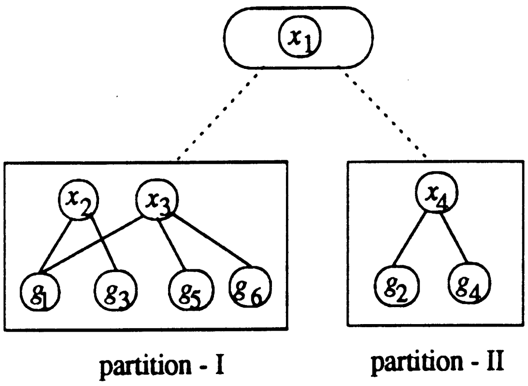
Example 1: Pressure Vessel Design

A GDP for pressure vessel design based on a model in Wilde (1978) is given in Table 1. The vessel is made of a cylindrical body and is welded on both sides by hemispherical heads.

Table 1 Pressure vessel problem [Example 1]

Functional Representation	Description of the Design Requirements	Matrix Representation			
		x_1	x_2	x_3	x_4
$g_1(x_1, x_2, x_3) \leq 0$	Cylinder mass limits	1	1	1	0
$g_2(x_1, x_4) \leq 0$	Hemisphere mass limits	1	0	0	1
$g_3(x_1, x_2) \leq 0$	Stress limits in cylinder walls	1	1	0	0
$g_4(x_1, x_4) \leq 0$	Stress limits in head walls	1	0	0	1
$g_5(x_1, x_3) \leq 0$	Volume requirement	1	0	1	0
$g_6(x_3) \leq 0$	Limit on cylinder length	0	0	1	0

The variables $x_1, \ldots, x_4$ are cylinder and head radii, cylinder thickness, cylinder length, and head thickness, respectively. Decomposition into a master problem and two subproblems is sought. If x_1 is chosen as the linking variable and removed, the bipartite graph of the GDP partitions into two connected components, partitions I and II, Fig. 11(a). A block-angular structure is identified, Fig. 11(b), with no function terms identified with Eq. 11(a), and $\mathbf{x}_0 = x_1$, $\mathbf{x}_1 = (x_2, x_3)$, and $\mathbf{x}_2 = x_4$. Functions g_1, g_2 are selected for a linearly weighted objective, i.e., minimize the volume of the cylinder and heads. The synthesized ODP, Fig. 12, is now ready to be decomposed into a master problem in the linking variables x_1, and two subproblems in the local variables (x_2, x_3) and x_4. Instead of g_1, g_2 one can use the stresses in the cylinder and heads to compose the objective.



x ->	x_1	x_2	x_3	x_4
g_1	1	1	1	0
g_3	1	1	0	0
g_5	1	0	1	0
g_6	0	0	1	0
g_2	1	0	0	1
g_4	1	0	0	1

(a) (b)

Fig. 11 (a) Partitioning of the GDP, (b) Block-angular GDP [Example 1]

minimize $f = f_1[= w_1\ g_1] + f_2[= w_2\ g_2]$ $(x_1,\ldots,x_4)$	
subject to:	
$g_1(x_1, x_2, x_3) \le 0$	$g_2(x_1, x_4) \le 0$
$g_3(x_1, x_2) \le 0$	$g_4(x_1, x_4) \le 0$
$g_5(x_1, x_3) \le 0$	$g_6(x_3) \le 0$

(a)

x ->	x_1	x_2	x_3	x_4
f_1	1	1	1	0
f_2	1	0	0	1
g_1	1	1	1	0
g_3	1	1	0	0
g_5	1	0	1	0
g_6	0	0	1	0
g_2	1	0	0	1
g_4	1	0	0	1

(b)

Fig. 12 Synthesized ODP: (a) Functional form, (b) Matrix form [Example 1]

Example 2: Resource Management

Consider the GDP developed based on a problem in the area of resource management of an economic system (Wong 1970, appearing as test problem #113, in Hock and Schittkowski 1982), Eq. (15).

$$
\begin{array}{ll}
g_1(x_1, x_2) \le 0 & g_2(x_3, x_4) \le 0 \\
g_3(x_5, x_6) \le 0 & g_4(x_7, x_8) \le 0 \\
g_5(x_9, x_{10}) \le 0 & g_6(x_1, x_2, x_7, x_8) \le 0 \\
g_7(x_1, x_2, x_9, x_{10}) \le 0 & g_8(x_1, x_2, x_3, x_4) \le 0 \\
g_9(x_1, x_2, x_3, x_4) \le 0 & g_{10}(x_1, x_2, x_5, x_6) \le 0 \\
g_{11}(x_1, x_2, x_9, x_{10}) \le 0 & \\
h_1(x_1, x_2, x_7, x_8) = 0 & h_2(x_1, x_2, x_5, x_6) = 0
\end{array}
\qquad (15)
$$

We seek a decomposition with a master problem and four subproblems. Repeating the steps of previous example we obtain the ODP in Fig. 13. The set of linking variables is $\{x_1, x_2\}$ and $\{g_1, g_8, g_{10}, g_6, g_7\}$ is the set of functions chosen to compose the objective. The synthesized ODP is decomposable into a master problem in the linking variables $\{x_1, x_2\}$, and four subproblems in the local variables $\{x_3, x_4\}$, $\{x_5, x_6\}$, $\{x_7, x_8\}$, $\{x_9, x_{10}\}$.

x->	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
f_0	1	1	0	0	0	0	0	0	0	0
f_1	1	1	1	1	0	0	0	0	0	0
f_2	1	1	0	0	1	1	0	0	0	0
f_3	1	1	0	0	0	0	1	1	0	0
f_4	1	1	0	0	0	0	0	0	1	1
g_1	1	1	0	0	0	0	0	0	0	0
g_8	1	1	1	1	0	0	0	0	0	0
g_9	1	1	1	1	0	0	0	0	0	0
g_2	0	0	1	1	0	0	0	0	0	0
g_{10}	1	1	0	0	1	1	0	0	0	0
h_2	1	1	0	0	1	1	0	0	0	0
g_3	0	0	0	0	1	1	0	0	0	0
g_6	1	1	0	0	0	0	1	1	0	0
h_1	1	1	0	0	0	0	1	1	0	0
g_4	0	0	0	0	0	0	1	1	0	0
g_7	1	1	0	0	0	0	0	0	1	1
g_{11}	1	1	0	0	0	0	0	0	1	1
g_5	0	0	0	0	0	0	0	0	1	1

Fig. 13 Synthesized ODP - Matrix Form [Example 2]

Example 3: Solar House Heating System

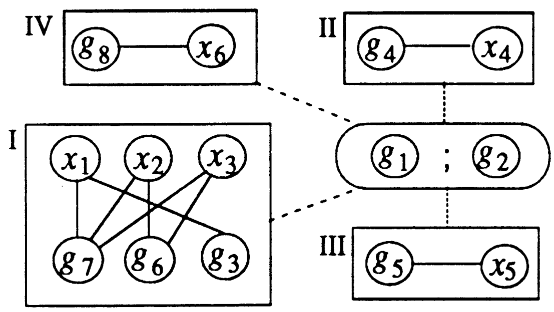
This GDP is based on a model proposed by Wilde (1978) and is presented in Table 2. Solar energy collected by a sheet collector is used to heat water stored in a tank. The heat from the tank is released in a controlled fashion to the rooms by free convention and radiation. The design problem is to select the collector area x_1, radius of storage tank x_2, height of storage tank x_3, wall insulation thickness x_4, roof insulation thickness x_5, and weather stripping length x_6, so that the house is maintained at about constant temperature at assumed cold weather conditions.

Table 2 Design problem of a solar house heating system and its FDT representation [Example 3]

GDP - Functional Description	Description of the Design Criteria	x_1	x_2	x_3	x_4	x_5	x_6
$g_1(x_1, x_2, x_3, x_4, x_5, x_6) \leq 0$	limits on cost	1	1	1	1	1	1
$g_2(x_1, x_4, x_5, x_6) \leq 0$	heat input into water	1	0	0	1	1	1
$g_3(x_1) \leq 0$	limits on collector size area	1	0	0	0	0	0
$g_4(x_4) \leq 0$	limits on wall insulation	0	0	0	1	0	0
$g_5(x_5) \leq 0$	limits on roof insulation	0	0	0	0	1	0
$g_6(x_2, x_3) \leq 0$	wall area of the water tank	0	1	1	0	0	0
$g_7(x_1, x_2, x_3) \leq 0$	heat storage capacity of water tank	1	1	1	0	0	0
$g_8(x_6) \leq 0$	limits on weather stripping length	0	0	0	0	0	1

We seek an ODP that can be decomposed hierarchically into a master problem and four subproblems in the dual space. We must first find coupling or linking functions. Selecting $\{g_1$,

$g_2\}$ as the coupling functions and removing them partitions the bipartite graph of the GDP as in Fig. 14(a). This partition would be acceptable if $\{g_1, g_2\}$ are additively separable in the partitioned variables $\mathbf{x}_i$, namely, $[\{x_1, x_2, x_3\}, \{x_4\}, \{x_5\}$, and $\{x_6\}]$. From the original model in Wilde (op. cit.), g_1 is additively separable in the functions $\{g_{11}(x_1), g_{12}(x_2, x_3), g_{13}(x_4), g_{14}(x_6)\}$, and g_2 is additively separable in the functions $\{g_{21}(x_1), g_{22}(x_4), g_{23}(x_5), g_{24}(x_6)\}$. The current partitioning is therefore acceptable. The dual-angular structure in the GDP identified is shown in Fig. 14(b). One could choose $\{g_3, g_4, g_5, g_8\}$ from the partitions and compose an ODP as shown in Fig. 15, to minimize the weighted sum of collector area, wall insulation, roof insulation, and weather stripping length.



(a)

x->	x_1	x_2	x_3	x_4	x_5	x_6
g_1	1	1	1	1	1	1
g_2	1	0	0	1	1	1
g_3	1	0	0	0	0	0
g_6	0	1	1	0	0	0
g_7	1	1	1	0	0	0
g_4	0	0	0	1	0	0
g_5	0	0	0	0	1	0
g_8	0	0	0	0	0	1

(b)

Fig. 14 (a) Partitioning of the GDP, (b) Dual-angular GDP [Example 3]

$$\underset{(x_1,\ldots,x_6)}{\textit{minimize}}\ f = f_1[= \mathrm{w}_1\ g_3] + f_2[= \mathrm{w}_2\ g_4] + f_3[= \mathrm{w}_3\ g_5] + f_4[= \mathrm{w}_4\ g_8]$$

subject to:

$g_1(x_1, x_2, x_3, x_4, x_5, x_6) \leq 0$

$g_2(x_1, x_4, x_5, x_6) \leq 0$

$g_3(x_1) \leq 0$ $\quad$ $g_4(x_4) \leq 0$

$g_5(x_5) \leq 0$ $\quad$ $g_6(x_2, x_3) \leq 0$

$g_7(x_1, x_2, x_3) \leq 0$ $\quad$ $g_8(x_6) \leq 0$

(a)

x->	x_1	x_2	x_3	x_4	x_5	x_6
f_1	1	0	0	0	0	0
f_2	0	0	0	1	0	0
f_3	0	0	0	0	1	0
f_4	0	0	0	0	0	1
g_1	1	1	1	1	1	1
g_2	1	0	0	1	1	1
g_3	1	0	0	0	0	0
g_6	0	1	1	0	0	0
g_7	1	1	1	0	0	0
g_4	0	0	0	1	0	0
g_5	0	0	0	0	1	0
g_8	0	0	0	0	0	1

(b)

Fig. 15 Synthesized ODP: (a) Functional form, (b) Matrix form [Example 3]

Example 4: Management of Water Resources

Consider the GDP developed based on a problem in the area of water resource management proposed by Sung (1978) (cited in an example problem by Haimes et al., 1990, exercise.# 1, section 9.5), Eq. (16). We seek an ODP that is decomposable into a master problem and three subproblems in the dual.

$$\begin{array}{lll} g_1(x_1, x_2) \le 0 & g_2(x_3, x_4) \le 0 & g_3(x_5, x_6, x_7) \le 0 \\ g_4(x_2) \le 0 & g_5(x_3, x_4) \le 0 & g_6(x_4, x_7) \le 0 \\ g_7(x_1) \le 0 & g_8(x_2) \le 0 & g_9(x_3) \le 0 \\ g_{10}(x_4) \le 0 & g_{11}(x_5) \le 0 & g_{12}(x_6) \le 0 \\ g_{13}(x_7) \le 0 & h_1(x_1, x_2, x_3) = 0 & h_2(x_5, x_6, x_7) = 0 \end{array} \tag{16}$$

All equality and inequality functions are linear and so additively separable in each variable. A hierarchically decomposed ODP obtained after going through steps as in the previous example is shown in Fig. 16. Coupling functions are $\{h_1, g_6\}$, and $\{g_1,..., g_3\}$ is the set of functions chosen to compose the objective.

x->	x_1	x_2	x_3	x_4	x_5	x_6	x_7
f_1	1	1	0	0	0	0	0
f_2	0	0	1	1	0	0	0
f_3	0	0	0	0	1	1	1
h_1	1	1	1	0	0	0	0
g_6	0	0	0	1	0	0	1
g_1	1	1	0	0	0	0	0
g_4	0	1	0	0	0	0	0
g_7	1	0	0	0	0	0	0
g_8	0	1	0	0	0	0	0
g_2	0	0	1	1	0	0	0
g_5	0	0	1	1	0	0	0
g_9	0	0	1	0	0	0	0
g_{10}	0	0	0	1	0	0	0
g_3	0	0	0	0	1	1	1
h_2	0	0	0	0	1	1	1
g_{11}	0	0	0	0	1	0	0
g_{12}	0	0	0	0	0	1	0
g_{13}	0	0	0	0	0	0	1

Fig. 16 Synthesized ODP - Matrix form [Example 4]

CONCLUDING REMARKS

In the methodology for hierarchical decomposition synthesis presented here the GDP partitioning to achieve a specific decomposition goal is not unique. There may also be several decomposition goals that need be explored and compared. Studying these alternatives is itself an optimization problem. Methods for "optimal" decomposition synthesis are currently under investigation.

The selection of a particular decomposition method depends on number of factors, and the following guidelines may be useful. Primal (dual) methods would be preferable if (a) the GDP can be partitioned with a small number of linking variables (functions), (b) constraint feasibility need (not) be maintained throughout the iterations, and (c) the GDP has a small (large) number of variables and a large (small) number of constraints. A hierarchically decomposed ODP for solution by both primal and dual methods can be synthesized from a GDP by introducing new variables, especially for highly coupled GDPs (Wismer and Chattergy 1978). The disadvantage is increase in the size of the problem. Such cases can be handled also by the methods presented here. As the GDP becomes more coupled, nonhierarchical decomposition may be the only way to partition the problem without increasing its size. It is not clear what should be the basis for synthesis in such a case.

Synthesizing an ODP that can be decomposed into a specified number of q subproblems may not be always possible. Also, a q value may not be determinable a priori. In such cases GDP partitions for different values of q must be studied and a suitable q value selected.

Only a two-level hierarchical decomposition synthesis was discussed here but the methodology can be extended to multi-level synthesis. Also, instead of a scalar substitute objective, one could treat the individual objectives as components of a vector and solve the vector minimization problem with a variety of techniques.

In conclusion, the proposed synthesis approach is an attractive, more systematic way to formulate and solve large optimal system design problems. The resulting optimization model need not be solved necessarily by decomposition methods.

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REFERENCES

Athan, T., 1994, *A Quasi-Random Method for Multicriteria Optimization,* Doctoral Dissertation, Department of Mechanical Engineering and Applied Mechanics, University of Michigan, Ann Arbor.

Azarm, S., Li, W., 1988, "A Two-Level Decomposition Method For Design Optimization," *Engineering Optimization*, Vol. 13, pp. 211-224.

Bazaraa, M., Sherali, H., Shetty., C., 1993, *Nonlinear Programming: Theory and Algorithms*, Second edition, Wiley, New York.

Dantzig, G., and Wolfe, P., 1960, "Decomposition Principles for Linear Programming," *Operations Research*, Vol. 8, pp. 101-111.

Deo, N., 1990, *Graph Theory With Applications to Engineering and Computer Science*, Prentice-Hall of India, New Delhi.

Haftka, R., 1984, "An Improved Computational Approach for Multilevel Optimum Design," *Journal of Structural Mechanics*, Vol. 12, No. 2, pp. 245-261.

Haimes, Y., Tarvainen, K., Shima, T., Thadathil, J., 1990, *Hierarchical Multiobjective Analysis of Large-Scale Systems*, Hemisphere Publishing, New York.

Hock, W., and Schittkowski, K., 1981, *Test Examples for Nonlinear Programming Codes,* Lecture Notes in Economics and Mathematical Systems, No. 187, (ed.) Beckman, M., and Kunzi, H., Springer-Verlag, Berlin.

Kirsch, U., 1981, *Optimal Structural Design,* McGraw-Hill, New York.

Lasdon, L., 1968, "Duality and Decomposition in Mathematical Programming," *IEEE, Trans. of Systems Science and Cybernetics*, Vol. SSC-4, No. 2, pp. 86-100.

Lootsma, F., 1990, "Exploitation of Structure in Nonlinear Optimization," *Parallel Computing*, Vol. 6, pp. 31-45.

Michelena, N., Papalambros, P., 1994, "A Network Reliability Approach to Optimal Decomposition of Design Problems," *Advances in Design Automation*, ASME, Vol. 2, pp. 195-204.

Michelena, N. and Papalambros, P., 1995a, "A Hypergraph Framework to Optimal Model-Based Decomposition of Design Problems," Technical Report UM-MEAM 95-02, Department of Mechanical Engineering and Applied Mechanics, University of Michigan, Ann Arbor, Michigan.

Michelena, N. and Papalambros, P., 1995b, "Optimal Model-Based Decomposition of Powertrain System Design," *J. of Mechanical Design*, ASME, to appear.

Papalambros, P., 1995, "Optimal Design of Mechanical Engineering Systems," *J. of Mechanical Design, and the J. of Vibration and Acoustics*, ASME, Vol. 117, pp. 55-62.

Sobieszczanski-Sobieski, J., 1982, *A Linear Decomposition Method For Large Optimization Problems - Blueprint for Development*, NASA TM 83248, Langley, Hampton, VA.

Sung, K., 1978, *Coordination of Overlapping Hierarchical Water Resources Systems*, Ph.D. Dissertation, Case Western Reserve University, Cleveland, Ohio.

Wagner, T., 1993, *A General Decomposition Methodology for Optimal Systems Design*, Ph.D. Dissertation, Department of Mechanical Engineering and Applied Mechanics, University of Michigan, Ann Arbor, MI.

Wagner, T., Papalambros, P., 1993a, "A General Framework for Decomposition Analysis in Optimal Design," *Advances in Design Automation*, ASME, DE-Vol. 65-2, pp. 315-325.

Wagner, T., Papalambros, P., 1993b, "Implementation of Decomposition Analysis in Optimal Design," *Advances in Design Automation,* ASME, DE-Vol. 65-2, pp. 327-335.

Wilde, D. J., 1978, *Globally Optimal Design*, Wiley, New York.

Wismer, D., Chattergy, R., 1978, *Introduction to Nonlinear Optimization: A Problem Solving Approach*, North-Holland, New York.

Wong, 1970, *Decentralization Planning by Vertical Decomposition of an Economic System: A Nonlinear Approach*, Ph.D. Dissertation, National Energy Planning, University of Birmingham, Birmingham, UK.