Sperner spaces and first-order logic

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We study the class of Sperner spaces, a generalized version of affine spaces, as defined in the language of point-line incidence and line parallelity. We show that, although the class of Sperner spaces is a pseudo-elementary class, it is not elementary nor even $\mathcal{L}_{\infty\omega}$-axiomatizable. We also axiomatize the first-order theory of this class.

E. Sperner [8] introduced a generalized notion of affine space, to be referred to as a *Sperner space*, which he showed how to represent algebraically by means of so-called quasimodules. His paper has generated a sizable literature. The axiom system given in [8] is phrased in a two-sorted language $\mathcal{L}$, with individual variables ranging over points and lines, and with two primitive notions, that of point-line incidence, and that of line parallelity.[1] The axioms are:

(1) For every two distinct points, there is a unique line joining them.

(2) There are at least two points.

(3) Every two lines are incident with the same (cardinal) number of points.

(4) Parallelity is an equivalence relation.

(5) For any line a and point P, there is exactly one parallel to a through P.

Notice that the last two axioms imply that distinct parallel lines have no common points. But nothing in the axioms requires the converse; it is entirely possible for two lines to be disjoint without being parallel. For example, the points and lines of an affine space of any positive dimension, with the usual notions of incidence and parallelity, consititute a Sperner space.

It is natural to ask whether the class of Sperner spaces is an elementary class. At first sight, axiom (3) seems to require an equicardinality quantifier, like the Härtig quantifier [4, 5], which gives rise to a much stronger logic than first order logic. However, one easily sees that the class of Sperner spaces is *pseudo-elementary* (cf. [6, §5.2]). Specifically, there is an extension $\mathcal{L}'$ of the language $\mathcal{L}$ by a quaternary relation R, whose first two arguments are line variables and whose last two arguments are point variables, such that Sperner spaces are exactly the $\mathcal{L}$-reducts of models of the first-order axioms (1), (2), (4), (5) and the conjunction of the following three sentences $(3')$[2], which say that, for any lines l and l', the binary relation $R(l, l', -, -)$ defines a bijection between

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[1] The axiom system may also be expressed in a one-sorted language with variables for points, and a quaternary relation of parallelity, the axioms being Ax 2.2.0 through A2.2.5 in [9], together with an axiom stating the existence of two different points, and an axiom corresponding to (3).

[2] Of course the method here used is not specific to Sperner spaces. Mere assertions of equicardinality, in contrast to more elaborate uses of the equicardinality quantifier, can always be given pseudo-elementary formulations in the same way.

the points on l and those on l':

$$(\forall l)(\forall l')(\forall A)\,(l \neq l' \wedge A \in l \to (\exists^{=1} B)\, R(l, l', A, B)),$$

$(3')$ $\qquad (\forall l)(\forall l')(\forall B)\,(l \neq l' \wedge B \in l' \to (\exists^{=1} A)\, R(l, l', A, B)),$

$$(\forall l)(\forall l')(\forall A)(\forall B)\,(R(l, l', A, B) \to A \in l \wedge B \in l').$$

One purpose of this note is to show that the class of Sperner spaces is not elementary, and in fact not even axiomatizable in the infinitary logic $\mathcal{L}_{\infty\omega}$. A second purpose is to axiomatize exactly what can be said of Sperner spaces in first-order terms.

Theorem 1 *The class of Sperner spaces is not axiomatizable in $\mathcal{L}_{\infty\omega}$.*

P r o o f. To prove the theorem, we shall construct two $\mathcal{L}$-structures, $\mathfrak{I}$ and $\mathfrak{I}'$, which are $\mathcal{L}_{\infty\omega}$-equivalent although the first one is a Sperner space and the second isn't. The construction will be a free construction in the sense of [2, 3]. It proceeds in countably many steps, each step adding more points and lines and defining their incidence and parallelity relations with each other and the previously added points and lines. Thus, $\mathfrak{I}$ will be the union of an increasing sequence of structures $\mathfrak{I}_i$, and similarly $\mathfrak{I}'$ will be the union of an increasing sequence $\mathfrak{I}'_i$.

The construction of $\mathfrak{I}$ begins with a system $\mathfrak{I}_0$ consisting of a line m with $\aleph_0$ points A_n on it, plus one more point P not incident with m.

After $\mathfrak{I}_{2i}$ has been constructed, obtain $\mathfrak{I}_{2i+1}$ as follows. For each pair of points $P_1, P_2 \in \mathfrak{I}_{2i}$ that are not yet joined by a line, add a new line $\varphi(P_1, P_2)$ incident with these two points (and with no other pre-existing points). Also, for each of the newly added lines l, add $\aleph_0$ additional points $\alpha_n(l)$ incident with this line (and no other lines). All the newly added points and lines are distinct, and none of the new lines are parallel to any old lines or to each other (except of course that each line is parallel to itself as required by Axiom (4)).

Then obtain $\mathfrak{I}_{2i+2}$ as follows. For each pair (P, l) of a point and a line in $\mathfrak{I}_{2i+1}$ such that l has as yet no parallel through P, add such a parallel; that is, add a line $\pi(P, l)$ and declare it to be incident with P (and no other pre-existing points) and to be parallel to l and (therefore) to all lines already parallel to l (and to no other pre-existing lines). Identify $\pi(P, l_1)$ with $\pi(P, l_2)$ if l_1 and l_2 were parallel, but otherwise let the new lines be distinct. Two of the new lines $\pi(P_1, l_1)$ and $\pi(P_2, l_2)$ are parallel just in case l_1 and l_2 are parallel. Finally, as in the preceding paragraph, add $\aleph_0$ new points $\alpha_n(l)$ on each newly added line l, the new points being distinct and not incident to any other lines.

This completes the definition of the increasing sequence $\mathfrak{I}_i$; we define $\mathfrak{I}$ to be its union. It follows immediately from the construction that $\mathfrak{I}$ is a Sperner space. In particular, each of its lines is incident with exactly $\aleph_0$ points.

The construction of $\mathfrak{I}'$ is entirely similar, except that we begin with $\mathfrak{I}'_0$ consisting of a line m' having $\aleph_1$ points A'_μ on it, plus one additional point P'. The rest of the construction is as above. The final structure $\mathfrak{I}'$ is not a Sperner space, for it has one line, namely m', with $\aleph_1$ points, while all other lines have only $\aleph_0$ points.

To complete the proof, we must show that $\mathfrak{I}$ and $\mathfrak{I}'$ are $\mathcal{L}_{\infty\omega}$-equivalent. For this purpose, we use the characterization of this infinitary equivalence in terms of Ehrenfeucht-Fraïssé games (see [1, Corollary 3.2.8 (2.2.8 in the 1995 edition)]). We must show that the Duplicator has a winning strategy in the ω-round game for the structures $\mathfrak{I}$ and $\mathfrak{I}'$.

To describe the strategy, we first observe that every element (point or line) in $\mathfrak{I}$ has a name, a closed term built from the initial elements m, A_n, and P, by means of the operations φ, α_n, and π. Except for the identification of $\pi(P, l_1)$ with $\pi(P, l_2)$ when l_1 and l_2 are parallel, distinct terms denote distinct elements. Furthermore, all relations of incidence and parallelity between elements can be read off from their names. Analogous observations apply to $\mathfrak{I}'$ except that the initial elements are m', A'_μ, and P'.

Now the Duplicator's winning (partial) strategy can be described as follows. ("Partial" means that the strategy may permit several choices; a partial strategy can always be narrowed down to a single choice, i. e., to a genuine strategy.) Play so that, after each of your moves, there is a bijection β between finitely many of the points A_n on m and finitely many of the points A'_μ on m', such that each of the chosen ("pebbled") elements of $\mathfrak{I}$ and the corresponding element of $\mathfrak{I}'$ are denoted by terms that differ only by replacing A_n's with their images under β and replacing m and P with m' and P'.

It is easy to see that the Duplicator can always play in accordance with this strategy. When the Spoiler chooses a new element in either structure, the Duplicator takes a term denoting this element, extends the bijection β from the previous move, if necessary, so that the domain or range of β contains all the A_n's or A'_μ's mentioned in this

term, and then applies this extended β to obtain a corresponding term denoting the element to be chosen in the other structure.

Finally, if the Duplicator follows this strategy, he wins. That is, the incidence and parallelity relations between the chosen elements are the same in both structures. This follows immediately from the fact that incidence and parallelity of elements of our two structures can be read off from the terms denoting them. □

The model-theoretic, Ehrenfeucht-Fraïssé argument for the $\mathcal{L}_{\infty\omega}$-equivalence of $\mathfrak{I}$ and $\mathfrak{I}'$ in the preceding proof can be replaced with a set-theoretic argument as follows. If these two structures were not $\mathcal{L}_{\infty\omega}$-equivalent, let θ be a sentence in this infinitary language that is true in one of the two structures and false in the other. This property of θ is preserved in all forcing extensions of the universe, in particular in an extension where $\aleph_1$ is collapsed to be countable. But in such an extension, there is a bijection between the points A_n on m and the points A'_μ on m', and such a bijection obviously extends to an isomorphism between $\mathfrak{I}$ and $\mathfrak{I}'$ (in the forcing extension). This gives a contradiction, since θ cannot be true in one of two isomorphic structures and false in the other.

We now turn to the first-order $\mathcal{L}$-theory of Sperner spaces. Let Σ be the axiom system consisting of the first-order axioms (1), (2), (4), (5) as above, and the first-order sentences $(3'' : k)$ stating, for each $k \in \mathbb{N}$, that, for any two lines l and l', if l has k different points on it, then there must be k different points on l' as well. Thus, the axiom schema $(3'')$ consisting of $(3'' : k)$ for all k can be regarded as Sperner's axiom (3) weakened by ignoring the differences between infinite cardinals.

Theorem 2 Σ *is an axiom system for the first-order $\mathcal{L}$-theory of Sperner spaces.*

P r o o f. Obviously, the axioms of Σ and (therefore) their logical consequences are among the first-order statements true in every Sperner space. To prove the converse, suppose toward a contradiction that $\mathfrak{M}$ were a model satisfying Σ but violating some other first-order $\mathcal{L}$-sentence θ that is satisfied by all Sperner spaces.

In particular, $\mathfrak{M}$ is not itself a Sperner space, so some of its lines have different numbers of points.

No line in $\mathfrak{M}$ can have a finite number k of points, since axioms $(3'' : k)$ and $(3'' : k+1)$ would then imply that all lines have exactly k points, contradicting the fact that $\mathfrak{M}$ is not a Sperner space. Thus all lines in $\mathfrak{M}$ have infinitely many points. Let κ be a cardinal number at least as large as the number of points on any line of $\mathfrak{M}$. (For example, κ could be the number of points in the whole structure $\mathfrak{M}$.) Notice that κ must be uncountable.

Consider a generic model of ZFC in which κ collapses onto ω (cf. [7, Theorem 39 and Lemma 9.9]). First-order truth is absolute between transitive models of set theory, so it is true in the forcing extension, as it was in the original universe, that $\mathfrak{M}$ satisfies Σ but violates θ. (Readers uncomfortable with forcing extensions of the whole universe can take "true in the forcing extension" to mean having Boolean value 1 in the corresponding Boolean-valued model.) Since it also satisfies the equicardinality axiom (3) thanks to the collapsing of cardinals, it is a Sperner space in the forcing extension. So, by enlarging the universe, we have obtained a Sperner space violating θ, something that didn't exist in the original universe.

However, this is impossible, given that Sperner spaces are pseudo-elementarily definable. Indeed, let Σ' be the the set of first-order statements (1), (2), (3'), (4), and (5); recall that Sperner spaces are exactly the $\mathcal{L}$-reducts of the models of Σ'. Therefore, in the original universe, θ holds in all models of Σ'. By the completeness theorem of first-order logic, there is a formal deduction of θ from Σ'. Such a formal deduction remains a formal deduction of θ from Σ' when we enlarge the universe by forcing. So θ must still be true in all Sperner spaces in the enlarged universe. This contradiction completes the proof. □

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