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SYNTHESIS OF LOSSLESS NETWORKS FOR PRESCRIBED TRANSFER IMPEDANCES
BETWEEN SEVERAL CURRENT SOURCES AND A SINGLE RESISTIVE LOAD

Technical Report No. 76

Electronic Defense Group
Department of Electrical Engineering

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Project 2262

TASK ORDER NO. EDG-7
CONTRACT NO. DA-36-039 sc-63203
SIGNAL CORPS, DEPARTMENT OF THE ARMY
DEPARTMENT OF ARMY PROJECT NO. 3-99-04-042
SIGNAL CORPS PROJECT 194B

November 1957

TABLE OF CONTENTS

	Page
LIST OF ILLUSTRATIONS	iii
ABSTRACT	iv
ACKNOWLEDGMENT	v
1. INTRODUCTION	1
2. THE SYNTHESIS METHOD	1
3. EXAMPLES OF THE METHOD	6
4. PRACTICAL CONSIDERATIONS AND CONCLUSIONS	11
REFERENCES	16
DISTRIBUTION LIST	17

LIST OF ILLUSTRATIONS

		Page
Figure 1a	Parallel Lossless Networks with a Single Resistive Load	3
Figure 1b	Parallel Lossless Networks Driven by a Resistive Source	3
Figure 2	The Situation for One Typical Lossless Network in Fig. 1a	4
Figure 3	Example of a Low-Pass, Band-Pass Frequency Multiplex Network	9
Figure 4	Example of a Two-Channel Bandpass Network Realization	12
Figure 5	Experimental Set-Up of a Two-Channel Low-Pass Partitioning Network	14

ACKNOWLEDGMENT

The writer is happy to acknowledge discussions with Dr. B. F. Barton which contributed to the ideas presented here.

ABSTRACT

A technique is presented for the synthesis of lossless networks open-circuited at one end and paralleled across a single resistance at the other end. The synthesis is for prescribed transfer impedances between the open-circuited terminals and the resistive termination. Such networks can be applied to a variety of frequency multiplexing problems, including the design of multi-channel amplifiers. Examples are included, and some practical limitations of such networks are considered.

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I. INTRODUCTION

The synthesis of a lossless network for a prescribed transfer impedance when terminated in a resistive load was developed by Cauer (Reference 1) and has been described by Guillemin and others (References 2 and 3). The purpose of this note is to present a generalization of that technique to the case of multiple current sources. The resulting networks are potentially useful in the synthesis of wideband multi-channel amplifiers, and in frequency multiplexing systems of all types (References 4 and 5).

2. THE SYNTHESIS METHOD

The circuit considered is illustrated in Figure 1 a and b. In Figure 1a n lossless networks driven by current sources $I_1^1, I_1^2, \dots, I_1^n$ are paralleled at their outputs across a single resistive load. This situation is of interest at the output of a multi-channel amplifier employing pentode tubes or in the multiplexing of a number of carrier channels to a single transmission facility. In Figure 1b a single resistive generator drives a number of lossless networks in parallel to produce output voltages $E_2^1, E_2^2, \dots, E_2^n$. This situation might be encountered at the input of a channelized wide-open receiver connected to a single antenna or in the input of a wideband multi-channel amplifier. In the

Figure 1a the output voltage E_2 is given by

$$E_2 = Z_{12}^1 I_1^1 + Z_{12}^2 I_1^2 + \dots + Z_{12}^n I_1^n \quad (1)$$

where Z_{12}^j is the transfer impedance from a current source I_1^j at the input to the j^{th} network to the voltage across the load resistance R when all networks are paralleled across R . As long as the lossless networks are also bi-lateral, the output voltages in Figure 1b are given by

$$\begin{aligned} E_2^1 &= Z_{12}^1 I_1 \\ E_2^2 &= Z_{12}^2 I_1 \\ E_2^n &= Z_{12}^n I_1 ; \end{aligned} \quad (2)$$

by reciprocity, the transfer impedances Z_{12}^j are the same as those in Eq. 1. A technique for synthesizing the lossless networks of Figure 1 for prescribed $Z_{12}^1, Z_{12}^2, \dots, Z_{12}^n$ will now be presented.

Since the inputs of the lossless network of Figure 1a and the outputs of the networks of 1b are open-circuited, one is naturally led to consider the open-circuit impedance parameters of these networks. These parameters are defined by the equations

$$E_1^j = z_{11}^j I_1^j + z_{12}^j I_2^j \quad (3)$$

$$E_2^j = z_{21}^j I_1^j + z_{22}^j I_2^j \quad (4)$$

for the positive voltage and current directions shown in Figure 2. With a load impedance Z_L connected at the output terminals, and a bi-lateral network, the transfer impedance for this network becomes

$$Z_{12}^j \triangleq \frac{E_2^j}{I_1^j} = \frac{z_{12}^j Z_L}{z_{22}^j + Z_L} \quad (5)$$

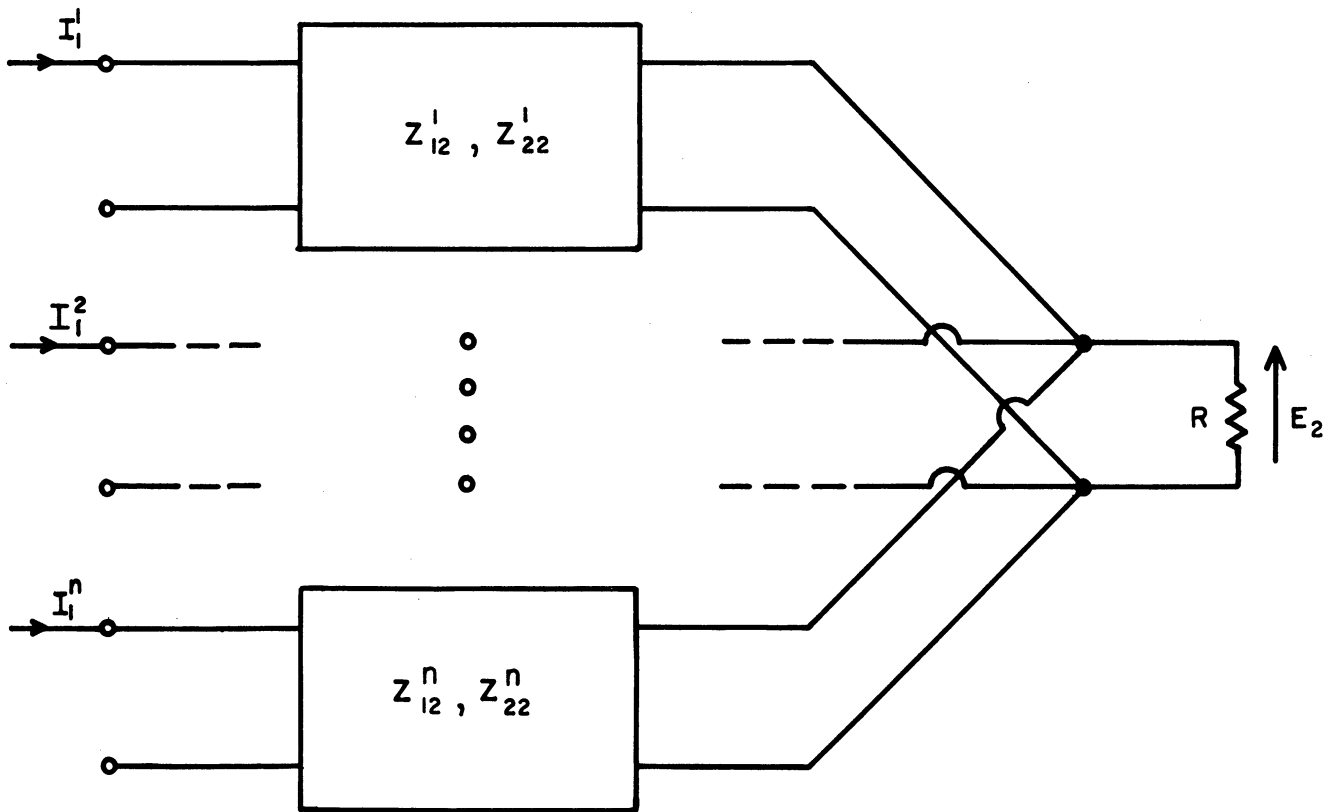


FIG. 1 (a). PARALLEL LOSSLESS NETWORKS WITH A SINGLE RESISTIVE LOAD.

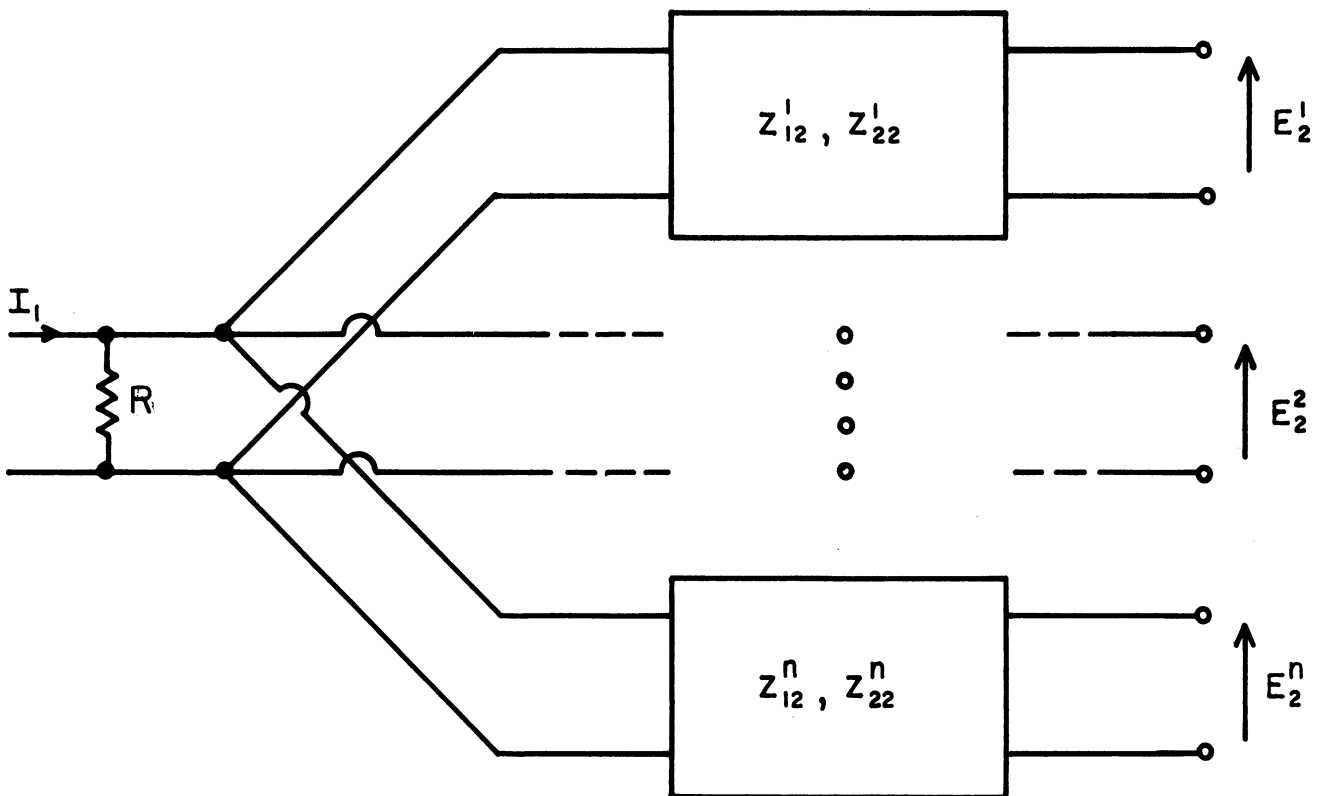


FIG. 1 (b). PARALLEL LOSSLESS NETWORKS DRIVEN BY A RESISTIVE SOURCE.

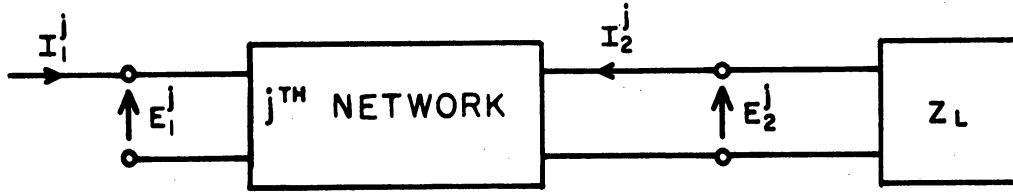


FIG. 2. THE SITUATION FOR ONE TYPICAL LOSSLESS NETWORK IN FIG. 1 (a).

For the j^{th} network in Figure 1a the load impedance is

$$Z_L = \frac{1}{\frac{1}{R} + \frac{1}{z_{22}^1} + \dots + \frac{1}{z_{22}^{j-1}} + \frac{1}{z_{22}^{j+1}} + \dots + \frac{1}{z_{22}^n}} \quad (6)$$

Dividing through by $z_{22}^j \cdot Z_L$ Eq 5 can be written

$$z_{12}^j = \frac{\frac{z_{12}^j}{z_{22}^j}}{\frac{1}{R} + \frac{1}{z_{22}^1} + \dots + \frac{1}{z_{22}^n}} \quad (7)$$

It is clear from this result that all the transfer impedances in Eqs 1 and 2 will have common poles. Thus Eq 1 can be written¹

$$E_2 = \frac{\sum_{j=1}^n \left(\frac{z_{12}^j}{z_{22}^j} \cdot I_1^j \right)}{\frac{1}{R} + \sum_{j=1}^n \left(\frac{1}{z_{22}^j} \right)} \quad (8)$$

The poles of z_{12}^j must lie in the left half of the complex p-plane. Given

$$E_2 = \frac{\sum_{j=1}^n N^j I_1^j}{A_0 + pB_0} \quad (9)$$

this means $A_0 + pB_0$ is a Hurwitz polynomial. To identify Eq 9 with Eq 8 two

possibilities exist: (1) split the odd part of the denominator pB_0 into n parts

1. This concise formulation was pointed out to the writer by B. F. Barton.

and write

$$E_2 = \frac{\frac{1}{A_0} \sum_{j=1}^n (N^j \cdot I_1^j)}{1 + \sum_{j=1}^n \left(\frac{pB_0^j}{A_0} \right)} \quad (10)$$

where $\sum_{j=1}^n pB_0^j = pB_0$; or (2) split the even part of the denominator A_0 into n parts giving

$$\frac{\frac{1}{pB_0} \sum_{j=1}^n (N^j \cdot I_1^j)}{1 + \sum_{j=1}^n \left(\frac{pB_0^j}{A_0} \right)}, \quad (11)$$

where $\sum_{j=1}^n A_0^j = A_0$. In case 1 one identifies

$$\frac{z_{12}^j}{z_{22}^j} = \frac{N^j}{A_0} \quad (12)$$

and

$$z_{22}^j = \frac{A_0}{pB_0^j}; \quad (13)$$

so that

$$z_{12}^j = \frac{N^j}{pB_0^j}. \quad (14)$$

Similarly in case 2

$$\frac{z_{12}^j}{z_{22}^j} = \frac{N^j}{pB_0^j}, \quad (15)$$

$$z_{22}^j = \frac{pB_0}{A_0^j}, \quad (16)$$

and

$$z_{12}^j = \frac{N^j}{A_0^j}. \quad (17)$$

Examination of Eqs 14 and 17 points up a limitation of restricting ourselves to lossless networks and serves as a basis for choosing between case 1 and 2. Since z_{12}^j must be purely imaginary for $p = j\omega$, we see that the numerator polynomials N_j must all be even or odd for this identification to be realizable. If the N_j are even polynomials, case 1 is applicable; if they are odd, case 2 would be chosen. In either case z_{12}^j and z_{22}^j have the same denominator and hence, in general, common poles. The only other requirement necessary to assure the success of this identification is to split

$$A_o = A_o^1 + A_o^2 + \dots + A_o^n$$

or

$$pB_o = pB_o^1 + pB_o^2 + \dots + pB_o^n$$

so as to make z_{22}^j a positive real function. A division that is always realizable is $A_o^j = A_o/n$ or $pB_o^j = pB_o/n$. In general, one has a considerable range of freedom in making this division which can be used to simplify some of the networks to be synthesized or to control other parameters of interest (such as the ratio of capacities at the inputs to the reactive networks).

Once a satisfactory division of A_o or pB_o has been chosen each lossless network is synthesized by expanding z_{22}^j for the prescribed z_{12}^j (Ref. 2). This can always be achieved to within a constant multiplier. The importance of this multiplier will depend upon the specific application of the multiplexing network.

3. EXAMPLES OF THE METHOD

To illustrate the ideas of the previous section consider a two channel example such that¹

1. For this function the components of $|E_2|^2$ due to I_1^1 and I_1^2 alone add up to a Tschebycheff lowpass filter characteristic with a maximum 1 db attenuation in the passband from zero to one radian per second.

$$E_2 = \frac{I_1^1 + 2 p^2 I_1^2}{1 + 3.200 p + 3.497 p^2 + 2.941 p^3} \quad (18)$$

Since the numerator polynomials $N^1 = 1$ and $N^2 = 2p^2$ are both even functions of p , we must use case 1. The most general split for pB_o is

$$pB_o^1 = k_o p + k_1 p^3 = k_o p \left(1 + \frac{k_1}{k_o} p^2\right) \quad (19)$$

$$pB_o^2 = (3.200 - k_o) p + (2.941 - k_1) p^3 \quad (20)$$

while

$$A_o = 1 + 3.497 p^2 \quad (21)$$

Therefore, according to Eqs 12 and 13, we identify

$$z_{22}^1 = \frac{1 + 3.497 p^2}{k_o p \left(1 + \frac{k_1}{k_o} p^2\right)} \quad (22)$$

$$z_{12}^1 = \frac{1}{k_o p \left(1 + \frac{k_1}{k_o} p^2\right)} \quad (23)$$

$$z_{22}^2 = \frac{1 + 3.497 p^2}{(3.200 - k_o) p \left[1 + \left(\frac{2.941 - k_1}{3.200 - k_o}\right) p^2\right]} \quad (24)$$

$$z_{12}^2 = \frac{2p^2}{(3.200 - k_o) p \left[1 + \left(\frac{2.941 - k_1}{3.200 - k_o}\right) p^2\right]} \quad (25)$$

The conditions imposed by the desired positive real character of z_{22}^1 are:

$$k_o > 0$$

$$k_1 \geq 0$$

and

$$\frac{k_1}{k_o} < 3.497 \quad (26)$$

Similarly the conditions for z_{22}^2 to be realizable are

$$\begin{aligned} k_0 &< 3.200, \\ k_1 &\leq 2.941, \end{aligned} \quad (27)$$

and $\frac{2.941-k_1}{3.200-k_0} < 3.497$.

Since z_{12}^2 is itself realizable, the bandpass network can be directly synthesized as an L network, with $z_{22}^2 - z_{12}^2$ in the series arm and z_{12}^2 as the shunt arm, provided

$$z_{22}^2 - z_{12}^2 = \frac{1 + 1.497 p^2}{(3.200-k_0) p \left[1 + \left(\frac{2.941-k_1}{3.200-k_0} \right) p^2 \right]} \quad (28)$$

is also realizable. This will be true if

$$\frac{2.941-k_1}{3.200-k_0} \leq 1.497. \quad (29)$$

A minimum of elements are required when condition 29 is met with the equal sign.

For this case the condition 29 becomes

$$1.497 k_0 - k_1 = 1.849. \quad (30)$$

The network parameters are then

$$z_{22}^2 - z_{12}^2 = \frac{1}{(3.200-k_0)p} \quad (31)$$

and

$$z_{12}^2 = \frac{2p}{(3.200-k_0)p (1+1.849 p^2)} \quad (32)$$

The network is shown in Fig. 3a.

Since all the zeros of z_{12}^1 lie at infinity, expanding z_{22}^1 in Cauer's first form (alternatively removing poles of admittance and impedance at infinity) produces a satisfactory ladder network. This realization of Eq 22 is shown in Fig. 3b. Within the conditions 26, 27, and 30 one still has a considerable

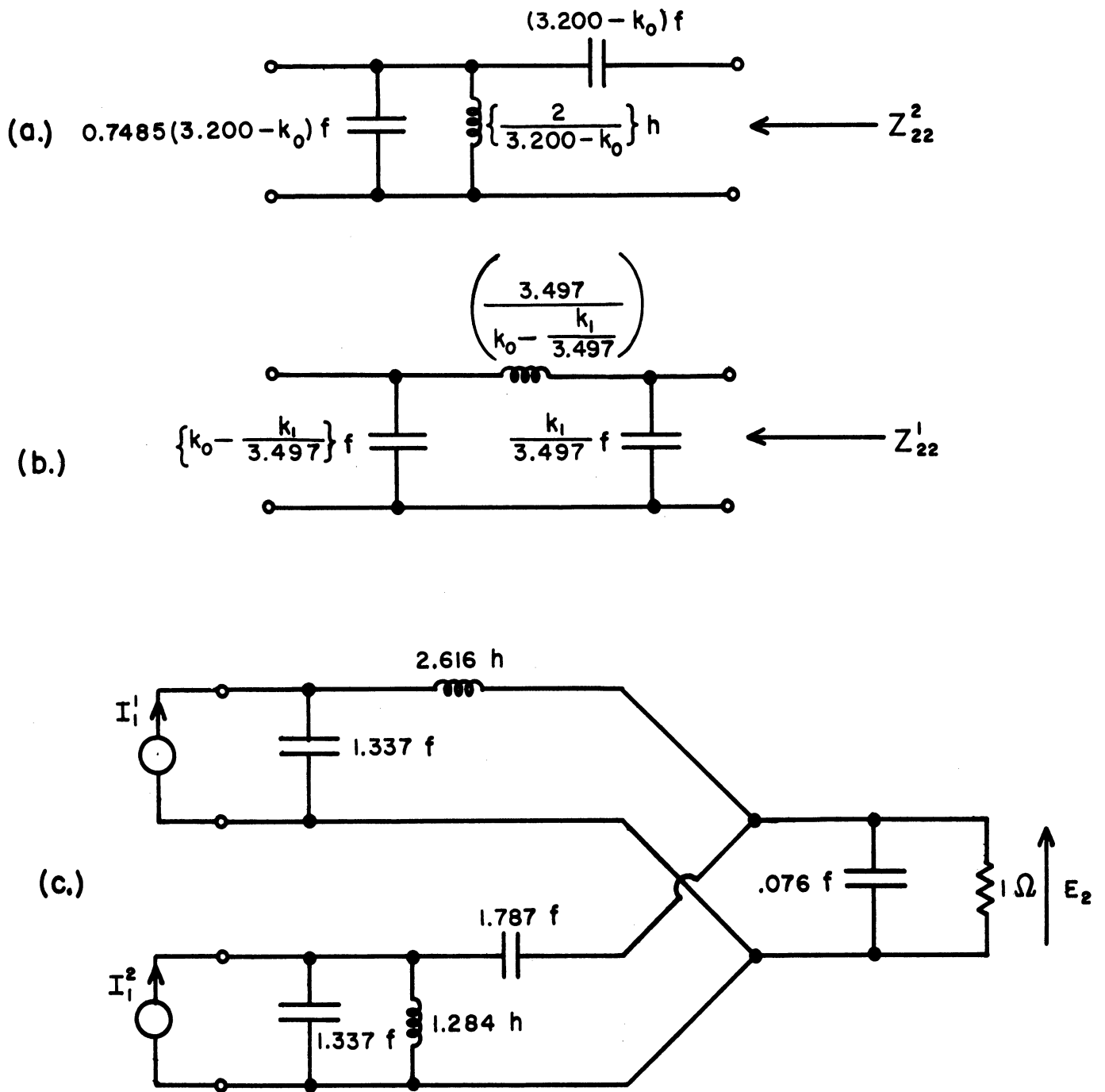


FIG. 3. EXAMPLE OF A LOW-PASS, BAND-PASS FREQUENCY MULTIPLEX NETWORK; (a) BAND-PASS NETWORK, (b) LOW-PASS NETWORK, AND (c) COMPLETE NETWORK WITH CONSTANTS ADJUSTED FOR EQUAL INPUT CAPACITIES.

freedom in the choice of k_1 or k_0 . Choosing $k_1 = 0$, for example, will reduce the number of reactive elements required.

Another choice of interest is to try to make the input capacities of the two networks equal. This will be true provided

$$1.7485 k_0 - 0.2859 k_1 = 2.395. \quad (33)$$

This condition together with Eq 30 solved simultaneously give the particular values $k_0 = 1.413$ and $k_1 = 0.2663$. Since none of the other conditions 26 and 27 are violated, this is a useful result. The complete network using these numerical values is given in Fig. 3c.

As a second example consider the synthesis of a two-channel bandpass circuit such that

$$E_2 = \frac{pI_1^1 + p^3I_1^2}{1 + 2p + 3p^2 + 2p^3 + p^4}, \quad (34)$$

The numerator polynomials are both odd in this example, so one is lead to case 2.

The odd part of the denominator is

$$pB_0 = 2p + 2p^3 \quad (35)$$

and the even part is split into two parts

$$A_0^1 = k_0 + k_1p^2 + k_2p^4 \quad (36)$$

and

$$A_0^2 = 1 - k_0 + (3-k_1)^2p + (1-k_0)p^4. \quad (37)$$

Then one identifies

$$z_{22}^1 = \frac{2p + 2p^3}{k_0 + k_1p^2 + k_2p^4}, \quad (38)$$

$$z_{12}^1 = \frac{p}{k_0 + k_1p^2 + k_2p^4}, \quad (39)$$

$$z_{22}^2 = \frac{2p + 2p^3}{1 - k_0 + (3 - k_1)p^2 + (1 - k_0)p^4},$$

and

$$z_{12}^2 = \frac{p^3}{1 - k_0 + (3 - k_1)p^2 + (1 - k_2)p^4}. \quad (41)$$

With constants k_0 , k_1 , and k_2 to be specified the possibilities open to the network designer are numerous. A choice which simplifies the structure of the number one channel considerably is to let $k_0 = k_2 = 0$. The network realization for this case is shown in Fig. 4a. The transfer impedance of the second channel has three zeros at zero frequency and one at infinity. A ladder structure having the correct poles and this distribution of zeros is obtained by removing one pole of admittance at infinity. The ladder obtained by this development is given in Fig. 4b. Checking, however, one finds that the transfer impedance for this ladder is

$$z_{12} = \frac{\left(\frac{2-2k_1}{2-k_1}\right)p^3}{1 + (3-k_1)p^2 + p^4} \quad (42)$$

which for all positive, non-zero k_1 is less than Eq 41 by the constant factor $2-2k_1/2-k_1$. Normally an overall scale factor times Eq 34 would not be serious, but the relative responses are important. These can be equalized by a tapping down on the output of the first channel. The result of making this adjustment and picking $k_1 = 0.5$ is the complete network of Fig. 4c¹. Alternately one might go back and make another choice for k_0 and k_2 .

4. PRACTICAL CONSIDERATIONS AND CONCLUSIONS

At the out-set it was hypothesized that the synthesis of a number of lossless channels with a single resistive load was a desirable objective. Such a

1. For the particular choices of $k_0 = k_2 = 0$ it is not possible to obtain equal input capacities in this case.²

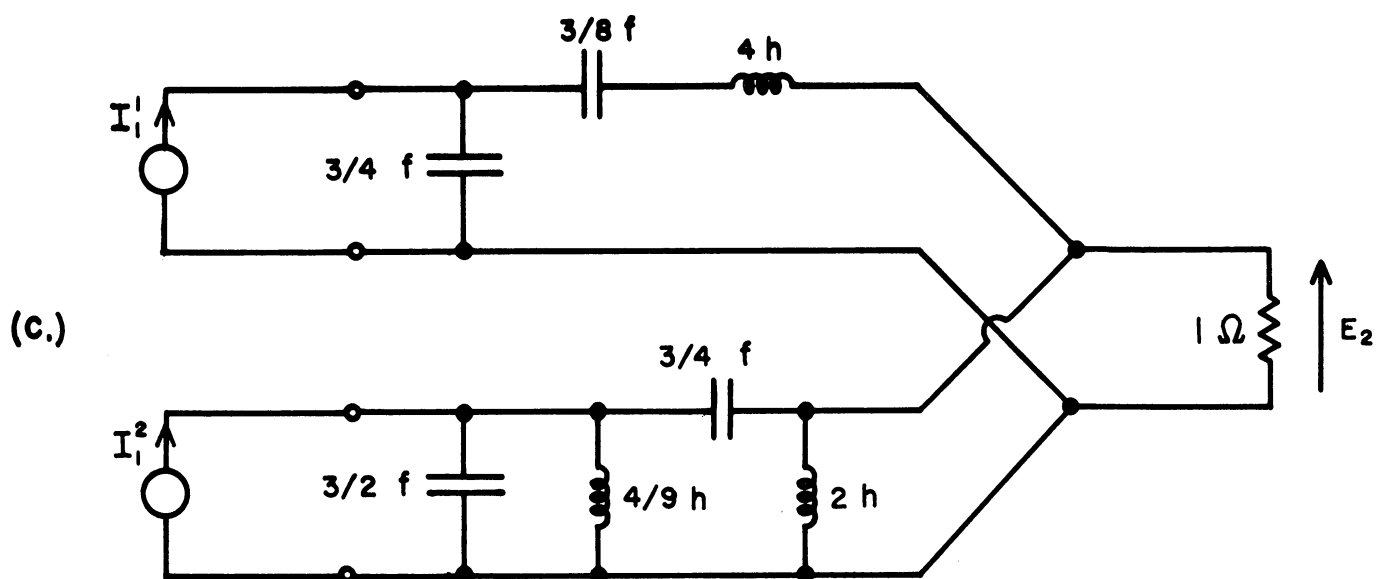
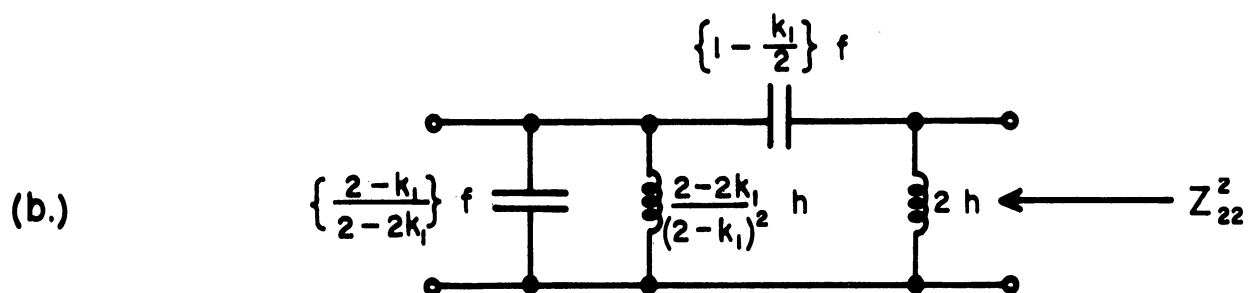
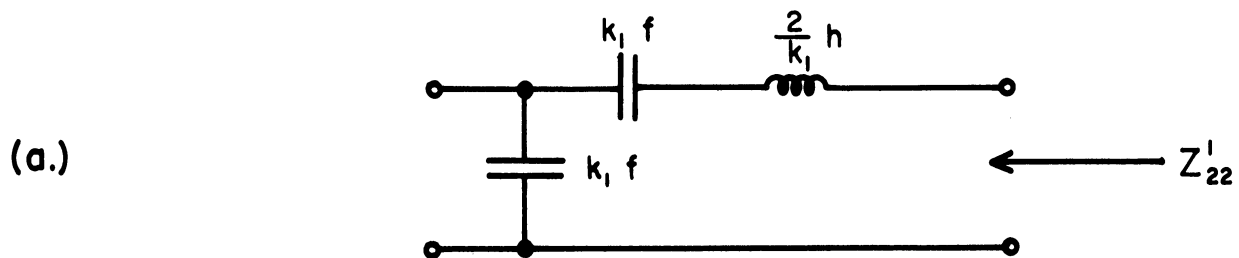
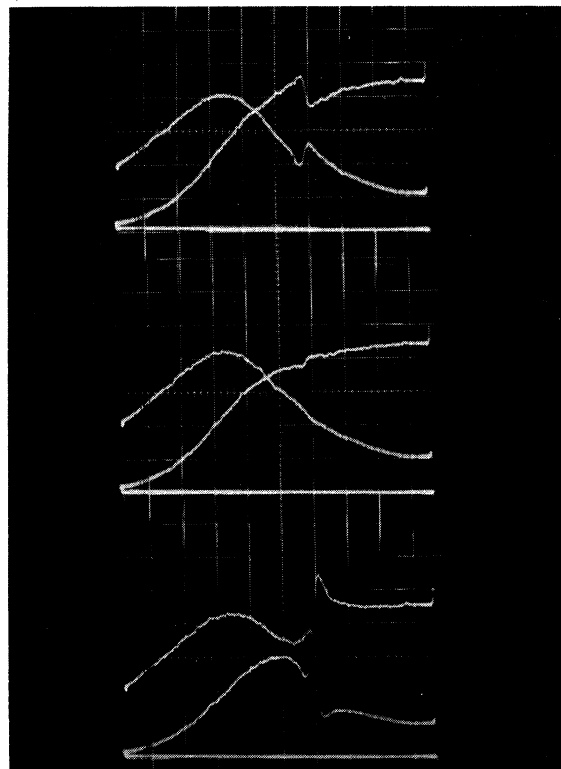
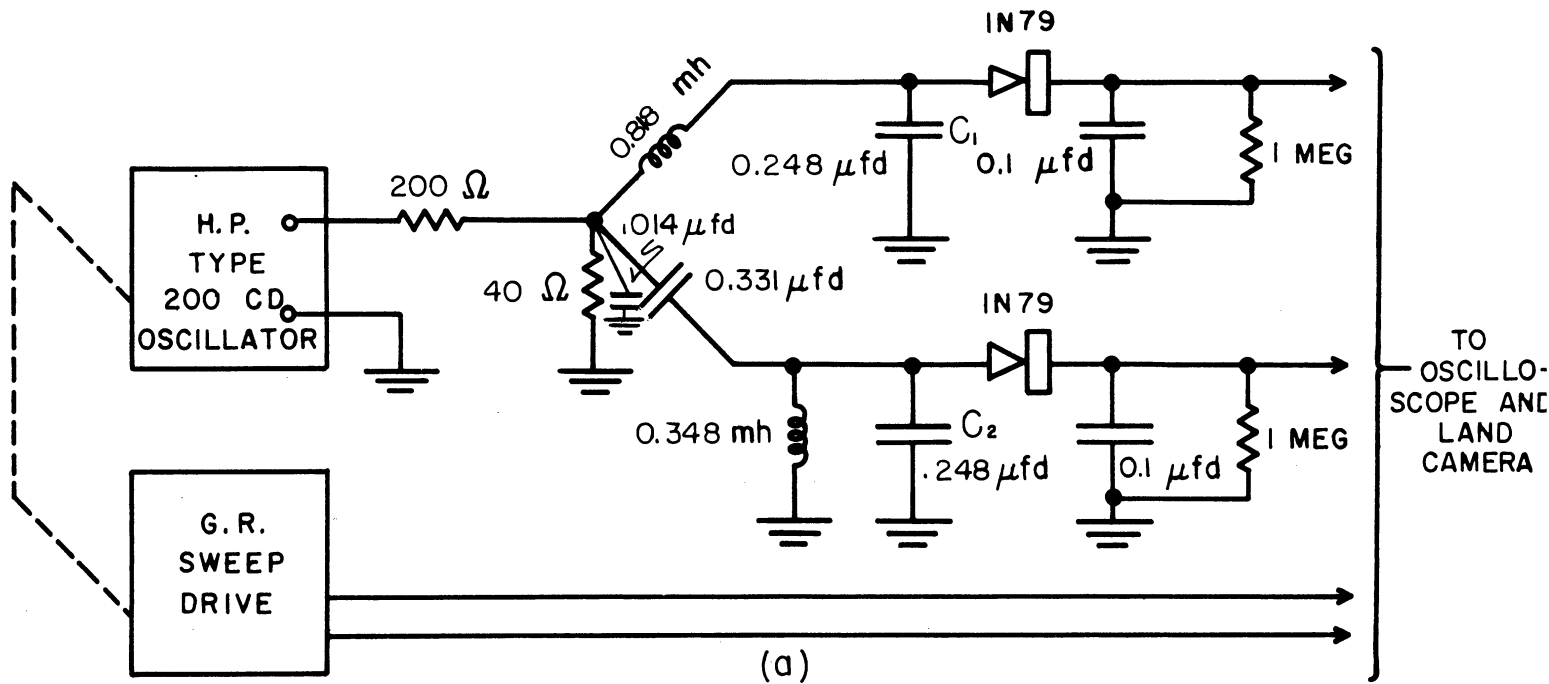


FIG. 4. EXAMPLE OF A TWO-CHANNEL BANDPASS NETWORK REALIZATION.

structure is attractive for power amplifiers employing pentodes, or the input to low-noise amplifiers. On the other hand, nature exacts a considerable price for this freedom from excess dissipation. One limitation, already indicated, is that the numerator polynomials used must always be either even or odd functions of p . This may considerably limit the approximation techniques one can employ.

Another practical limitation can be best illustrated by considering the network of Fig. 3c. The load impedance of the lowpass portion of this network is the parallel combination of the one ohm load resistor and the output impedance of the bandpass network, z_{22}^2 . According to Eq 34 this impedance has a zero at $\omega = \pm 0.535$ radians per second. This zero is not, however, a zero of the overall transfer impedance from the low-pass input, z_{12}^1 , specified by Eq 18. This is physically accomplished by having a zero of y_{11}^1 coincide with the zero of z_{22}^2 . This means that when the output of the lowpass network is short-circuited by the zero of z_{22}^2 , the impedance seen by the current source I_1^1 becomes infinite. The resulting infinite current through the 2.616 henry inductance is just sufficient to produce the desired finite voltage across the short-circuited load resistor. It is clear that such a pole-zero cancellation can be expected to require very critical trimming of at least one component in a practical network.

Experimental investigations reveal that these lossless networks terminated in a single resistance can indeed be rather critical of adjustment. Figure 5a illustrates the result of a 10% change in the shunt capacity of each channel on the individual responses E_2/I_1^1 and E_2/I_1^2 for the network of Fig. 3. While the resulting distortion of the responses are severe, it was also found that the shunt inductance of the bandpass channel always could be used to trim out the shape distortion.



$$C_1 = 0.248 \mu \text{fd}$$

$$C_2 = 0.273 \mu \text{fd}$$

$$C_1 = C_2 = 0.248 \mu \text{fd}$$

$$C_1 = 0.273 \mu \text{fd}$$

$$C_2 = 0.248 \mu \text{fd}$$

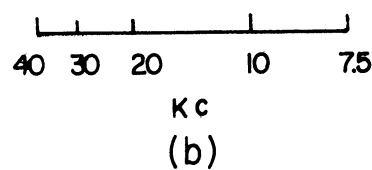


FIG. 5. EXPERIMENTAL SET-UP OF A TWO-CHANNEL LOW-PASS PARTITIONING NETWORK; (a) SET-UP; (b) CHANGE IN (E_1^1) AND (E_2^2) VERSUS FREQUENCY FOR 10 PERCENT CHANGES IN $0.248 \mu \text{fd}$ CONDENSERS.

The fact that these networks are critical of adjustment is not surprising; this is a well known drawback of a double-tuned circuit only loaded on one side, see for example Reference 6. The network designer can reduce these effects by employing Barton's insertion loss design method or by introducing some dissipation into the lossless structures through Darlington's predistortion techniques (References 6 and 7). In each case the effect is to move the pole-zero cancellations into the interior of the complex p -plane where they have less influence on the response along the $j\omega$ -axis. Another approach is to control the approximation problem so as to move these pole-zero cancellations along the $j\omega$ -axis to some region of less importance to the desired overall response.

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