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Research

Identifying the Period of a Step Change in High-Yield Processes

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Quality control charts have proven to be very effective in detecting out-of-control states. When a signal is detected a search begins to identify and eliminate the source(s) of the signal. A critical issue that keeps the mind of the process engineer busy at this point is determining the time when the process first changed. Knowing when the process first changed can assist process engineers to focus efforts effectively on eliminating the source(s) of the signal. The time when a change in the process takes place is referred to as the change point. This paper provides an estimator for a period of time in which a step change in the process non-conformity proportion in high-yield processes occurs. In such processes, the number of items until the occurrence of the first non-conforming item can be modeled by a geometric distribution. The performance of the proposed model is investigated through several numerical examples. The results indicate that the proposed estimator provides a reasonable estimate for the period when the step change occurred at the process non-conformity level. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Control charts are used for monitoring the performance of a quality characteristic. They assist process engineers to distinguish random causes from assignable causes. When a control chart signals, a search is initiated to identify and eliminate the source of the assignable cause. The nature of the search depends on the experience and the knowledge of the process engineer. The more knowledge and experience the person has about the process, the faster he/she can pinpoint the source(s) of the assignable cause. However, if the time when the change began could be determined with a reasonable level of accuracy, then the engineer could search within a smaller window of observations to identify the possible source(s) of the assignable cause.

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The issue of step changes in the parameters of various distributions in the context of statistical process control has been addressed by different researchers. Samuel *et al.*[1,2] considered step changes in the mean, μ, and variance, σ^2, of a normal distribution, respectively. Samuel and Pignatiello[3] compared the performance of the maximum likelihood estimator (MLE) with the built-in estimators of EWMA and CUSUM in determining the time of a step change in a normal process mean and showed the superiority of the MLE. Pignatiello and Simpson[4] proposed a magnitude-robust control chart not only for monitoring a normal process mean, but estimating the time of step changes in the mean as well. Perry and Pignatiello[5] considered a linear trend change in the mean of a normal process. Samuel and Pignatiello[6] studied a step change in the parameter of a Poisson process, λ. Perry *et al.* proposed methods for determining the change point in the rate of a Poisson distribution[7,8] with a linear trend and monotonic change, respectively. Pignatiello and Samuel[9] and Perry *et al.*[10] considered a step change and monotonic change in the non-conformity level, p, of a binomial process, respectively. Perry and Pignatiello[11] showed that the performance of the MLE is better than the built-in estimators of EWMA and CUSUM in identifying the change point of a binomial distribution. Nedumaran *et al.*[12] identified the time of a step change in a χ^2 control chart. Timmer and Pignatiello[13] provided change point estimates for the parameters of an autoregressive process of order 1. Zou *et al.*[14] proposed a change point model for monitoring linear profiles with unknown parameters. In this paper, we consider a step change in the non-conformity level of a geometric process. In high-yield processes where the level of non-conformity is in the range of part per million, e.g. 1000 PPM, it is common to consider the number of conforming items until the occurrence of a non-conforming item as a geometric random variable with parameter p. The non-conformity level p indicates the probability that a randomly selected item does not conform to predefined specifications. This paper provides an MLE for the period when a change in the process non-conformity level, p, takes place. The process change model is discussed in Section 2. An illustrative example is presented in Section 3. Performance of the estimator is investigated in Section 4. The final section provides our concluding remarks.

2. STATISTICAL MODEL

Consider a process that is initially under statistical control and observations are generated according to a Bernoulli process with fraction non-conforming p_0. It is assumed that the fraction non-conforming p_0 is in the range of part per million. Many researchers including Xie and Goh[15] and Woodall[16] have warned about using the traditional p chart when fraction non-conforming is very low. The reason lies in the fact that low level of non-conformity requires a relatively large sample size in order to detect shifts in the process non-conformity level promptly. To overcome the large sample size dilemma, it is recommended to consider the number of conforming units until a non-conforming unit is generated as a geometric random variable and apply a geometric control chart to control the process non-conformity level, i.e.

$$\begin{aligned} g(x;p) &= p(1-p)^{x-1}, \quad x \geq 1 \\ G(x;p) &= 1-(1-p)^x \end{aligned} \tag{1}$$

where $g(\cdot)$ and $G(\cdot)$ are the p.d.f. and c.d.f. of a geometric random variable, respectively, and p is the process non-conformity proportion.

Control charts based on the geometric distribution, which were initially developed by Calvin[17], were further studied by several researchers such as Kaminsky *et al.*[18], Nelson[19], Quesenberry[20], Xie and Goh[21], McCool and Joyner-Moltey[22] and Xie *et al.*[23].

Suppose that at an unknown point in period τ, referred to as process change point, process non-conformity level changes from p_0 to $p_1 = \delta p_0$, where δ is the magnitude of the change. Values of $\delta > 1$ represent deterioration in the non-conformity level and values of $\delta < 1$ represent an improvement in the non-conformity level. We also assume that once the process shifts from the in-control state p_0 to an out-of-control state p_1, it remains at the new level until the source of the assignable cause is identified and eliminated.

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Let us define period i as the time interval between the $(i-1)$st and ith non-conforming units and X_i as the number of inspected units until a non-conforming unit is observed in the period i. It is well known that X_i follows a geometric distribution with parameter p_0, where p_0 represents the in-control non-conformity level. Suppose at period T a signal is generated by a geometric control chart. Hence, $LCL \leq X_i \leq UCL$ for $i<T$ and $X_i<LCL$ or $X_i>UCL$ for $i=T$. It is assumed that no false alarm is generated by the chart. Therefore, observations $X_1, X_2, \ldots, X_\tau$ belong to the in-control process with fraction non-conforming p_0, while observations $X_{\tau+1}, X_{\tau+2}, \ldots, X_T$ come from the out-of-control process with fraction non-conformity level p_1. It is now desired to identify the period τ at which the change has taken place in the process. The following section provides a numerical example through which the procedure for estimating the period τ is presented.

3. AN ILLUSTRATIVE EXAMPLE

Consider a process with an in-control non-conformity level of $p_0=0.0005$. Since the process non-conformity level is very low it would be appealing to use a geometric control chart. Based on the control limits provided by Xie and Goh[21] for geometric control charts, we obtain the following values for the upper and lower control limits when type one error probability (α) is equal to 0.0027:

$$\begin{aligned} UCL &= \frac{\mathrm{Ln}(\alpha/2)}{\ln(1-p_0)} = \frac{\mathrm{Ln}(0.0027/2)}{\mathrm{Ln}(1-0.0005)} = 13\,211.99 \\ LCL &= 1+\frac{\mathrm{Ln}(1-\alpha/2)}{\mathrm{Ln}(1-p_0)} = 1+\frac{\mathrm{Ln}(1-0.0027/2)}{\mathrm{Ln}(1-0.0005)} = 3.70 \end{aligned} \tag{2}$$

Thus, if $X_i<4$ or $X_i>13\,211$ the chart signals an out-of-control condition, indicating a change in the process non-conformity level.

The change point estimator proposed here for the period when the step change occurred in the process non-conformity level is based on the MLE method. When the geometric control chart signals an out-of-control condition, the proposed method can be applied to determine the period when the step change occurred in the process parameter p_0. The period when the step change occurred is that value of i $(0 \leq i < T)$ that maximizes

$$L_i = i\,\mathrm{Ln}\left(\frac{p_0(1-\hat{p}_{1,i})}{\hat{p}_{1,i}(1-p_0)}\right) - SX_{i,T}\,\mathrm{Ln}\left(\frac{1-p_0}{1-\hat{p}_{1,i}}\right) + T\,\mathrm{Ln}\left(\frac{\hat{p}_{1,i}}{1-\hat{p}_{1,i}}\right) \tag{3}$$

where $\hat{p}_{1,i}=(T-i)/SX_{i,T}$ is the estimate of process fraction non-conformity level and $SX_{i,T}=\sum_{j=i+1}^{T} x_j$ is the sum of inspected units in the period $i+1, i+2, \ldots, T$, respectively (see Appendix A for details).

The value of i that maximizes L_i would be the estimate of the last period from the in-control process and $\hat{p}_{1,i}$ would be its corresponding estimate of the changed fraction non-conformity level.

Now we present a simple demonstration of the method through a numerical example. Table I gives the number of inspected units, X_{i+1}, the values of $\hat{p}_{1,i}$ and $SX_{i,T}$, respectively, for $i=0,1,\ldots,23$. According to this table, the geometric control chart signals a change in the process non-conformity level at period $T=24$.

To determine the period when the step change occurred, one needs to check the above table for the largest value of L_i. The largest value of L_i corresponds to period 10, indicating that the change in the non-conformity level has most likely occurred at this step. This means that our estimate of τ, the period when the step change occurred in the process non-conformity level, would be $\hat{\tau}=10$. Hence, the process engineer should check his/her records for an assignable cause that occurred most likely around period 10.

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Table I. Change point estimator computations

Period no.	i	X_i	$SX_{i,T}=\sum_{j=i+1}^{T} x_j$	$\hat{p}_{1,i}=(T-i)/SX_{i,T}$	L_i
1	0	3070	21 818	0.0011	−176.6
2	1	1345	19 167	0.0012	−175.5
3	2	679	16 923	0.0013	−175.4
4	3	5378	17 500	0.0012	−175.8
5	4	2345	11 765	0.0017	−171.6
6	5	2188	9048	0.0021	−169.7
7	6	1954	6923	0.0026	−167.2
8	7	843	5000	0.0034	−164.2
9	8	1506	4103	0.0039	−163.5
10	**9**	**280**	2586	**0.0058**	**−159.4**
11	10	293	2333	0.006	−160.4
12	11	28	2031	0.0064	−161.2
13	12	131	2000	0.006	−163.6
14	13	300	1864	0.0059	−165.3
15	14	154	1563	0.0064	−166.2
16	15	327	1429	0.0063	−167.8
17	16	211	1096	0.0073	−168.4
18	17	302	875	0.008	−169.6
19	18	15	577	0.0104	−169.9
20	19	221	562	0.0089	−172.7
21	20	242	342	0.0117	−173.6
22	21	30	100	0.03	−173
23	22	68	70	0.0286	−176.3
24	23	2	2	0.5	−176.2

4. PERFORMANCE EVALUATION

The performance of the proposed estimator for the period when the step change occurred in the non-conformity level is investigated through a Monte Carlo simulation study. Using a geometric distribution, 100 observations from an in-control process with $p=p_0$ are first generated.

Observations with X_i value exceeding either of the control limits are considered as false alarms, since the process is in-control. In case a false alarm is encountered in a simulation run, it is treated in the same way that one would treat it in an actual process. In other words, when a false alarm is encountered at a period $i<\tau$, the control chart is restarted and all previous observations are omitted, while the scheduled change point does not change. It is rational to exclude such observations from the calculation since clearly the period of change is not prior to the false alarm period. Therefore, in simulation runs where a false alarm occurs, the number of observations considered in estimating τ is less than 100.

Starting with period 101, a shock was induced to the process, changing the non-conformity level from p_0 to $p_1=\delta p_0$. We then generated enough observations until an out-of-control signal was detected. From this point on, the estimate for the period when the step change occurred, τ, which should be close to 100, was computed and recorded. Furthermore, to estimate the expected period when the first alarm is given by the geometric control chart ($\hat{E}(T)$), we recorded the period number in which the signal had been detected. This process was repeated 10 000 times for $p_0=0.0005$ and for different values of δ. The average and standard error of the 10 000 estimates of the period when the step change occurred, namely $\bar{\hat{\tau}}$ and $Se(\bar{\hat{\tau}})$, along with the estimation of the expected period when the first alarm is given, $\hat{E}(T)=ARL+\tau$, are presented in Tables II and III for different magnitudes of increases ($\delta>1$) and decreases ($\delta<1$) in fraction non-conformity level, respectively. The average run length (ARL) in the above expression for $\hat{E}(T)$ refers to the number of points

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Table II. Average change point estimates ($\bar{\hat{\tau}}$) and standard error using $p_0 = 0.0005$, $\tau = 100$, $p_1 > p_0$ and $N = 10000$ independent simulation runs

p_1	0.0006	0.0007	0.0008	0.0009	0.001
$\bar{\hat{\tau}}$	164.50	106.38	99.72	98.27	98.46
$Se(\bar{\hat{\tau}})$	1.223	80.4462	0.2629	0.1744	0.1348
$\hat{E}(T)$	565.85	543.45	510.75	477.38	452.61

Table III. Average change point estimates ($\bar{\hat{\tau}}$) and standard error using $p_0 = 0.0005$, $\tau = 100$, $p_1 < p_0$ and $N = 10000$ independent simulation runs

p_1	0.0004	0.0003	0.0002	0.0001
$\bar{\hat{\tau}}$	149.09	105.47	100.65	99.59
$Se(\bar{\hat{\tau}})$	0.8058	0.2152	0.1028	0.0669
$\hat{E}(T)$	248.81	149.29	113.69	103.70

plotted on the control chart before a signal is observed and it can easily be obtained by subtracting τ from $\hat{E}(T)$.

Table II reveals that a 40% increase in the fraction non-conformity level ($p_1 = 0.0007$) would be detected by the geometric control chart on average of 443.45 periods after the change has actually occurred in the process. However, the *MLE* provides an average estimate of 106.38 for the period when the step change occurred in fraction non-conforming, which is very close to the actual change point of 100. The standard error of the estimates for this case is $Se(\bar{\hat{\tau}}) = 0.4462$, which is relatively small. The results in Table II indicate that the estimates for period when the step change occurred get closer to the true value as the size of the shift in the process non-conformity level increases.

According to the results in Table III, the geometric control chart signals on average 49.29 periods after the process fraction non-conforming drops by 40%, i.e. $p_1 = 0.0003$. However, the change point estimator performs relatively well by yielding 105.47 as the average estimate for the period when the step change occurred. The results in Table III reveal that the performance of the *MLE* improves as the value of the fraction non-conforming decreases. In other words, as the deviation from the in-control value of fraction non-conformity level increases, the standard error of the estimates decreases.

We now consider the frequency with which the change point estimate is within m periods of the true change point for $m = 1, 2, 3, 4, 5, 10, 15, 20, 25, 30, 35, 40$ and 45. The results from the same simulation study for the increases and decreases in the process fraction non-conformity levels are given in Tables IV and V, respectively.

Table II shows that for a 40% increase in the fraction non-conformity level ($p_1 = 0.0007$), the control chart yields an *ARL* of 443.45. According to Table IV, the proposed *MLE* estimates the true change point 4% of the times correctly when p_1 is equal to 0.0007. The change point is estimated 25.88% of the times within five periods of the process change point. Similarly, for a 60% increase in the fraction non-conformity level ($p_1 = 0.0008$), the control chart yields an *ARL* of 410.75. For this step change, the proposed *MLE* estimates the process change point 7.54% of the times correctly. The change point is estimated 67.03% of the times within 15 periods of the process change point. The results in Table IV indicate that the performance of the estimator improves as the magnitude of the change increases.

Table II indicates that for a 40% decrease in the process fraction non-conformity level ($p_1 = 0.0003$), the control chart *ARL* drops to 49.29. For a step change of this magnitude, according to Table V, the true process change point is estimated 8.13% of the times correctly and in 41.48% of the times the change point is estimated within five periods of the true process change point. For a 60% decrease ($p_1 = 0.0002$) in the fraction non-conformity level, the control chart yields an *ARL* of 13.69 and the true process change point

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Table IV. Precision of the estimator with $p_0 = 0.0005$, $\tau = 100$ and $p_1 > p_0$ using 10 000 independent simulation runs

p_1	0.0006	0.0007	0.0008	0.0009	0.001
$\hat{P}(\hat{\tau}=\tau)$	0.0124	0.0400	0.0754	0.1071	0.1455
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 1)$	0.0318	0.0978	0.1701	0.2268	0.2946
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 2)$	0.0528	0.1440	0.2424	0.3239	0.4041
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 3)$	0.0702	0.1884	0.3063	0.3973	0.4859
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 4)$	0.0869	0.2247	0.3592	0.4588	0.5513
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 5)$	0.1038	0.2588	0.4050	0.5081	0.6066
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 10$	0.1673	0.4824	0.5652	0.6896	0.7698
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 15)$	0.2247	0.5578	0.6703	0.7871	0.8561
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 20)$	0.2726	0.6151	0.7425	0.8472	0.9087
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 25)$	0.3203	0.6658	0.7979	0.8857	0.9385
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 30)$	0.3611	0.7060	0.8386	0.9146	0.9580
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 35)$	0.3947	0.7060	0.8686	0.9376	0.9696
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 40)$	0.4259	0.7419	0.8910	0.9523	0.9773
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 45)$	0.4619	0.7691	0.9122	0.9632	0.9827

Table V. Precision of the estimator with $p_0 = 0.0005$, $\tau = 100$ and $p_1 < p_0$ using 10 000 independent simulation runs

p_1	0.0004	0.0003	0.0002	0.0001
$\hat{P}(\hat{\tau}=\tau)$	0.0179	0.0813	0.2254	0.4514
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 1)$	0.0433	0.1779	0.4119	0.6915
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 2)$	0.0668	0.2501	0.5311	0.8106
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 3)$	0.0875	0.3138	0.6197	0.8728
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 4)$	0.1070	0.3708	0.6829	0.9125
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 5)$	0.1238	0.4148	0.7302	0.9359
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 10)$	0.2042	0.5860	0.8752	0.9695
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 15)$	0.2725	0.6957	0.9315	0.9780
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 20)$	0.3304	0.7747	0.9574	0.9823
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 25)$	0.3806	0.8277	0.9719	0.9860
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 30)$	0.4266	0.8703	0.9785	0.9880
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 35)$	0.4700	0.8996	0.9826	0.9894
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 40)$	0.5050	0.9228	0.9859	0.9912
$\hat{P}(\lvert\hat{\tau}-\tau\rvert \leq 45)$	0.5379	0.9396	0.9878	0.9929

is estimated 22.54% of the times correctly. Simulation results indicate that the change point is estimated 73.02% of the times within five periods of the true process change point. We can see that by increasing (or decreasing) the magnitude of a change, the probability of accurate estimation of the true change point increases.

5. CONCLUSIONS

When a control chart signals the presence of a shift in a process, process engineers begin a search to identify the source(s) of the assignable cause. A dilemma for the process engineer at this point is to identify the time when the process actually changed. Unfortunately the time of the change is not usually known to the process engineer. However, if he/she knew when the actual step change occurred in the process level, it not

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only limits the scope of the search window to just a few observations but also helps in the effective use of resources to eliminate the source(s) of the assignable cause.

In this paper, an estimator based on MLE method and geometric distribution was developed to identify the period when the step change occurred in the fraction non-conformity level in high-yield processes. The performance of the proposed estimator was investigated numerically using shifts of different magnitudes in the non-conformity level. The results indicate that the estimator is reasonably accurate and precise for estimating the period when the step change occurred in the fraction non-conformity level.

The proposed method is simple, effective and can be easily implemented. This makes the method appealing to process engineers who work with high-yield processes.

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APPENDIX A

In this Appendix, we consider the derivation of the *MLE* of τ, the process fraction non-conforming change point. MLE techniques are discussed in Casella and Berger[24], for example. We will denote the *MLE* of the change point τ as $\hat{\tau}$ $(0 \le \tau < T)$. Given the number of conforming items in the observations $X_1, X_2, \ldots, X_T$, the *MLE* of τ is the value of τ that maximizes the likelihood function or, equivalently, its logarithm. The likelihood function is

$$L(\tau, p_1|X) = \prod_{j=1}^{\tau} p_0(1-p_0)^{x_j-1} \prod_{j=\tau+1}^{T} p_1(1-p_1)^{x_j-1}$$

$$L(\tau, p_1|X) = p_0^{\tau}(1-p_0)^{\sum_{j=1}^{\tau} x_j-\tau} p_1^{T-\tau}(1-p_1)^{\sum_{j=\tau+1}^{T} x_j-T+\tau}$$

The logarithm of the likelihood function is

$$\operatorname{Ln} L(\tau, p_1|X) = \tau \operatorname{Ln} p_0 + \left(\sum_{j=1}^{\tau} x_j - \tau\right) \operatorname{Ln}(1-p_0) + (T-\tau)\operatorname{Ln} p_1 + \left(\sum_{j=\tau+1}^{T} x_j - T + \tau\right) \operatorname{Ln}(1-p_1)$$

$$\operatorname{Ln} L(\tau, p_1|X) = \tau \operatorname{Ln} p_0 + \left(\sum_{j=1}^{T} x_j - \sum_{j=\tau+1}^{T} x_j\right) \operatorname{Ln}(1-p_0) - \tau \operatorname{Ln}(1-p_0)$$

$$+(T-\tau)\operatorname{Ln} p_1 + \left(\sum_{j=\tau+1}^{T} x_j - T + \tau\right) \operatorname{Ln}(1-p_1)$$

The value of p_1 that maximizes the likelihood function is $\hat{p}_{1,\tau} = T - \tau / \sum_{j=\tau+1}^{T} x_j$.

The maximum likelihood estimate of the change point (the last period from the in-control process) τ is

$$\hat{\tau} = \underset{0 \le i < T}{\arg\max}\{L_i\}$$

where

$$L_i = i \operatorname{Ln}\left(\frac{p_0(1-\hat{p}_{1,j})}{\hat{p}_{1,j}(1-P_0)}\right) - \sum_{j=t+1}^{T} x_j \operatorname{Ln}\left(\frac{1-p_0}{1-\hat{p}_{1,j}}\right) + T \operatorname{Ln}\left(\frac{\hat{p}_{1,j}}{1-\hat{p}_{1,j}}\right) \quad \text{and}$$

$$\hat{p}_{1,i} = \frac{T-i}{\sum_{j=i+1}^{T} x_j}$$

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