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Mortality Contingent Claims: Impact of Capital Market, Income, and Interest Rate Risk

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Abstract

In this paper, we consider optimal insurance, portfolio allocation, and consumption rules for a stochastic wage earner with CRRA preferences whose lifetime is random. In a continuous time framework, the investor has to decide among short and long positions in mortality contingent claims a.k.a. life insurance, stocks, bonds, and money market investment when facing a risky stock market and interest rate risk. We find an analytical solution for the complete market case in which human capital is exactly priced. We also extend the analysis to the case where income is unspanned. An illustrative analysis shows when the wage earner's demand for life insurance switches to the demand for annuities.

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1 Introduction

In this paper, we analyze how income, stock market, and interest rate risk affect the purchase of life insurance and pension annuities over the life cycle in a continuous time setting. We expand prior literature on pension annuities and life insurance by simultaneously considering income, stock market, and interest rate risk as the main sources of uncertainty over the life cycle apart from mortality risk. Mortality risk has two different outcomes. On the one hand, the investor can run out of savings and fall into poverty before dying. In this case, the investor can neither consume nor bequeath her heirs (longevity risk). On the other hand, the investor can die too early without consuming enough of her savings in case she has not a bequest motive or she can perish without having built up sufficient wealth to leave an appropriate legacy for her heirs. The latter is also known as brevity risk.

While in the past only wealthy individuals were concerned with personal financial planning, today everyone has to accumulate wealth and manage her assets effectively to meet the consumption and bequest liabilities that arise during her lifetime. Personal finance particularly becomes critical for everyone's individual welfare in light of the demographic shifts eroding and undermining public PAYGO systems. Until recently several employers have taken on longevity and investment risk by providing defined benefit solutions to their workers. Today's employees have to increasingly bear the investment risk when electing a certain defined contribution 401k plan from their employer's menu. Ideal candidates for the efficient management of consumption (bequest-) liabilities are pension annuities (life insurance). Both products have appealing return and payoff characteristics since their pricing is either contingent on surviving (pension annuities) or on dying (life insurance). Compared to an investment in mutual funds, the consumption possibilities are higher when annuitizing, while the bequest potential increases when purchasing life insurance. In case the individual annuitizes, the investor surrenders bequest potential, while the investor gives up consumption possibilities when purchasing life insurance. If the investor purchases a life insurance, the premium is taken out of the investor's financial wealth. As already demonstrated by Richard (1975), the investor controls her legacy by finding the right balance between savings and life insurance early in life and seeks the appropriate split between financial wealth and pension annuities during the later stage of her life cycle. So the key point of interest is to know when the investor should switch from life insurance to pension annuities in order to satisfy both her desire to consume and her wish to bequeath her heirs and what wealth level triggers the short position in life insurance.

Even though the selection of life insurance and pension annuities is a challenging task in a world where mortality is the sole source of risk (see Yaari, 1965 or Pliska and Ye, 2007), there are many other sources of risk influencing both the timing and the extent to which life insurance and pension annuities are purchased. Labor income is probably the most influential exogenous factor determining the timing of pension annuities (see e.g. Horneff, Maurer, and Stamos, 2008a). By the same token, labor income also influences the level of savings and therefore the overall demand for life insurance.

Not only does labor income have a considerable impact on the purchase of life insurance and pension annuities, but the influence on the asset allocation within the investor's financial wealth is also considerable. Studies such as Cocco, Gomes, and Maenhout (2005), Duffie, Fleming, Soner, and Zariphopoulou (1997), Gomes and Michaelides (2005), Heaton and Lucas (1997), Koo (1998), Viceira (2001), and several more consider stochastic income. Several of these studies find labor income to be more closely related to bonds than to stocks. Therefore, human capital acts partly as a substitute for bonds. In turn, the optimal stock fraction of financial wealth decreases over the life cycle as the fraction of human capital declines in the investor's total augmented wealth. Augmented wealth is composed of both financial wealth and human capital. This is why we pay particular attention to modeling the investor's labor income.

Correlation between the innovations of the stock return and the income growth had only a limited impact on the overall asset allocation because previous studies considered a single risky asset and a constant investment opportunity set. Munk and Sørensen (2007) show that correlation particularly matters if the investor faces a stochastic investment opportunity set with correlated interest rate risk.

There are numerous other studies looking only at interest rate risk in isolation. Other studies analyzing dynamic asset allocation in the presence of interest rate risk include Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (2001), Deelstra, Graselli and Koehl (2000), Liu (2007), Sangvinatsos and Wachter (2005), and many more. Since the interest rate dynamics are also important for determining the value of the investor's human capital, we find it particularly interesting to analyze its impact on the decision when to buy pension annuities and when to purchase life insurance.

Indeed, the literature on life insurance and pension annuities is vast. Early work on life insurance in a financial context includes Campbell (1980), Fischer (1973), Hakansson (1969), Hurd (1989), Lewis (1989), Yaari (1965). While previous studies analyze in which way mortality, utility, strength of bequest, risk aversion, and intertemporal elasticity of substitution influence the demand for life insurance, the articles are not written in the spirit of modern portfolio choice models.

In many studies, pension annuities are analyzed separately as most products involve life long payments. For portfolio choice studies including gradual annuitization only we refer the interested reader to Milevsky and Young (2007) and Horneff, Maurer, and Stamos (2008a, 2008b)

Only a few articles consider the timing of life insurance and pension annuities simultaneously. Richard (1975) is a pioneering study modeling life insurance in the framework of modern portfolio choice. Dybvig and Liu (2004) consider a model with flexible retirement dates by endogenizing the leisure decision while the investor can purchase life insurance and pension annuities. Huang, Milevsky, and Wang (2005) as well as Huang and Milevsky (2008) consider a consumption function including the bread winner and her respective heirs, where the model includes the bread winner's risky labor income and stochastic capital markets. Pliska and Ye (2007) model an economy with no risk sources except the uncertainty surrounding the investor's remaining lifetime. However, their study looks at pre-retirement behavior only. Purcal (2003) analyzes the interaction of life insurance and unspanned income in the presence of a stochastic stock market. Kraft and Steffensen (2008) generalize the Richard's (1975) results by assuming a multistage Markovian framework including uncertain lifetime and disability but they do not consider capital market risks and therefore disregard the individual's investment decision. In essence, we extend the analysis of Munk and Sørensen (2007) to uncertain lifetime and insurance products and the article by Richard (1975) to stochastic labor income and interest rate risk by using Pliska and Ye's (2007) model which only considers riskless bonds and life insurance. Compared to Huang and Milevsky (2008), we assume an age dependent income process and introduce interest rate risk. Interest rate risk is important because it influences the demand for mortality contingent claims, expands the stochastic investment opportunity set for capital markets, and influences the pattern of the labor income process.

First, we develop our stochastic model before we analyze the case of spanned income. Then we proceed to the case of unspanned income. A final chapter concludes.

2 The Model

2.1 Uncertain Lifetime and Preferences

We assume that the investor is alive at time $t = 0$ and the investor's age at death (lifetime) is a non-negative continuous random variable T on the probability space $(\Omega, \mathcal{F}, P)$ (c.f. to Pliska and Ye, 2007). In our model, we suppose that the random variable T has a probability distribution with underlying probability density function $\bar{f}(t)$ and a cumulative distribution function $\bar{F}(t)$:

$$\bar{F}(t) = P(T < t)) = \int_0^t \bar{f}(u)du. \tag{1}$$

The function $S(t)$ is also known as the survivor function and is therefore defined as the probability that the age at death (survival time) is greater than or equal to t:

$$S(t) = P(T \geq t)) = 1 - \bar{F}(t). \tag{2}$$

The survivor function can be used to compute the probability that the individual lives from time zero to some time beyond t. The force of mortality or hazard rate is the instantaneous 'death rate' for the investor when she has survived until time t and is given by:

$$\lambda(t) \equiv \lim_{\Delta t \to 0} \frac{P(t \leq T < t + \Delta t \mid T \geq t)}{\Delta t} = \frac{\bar{f}(t)}{S(t)} = -\frac{d}{dt} \ln(S(t)). \tag{3}$$

Thus, the survivor function can be simply restated as:

$$S(t) = \exp\left(-\int_0^t \lambda(u)du\right). \tag{4}$$

In the remainder, we use a parametric function for the force of moratlity. We choose the Makeham-Gompertz law of mortality because of its widespread use in the finance and insurance literature (see e.g. Milevsky and Young, 2007).

$$\lambda(t) = \vartheta + \frac{1}{\chi}\exp\left(\frac{t-\xi}{\chi}\right). \tag{5}$$

The parameters ϑ, ξ, and χ determine the shape of the force of mortality function. While ξ is the mode of the remaining lifetime, the parameter χ can be interpreted as a dispersion parameter of the distribution. The constant ϑ aims at capturing the component of the death rate that can be attributed to accidents. The exponentially increasing portion of (5) reflects natural causes of death over the life cycle.

Under the risk neutral measure λ_t turns out to be equal to the hazard rate μ_t. Interestingly, the hazard rate μ_t is used by the insurance companies to price their products. Therefore, the heterogeneous beliefs about individual survival probabilities affect the value of human capital in the market place.[1]

In the setup of our optimization problem, we are concerned with maximizing the investor's expected utility from lifetime consumption $U(c_t,t)$, from bequest $V(\upsilon_t,t)$, and from final wealth $L(W_\omega,\omega)$. Here, ω represents the truncated lifetime and can be interpreted as the last year of financial planning during the investor's life. In turn, the indirect utility function J is given by:

$$\begin{aligned} &J(W,y,r,t) = \\ &\max_{(c,\theta_Q,\theta_B,\theta_M,x)_t^{\omega\wedge T}} E_t\left[\int_t^{\omega\wedge T} U(c_s,s)ds + V(\upsilon_T,T)I_{\{T\leq\omega\}} + L(W(\omega))I_{\{T>\omega\}} \mid \mathcal{F}_t\right], \end{aligned} \tag{6}$$

where c_s denotes the consumption at time s, υ_T is the legacy the investor leaves at death, and W_ω is the final wealth.

[1] Insurance companies usually assume a different mortality table to adjust the survival probabilities to reflect adverse selection. While λ_t is used to weight the utility from consumption and bequest potential in the optimization problem, μ_t describes the force of mortality used to price insurance products.

The optimization problem is dependent on several state variables such as the level of cash on hand W, income y, the short rate r, and age t. The state variable cash on hand W can be influenced by the controls consumption c, stock investment θ_Q, bond holdings θ_B, cash or money market holdings θ_M, and the amount of life insurance x. The controls are both time and state dependent.

We have to include the final wealth or cash on hand W_ω in order to find a closed form solution for our problem.

Richard (1975) chooses an artificial terminal condition for his optimization problem. In his case, the integral for determining the indirect utility function is not defined at the very end because the integrand becomes infinite. Technically, the resulting closed form expression is not the solution to the stated optimization problem. Yet, there are different ways of addressing this issue. For instance, Pliska and Ye (2007) state their problem in a mathematically correct manner by introducing some fixed investment horizon. They interpret this particular point in time as the beginning of the retirement period.[2] If the investor considered in Pliska and Ye (2007) lives up to her terminal date, she is granted the remaining wealth for the periods to come. We have also thought of another solution to the problem in as much as we can assume that the investor leaves neither bequest nor wealth to live on for the periods after the financial planning horizon. The resulting solution would be very similar to the one of Richard's (1975) article.[3] However, we prefer the final wealth condition as it appears to be more economically sound. Most probably, the investor is happy to receive a lump sum to cover her care giving expenses at the very end of the financial planning horizon.

In the literature, Dybvig and Liu (2004), for instance, evade any problems related to the indirect utility function by using a fixed hazard rate. However, a fixed hazard rate is a questionable assumption both from an economic and biological standpoint as the investor cannot age in their model and has the same expected lifetime throughout her entire existence. Huang, Milevsky, and Wang (2005), for example, avoid the bequest completely by modeling a consumption function including the bread winner as well as her heirs.

Turning to our original problem, we can rewrite (6) in the following way because

[2] A detailed derivation can be found in Ye (2006).

[3] The results of this alternative problem are available upon request.

we assume that T is independent of the filtration $\mathbb{F}$:

$$J(W,y,r,t) = \max_{(c,\theta_Q,\theta_B,\theta_M,x)_t^\omega} E_t \left[\int_t^\omega \frac{S(\omega)}{S(t)} U(c_s,s)ds \right. \\ \left. + \int_t^\omega \frac{S(u)}{S(t)} \lambda(u) \int_t^u U(c_s,s)dsdu + \int_t^\omega \frac{S(u)}{S(t)} \lambda(u) V(\upsilon_u,u)du + \frac{S(\omega)}{S(t)} L(W(\omega)) \mid \mathcal{F}_t \right]. \tag{7}$$

Applying Fubini Tonelli's theorem and swapping the order of integration, we get:

$$J(W,y,r,t) = \max_{(c,\theta_Q,\theta_B,\theta_M,x)_t^\omega} E_t \left[\int_t^\omega \frac{S(s)}{S(t)} \left[U(c_s,s)ds + \lambda(s) V(\upsilon_s,s) \right] ds \right. \\ \left. + \frac{S(\omega)}{S(t)} L(W(\omega)) \mid \mathcal{F}_t \right]. \tag{8}$$

In the remainder, we assume that the investor has $CRRA$ preferences for consumption, bequest, and final wealth.

$$U(c_t,t) = m(t)\frac{c_t^{1-\gamma}}{1-\gamma}, \quad V(\upsilon_t,t) = m(t)\frac{\upsilon_t^{1-\gamma}}{1-\gamma}, \quad L(W_\omega,\omega) = m(\omega)\frac{W_\omega^{1-\gamma}}{1-\gamma}, \tag{9}$$

where γ is the level of risk aversion which is greater than one. The subjective discount rate $m(t)$ is equal to $e^{-\varphi t}$, where the rate of time preference is φ.

In the following sections, we will define the dynamics of each state variable separately. One state variable can be omitted because of scale independence. Later on, we will reduce the state space to normalized cash on hand as an income multiple, the level of the short rate r, and age t.

For now, we consider the state variable cash on hand W before normalization because we want to express the short positions in terms of absolute values rather than relative figures.

2.2 Financial Markets

The bond market is modeled according to Vasicek (1977) in which the short rate spans the whole term structure of interest rates.

$$dr = \kappa(\bar{r} - r_t)dt - \sigma_r dz_r, \tag{10}$$

where κ determines the speed driving the current short rate r_t to its long run mean $\bar{r}$ and where z_r is a one dimensional standard Brownian motion and σ_r is the volatility of the short rate process. The price of a zero bond B_t^s can be represented by an affine function:

$$B_t^s \equiv B^s(r_t, t) = e^{-a(s-t)-b(s-t)r_t}, \tag{11}$$

where s is the initial maturity.

$$\begin{aligned} b(\tau) &= \tfrac{1}{\kappa}(1 - e^{-\kappa\tau}) \\ a(\tau) &= R_\infty(\tau - b(\tau)) + \tfrac{\sigma_r^2}{4\kappa} b(\tau)^2 \\ R_\infty &= \bar{r} + \tfrac{\sigma_r \phi_r}{\kappa} - \tfrac{\sigma_r^2}{2\kappa^2}, \end{aligned} \tag{12}$$

where τ determines the time to maturity of a bond at time t, R_∞ is the asymptotic long rate representing the yield of a zero coupon bond as maturity goes to infinity, and ϕ_r is the market price of risk. The dynamics of the bond price can be stated as:

$$\frac{dB_t}{B_t} = (r_t + \sigma_B(r_t, t)\phi_r)dt + \sigma_B(r_t, t)dz_r, \tag{13}$$

where $\sigma_B(r_t, t)$ is the bond return volatility. In general, the bond return volatility depends on both the level of the short rate and the time-to-maturity. In this model, the bond return has a perfect negative instantaneous correlation with the interest rate $\rho_{Br} = -1$. Even though the affine interest model has only one source of uncertainty, the Vasicek model allows for all shapes of term structures observed empirically: normal, flat, and inverted term structures.

As far as stock markets are concerned, we assume that the individual can invest in a single non-dividend paying stock index that obeys a price process Q of the following nature:

$$\frac{dQ_t}{Q_t} = (r_t + \psi)dt + \sigma_Q(\rho_{QB} dz_r + \sqrt{1 - \rho_{QB}^2} dz_Q) \tag{14}$$

The expected return on equity is equal to r_t plus the constant equity premium ψ. The innovations of the stock and the bond process are correlated where the correlation is given by ρ_{QB}. Here, σ_Q represents the volatility of the stock return. To simplify the notation for the sections to come, we combine the bond and stock price dynamics in $P_t = (B_t, Q_t)^T$ by using matrix notation. We have to multiply the lower triangular matrix found by Cholesky decomposition by the vector of the two standard Brownian motions $Z = (z_r, z_Q)^T$.

$$\Sigma(r_t, t) = \begin{pmatrix} \sigma_B(r_t, t) & 0 \\ \sigma_Q \rho_{QB} & \sigma_Q \sqrt{1 - \rho_{QB}^2} \end{pmatrix} \tag{15}$$

$$dP_t = diag(P_t)[(r_t 1 + \Sigma(r_t, t)\Phi)dt + \Sigma(r_t, t)dZ_t], \tag{16}$$

where $\Phi = (\phi_r, \phi_Q)^T$ is the vector of market risk premiums and ϕ_Q is the market risk premium of stocks which is defined as:

$$\phi_Q = \frac{1}{\sqrt{1 - \rho_{QB}^2}} \left(\frac{\psi}{\sigma_Q} - \rho_{QB}\phi_r \right) \tag{17}$$

2.3 Insurance Markets

In the insurance market used in this model, the investor can buy or sell instantaneous term life insurance as in Richard (1975). Instantaneous term life insurance means that the investor can only purchase death insurance for the next second. The investor has to pay a premium of x_t to ensure that her heirs receive an amount of $\frac{x_t}{\mu_t}$ at her death, where μ_t is the hazard rate the insurance company uses to price the term life insurance:

$$\mu_t = \lambda_t + \iota_t, \tag{18}$$

where ι is the incremental load factor increasing the standard hazard rate at time t. The incremental load factor is supposed to fulfill the following condition:

$$D(t) = \int_t^{\omega} \iota(s)ds < \infty, \tag{19}$$

where D are the total costs. If the investor dies immediately, her heirs receive her accumulated cash on hand plus the payout from the instantaneous term life insurance as a legacy υ_t.

$$\upsilon_t = W_t + \frac{x_t}{\mu_t}, \tag{20}$$

where W_t is the cash on hand. Again, we show the dynamics of W_t after introducing all other decision and state variables. Surviving the next second, the investor has to repurchase instantaneous term life insurance in order to hedge the liability arising from her bequest motive for the following second. Indeed, life insurance is cheap for a wage earner. Here is an example: The US female investor is 35 years old and she has just purchased a life insurance for USD 100. Her hazard rate is 0.265 percent at that time. In turn, her death benefit amounts to USD 37,736. So we do not expect an extensively large cash outflow for life insurance at the early stage of her life cycle. Only a small fraction of her wealth goes towards funding the additional legacy. Modeling life insurance in this way does not only cover death insurance but also annuities. Theoretically, death insurance is just the flip side of an annuity. The investor has to pay a premium when she is alive while her heirs receive a death benefit in case she perishes. If the investor shorts the term life insurance, she will receive the premium as a benefit when she is alive and will have to pay the death benefit as the annuity premium from her legacy. In real world, a life annuity is a financial contract between a buyer (annuitant) and a seller (insurer) that pays out a periodic amount for as long as the buyer is alive, in exchange for an initial premium (Brown et al., 2001). However, there is a similar product out in the insurance market known as reverse mortgage. A reverse mortgage works as follows: A reverse mortgage is used to release the home equity in a property as a loan given in multiple payments to the home owner. Indeed, the multiple payments can be a life annuity spelling out how much the house owner receives for as long as she stays alive. The homeowner's obligation is deferred until the owner dies. So the premium is paid only at death. The home equity value works as collateral for the remaining time. Still the annuity payments are life long to limit the adverse selection in the insurance market (see Brugiavini, 1993). Arguably one cannot find the exact same payoff structure of short positions in term life insurance in real life. However, it works as an excellent simplification and shows similar characteristics of a reverse mortgage with life annuity payments. In turn, the introduction of life long payments necessitates the numerical solution of a combined optimal control and stopping time problem (see Milevsky and Young, 2007).

2.4 Labor Income

In the remainder, we use a tractable model of labor income by assuming that the investor as a wage earner receives a continuous non-negative income from non-financial sources throughout her lifetime. The income rate is y_t at time t. Identical to Munk and Sørensen (2007) we assume that y_t evolves according to the stochastic differential equation:

$$\begin{aligned} dy_t = y_t \left[(\zeta_0(t) + \zeta_1 r_t)dt + \sigma_y \left[\rho_{yP}^T dZ + \sqrt{(1 - \rho_{yB}^2 - \hat{\rho}_{yQ}^2)} dz_y \right] \right] \\ \text{for } t \in [0, \min\{\omega, T\}], \end{aligned} \tag{21}$$

where ζ_0 is the time dependent drift term and ζ_1 denotes the the sensitivity of the wage rate with respect to the short rate. Here z_y is a one-dimensional standard Brownian motion, $\rho_{yP} = (\rho_{yB}, \hat{\rho}_{yQ})^T$ is the vector of correlations, and where

$$\hat{\rho}_{yQ} = \frac{\rho_{yQ} - \rho_{QB}\rho_{yB}}{\sqrt{1 - \rho_{QB}^2}}. \tag{22}$$

The coefficients in front of the standard Brownian motions can be found by applying the Cholesky-Crout algorithm to the covariance matrix. The drift term of the income process is also influenced by the level of the short rate to reflect the impact of the business cycles on the general wage level.

3 Spanned Income

3.1 Optimal Policies

In order to derive the optimal policies for the individual, we need to consider the dynamic budget constraint. The holdings in the money market account θ_M are given as the residual after subtracting other investments from the current cash on hand defined below. The residual value is determined by $\theta_M = W_t - \theta_Q - \theta_B$. Given the optimal consumption, investment, and insurance strategy, the wealth evolution can be written as:

$$
\begin{aligned}
&dW_t = (W_t r_t + \Theta^T \Sigma(r_t,t)\Phi - c_t - x_t + y_t)dt + \Theta^T\Sigma(r_t,t)dz \\
&\text{for } t \in [0, \min\{\omega, T\}],
\end{aligned}
\tag{23}
$$

where $\Theta = (\theta_B, \theta_Q)^T$. Applying Ito's Lemma to (8), we find the following Hamilton-Jacobi-Bellman (HJB) equation. For simplicity, state dependencies are dropped.

$$
\begin{aligned}
\lambda_t J = &\sup_{(c,\theta_Q,\theta_B,\theta_M,x)} \{U(c_t,t) + \lambda_t V(\upsilon_t,t) + J_t + J_W[r_t W_t + \Theta^T\Sigma\Phi - c_t - x_t + y_t] \\
&+\tfrac{1}{2}J_{WW}\Theta^T\Sigma\Sigma^T\Theta + J_r\kappa[\bar{r} - r_t] + \tfrac{1}{2}J_{rr}\sigma_r^2 + J_y y(\zeta_0 + \zeta_1 r_t) \\
&+\tfrac{1}{2}J_{yy}y^2\sigma_y^2 - J_{Wr}\Theta^T\Sigma e_1\sigma_r + J_{Wy}y\sigma y\Theta^T\Sigma\rho_{yP} + J_{ry}y\rho_{yr}\sigma_y\sigma_r\ \} \,,
\end{aligned}
\tag{24}
$$

where $e_1 = (1,0)^T$ and the terminal condition is $J(W,y,r,\omega) = L(W,\omega)$.

The first order condition with respect to consumption is the typical standard envelope condition: the incremental utility from saving cash on hand is equal to the incremental value of consuming cash on hand immediately. A similar envelope condition holds for the bequest.

$$
\begin{aligned}
U'(c_t) &= J_W(W_t, y_t, r_t, t) \\
\tfrac{\lambda_t}{\mu_t}\cdot V'(\upsilon_t) &= J_W(W_t, y_t, r_t, t).
\end{aligned}
\tag{25}
$$

The individual purchases as much insurance from the current cash on hand as necessary to equate the incremental utility increase relative to the gain of having more cash on hand available for future periods to come. It is a surprising but well known result that the new asset class life insurance is more closely related to the optimal consumption strategy than it is to alternative investment strategies such as stocks, bonds, and money market. The optimality condition suggests that the individual reacts to changes in the state variables. Solving for the optimal consumption rate and level of life insurance, we obtain the following optimal controls:

$$
\begin{aligned}
c^* &= (J_W/m(t))^{-\frac{1}{\gamma}} \\
x^* &= \mu_t\left(\tfrac{\mu_t J_W}{\lambda_t m(t)}\right)^{-\frac{1}{\gamma}} - \mu_t W_t.
\end{aligned}
\tag{26}
$$

The optimal investment strategy is given by:

$$\Theta^* = -\frac{J_W}{J_{WW}}(\Sigma(r_t,t)^T)^{-1}\Phi - \frac{J_{Wy}}{J_{WW}}y\sigma_y(\Sigma(r_t,t)^T)^{-1}\rho_{yP} + \frac{J_{Wr}}{J_{WW}}\frac{\sigma_r}{\sigma_B(r_t,t)}e_1. \quad (27)$$

The first part of the optimal investment controls corresponds to the standard mean-variance optimal portfolio. The second term of (27) refers to the hedge against variations of the income, while the last part of the optimal control stems from the hedge against interest rate changes. Given the optimality conditions, we can exactly determine the controls for a certain (W, y, r, t) quadruple in the state space if we obtain a functional expression for $J(W, y, r, t)$. The value function is homogeneous of the degree $(1-\gamma)$ in wealth. In case of the Vasicek interest rate specification, a closed form solution can be found. Whenever income is spanned and no portfolio constraints are in place, income can be replicated by financial assets. The present value of future income is simply considered as part of the investor's augmented wealth. The individual has a total augmented wealth of $W_t + H(y_t, r_t, t)$. Here, the income rate is spanned by the financial assets available if the correlations obey the following relationship:

$$\begin{gathered}\rho_{yB}^2 + \hat{\rho}_{yQ}^2 = 1 \\ \rho_{yQ} + \rho_{QB}\rho_{yr} = \pm\sqrt{(1-\rho_{yr}^2)(1-\rho_{QB}^2)}.\end{gathered} \quad (28)$$

Due to the CRRA function and the assumptions made previously, we can write the indirect utility function with labor income as:

$$J(W, y, r, t) = J(W + H(y, r, t), r, t). \quad (29)$$

$$J(W, y, r, t) = \frac{1}{1-\gamma}m(t)g(r,t)^{\gamma}(W + H(y, r, t))^{1-\gamma}. \quad (30)$$

In addition, we can propose the following guess (30) for the function $J(W, y, r, t)$. What remains to be done is to find a functional expression for $g(r,t)$ and to price the individual's income. First, we derive the function $g(r,t)$ for our guess in equation (30) in appendix (A) *Value Function.* We also price the income stream under the risk neutral measure (see appendix (B) *Human Capital* for details). The function

$g(r,t)$ can be stated in the following way:

$$
\begin{aligned}
&g(r,t) = \int_t^\omega k(s) f(s-t) B^s(r_t,t)^{\frac{\gamma-1}{\gamma}} ds + f(\omega - t) B^\omega(r_t,t)^{\frac{\gamma-1}{\gamma}},\\
&f(\hat{\tau}) = \exp\left(-\tfrac{\varphi}{\gamma}(\hat{\tau}) + \tfrac{1-\gamma}{\gamma}\bar{D}(\hat{\tau}) + \tfrac{1-\gamma}{2\gamma^2}\|\Phi\|^2\hat{\tau}\right.\\
&\left.+\tfrac{1-\gamma}{\gamma^2}\left((\bar{r} - R_\infty)(\hat{\tau} - b(\hat{\tau})) - \tfrac{\sigma_r^2}{4\kappa} b(\hat{\tau})^2\right)\right),
\end{aligned}
\tag{31}
$$

$$
k(s) = \left[\left(\tfrac{1}{\mu_s}\right)^{\frac{1-\gamma}{\gamma}} \lambda_s^{\frac{1}{\gamma}} + 1\right], \bar{D}(\hat{\tau}) = \int_0^{\hat{\tau}} \left(\mu_u - \tfrac{\lambda_u}{1-\gamma}\right) du,
$$

where $\hat{\tau}$ is the time horizon. The function $g(r,t)$ looks similar to a bond pricing formula. The first part is scaled by the time dependent coefficients $k(s)$ and $f(\hat{\tau})$, and nonlinearly transformed by $\frac{\gamma-1}{\gamma}$. The last part of the sum is multiplied by the function $f(\hat{\tau})$ only, while the zero bond B^ω is raised to the power of $\frac{\gamma-1}{\gamma}$.

$$
\begin{aligned}
&H(y,r,t) = E^Q\left[\int_t^\omega \tfrac{\bar{S}(s)}{\bar{S}(t)} y_s \exp^{-\int_t^s r_u du} ds\right]\\
&H(y,r,t) = \int_t^\omega \tfrac{\bar{S}(s)}{\bar{S}(t)} y_t h(t,s) (B^s(r,t))^{1-\zeta_1} ds\\
&\ln h(t,s) = \int_t^s \left(\zeta_0(u) - \sigma_y(u)\rho_{yP}^T\Phi - (\zeta_1 - 1)\sigma_y(u)\rho_{yB}\sigma_r b(s-u)\right) du\\
&+\zeta_1(\zeta_1 - 1)\tfrac{\sigma_r^2}{2\kappa^2}((s-t) - b(s-t) - \tfrac{\kappa}{2} b(s-t)^2),
\end{aligned}
\tag{32}
$$

where $\bar{S}$ is the survivor function including the hazard rate under the risk neutral measure. Human capital shows similarity to a defaultable bond. The bond price has been scaled by the function $h(t,s)$ and transformed by $(1-\zeta_1)$.

3.2 Optimal Demand for Life Insurance or Pension Annuity

Even though the closed form solution is found for the special case when income can be replicated by financial assets, it is instructive to analyze the solution more carefully to gain some economic insight.

Equation (33) shows that the optimal insurance demand does not only depend on all four state variables (W, y, r, t) but also on all parameters involved in the entire optimization problem. Augmented wealth and the function $g(r,t)$ determine the optimal insurance strategy to a large extent. The overall absolute insurance demand heavily depends on the weights assigned by the hazard rate μ to price the life insurance. The older the investor and the higher the hazard rate is, the larger are the cash flows related to the demand for life insurance.

$$x^* = \mu_t \left(\frac{\mu_t}{\lambda_t}\right)^{-\frac{1}{\gamma}} \frac{W_t + H(y,r,t)}{g(r,t)} - \mu_t W_t. \tag{33}$$

The value of x^* can be positive or negative. If the sign of x^* is negative, the individual has a positive demand for pension annuities. The demand for pension annuities will be positive if the investor shorts life insurance. The investor is induced to short life insurance whenever:

$$\mu_t \left(\frac{\mu_t}{\lambda_t}\right)^{-\frac{1}{\gamma}} \frac{W_t + H(y,r,t)}{g(r,t)} < \mu_t W_t. \tag{34}$$

Wealth has to be larger than the scaled value of augmented wealth consisting of human capital and financial wealth. The annuity demand becomes greater whenever either W_t rises or the value of human capital declines. In general it is not clear which effect the short rate has on the demand for life insurance because in equation (33 the nominator as well as the denominator are dependent on the short rate. For two cases, the direction is obvious. If the coefficient of relative risk aversion $\gamma > 1$ and the short rate sensitivity of the wage rate $\zeta_1 < 1$, than the demand for life insurance decreases for increasing short rates. Further, if $\gamma > 1$ and $\zeta_1 > 1$, than the demand for life insurance is increasing for rising short rates.

4 Unspanned Income

4.1 Optimal Policies

For computational convenience only, we rewrite the HJB from (24) by considering the following changes: First, we take the rate of time preference φ to the left hand side of the equation. Second we have to redefine the utility function as:

$$\begin{aligned} \bar{U}(c_t,t) &= \tfrac{c_t^{1-\gamma}}{1-\gamma} \\ \bar{V}(\upsilon,t) &= \tfrac{\upsilon_t^{1-\gamma}}{1-\gamma} \end{aligned} \tag{35}$$

Third, we replace the final condition by $J(W, y, r, \omega) = \frac{W_\omega{}^{1-\rho}}{1-\rho}$. The new HJB is in line with Purcal (2003) and similar to the one used in Milevsky and Young (2007).

Since we use $CRRA$ utility, we find that the utility function is homogeneous of degree $1-\gamma$ in $(W,\, y)$ and therefore we can rewrite the indirect utility function as:

$$J(W, r, y, t) = y^{1-\gamma} F\left(\frac{W}{y}, r, t\right) \tag{36}$$

We define η as the wealth-to-income ratio $\frac{W}{y}$ and express controls c and x normalized with income as $\hat{c}$ and $\hat{x}$. In order to give some illustrative examples for the case of unspanned income, we have to solve and optimize the PDE in (37) after replacing the indirect utility function $J(W, r, y, t)$ by $F(\eta, r, t)$.

$$\begin{aligned}
((\lambda_t + \varphi) - (1-\gamma)(\zeta_0 + \zeta_1 r) + \tfrac{1}{2}\gamma(1-\gamma)\sigma_y^2)\ F = \sup_{(\hat{c},\Pi,\hat{x})} \{&\bar{U}(\hat{c}, t) + \lambda_t \bar{V}(\hat{v}, t) \\
+F_t + F_r(\kappa[\bar{r} - r] + (1-\gamma)\rho_{yr}\sigma_y\sigma_r] + F_\eta([1 - \hat{c} - \hat{x} + r\eta + \eta\Pi^T\Sigma\Phi] \\
-\zeta_0\eta - \zeta_1 r\eta + \gamma\eta\sigma_y^2 - \gamma\eta\sigma_y\Pi^T\Sigma\rho_{yP}) + \tfrac{1}{2}\eta^2 F_{\eta\eta}[\Pi^T\Sigma\Sigma^T\Pi + \sigma_y^2(t) - 2\sigma_y\Pi^T\rho_{yP}] \\
+\tfrac{1}{2}F_{rr}\sigma_r^2 - \eta F_{\eta r}\sigma_r(\Pi^T\Sigma e_1 + \rho_{yr}\sigma_y)\ \}\,,
\end{aligned} \tag{37}$$

where Π is the vector of portfolio weights. We also stipulate that the bond and equity weights have to add up to one ($\pi_Q + \pi_B = 1$) to avoid excessive money market borrowings. We optimize and solve the HJB (37) by adopting the optimization method used in Brennan et al. (1997).[4]

4.2 Calibration

We calibrate our entire model to asset returns, income profiles, and survival probabilities found in US data. In our stylized analysis, we assume symmetric mortality beliefs $\mu_t{=}\lambda_t$ by fitting the standard hazard rate λ to the 2000 Population Basic mortality table for US females. Applying non-linear least square, we fit the Makeham-Gompertz force of mortality so that the estimated parameters turn out to be $\xi = 87.24$, $\chi = 10.54$, and $\vartheta = 0.001$. In terms of the preference function, we choose a moderately risk averse investor with a coefficient of risk aversion equal to $\gamma = 4$. Her rate of time preference is $\varphi = 0.03$.

[4]Greater detail on the numerical methods can be found in Appendix C.

						Parameters						
Utility		Interest Rate		Stocks		Income		Covariance		
								B	Q	Y
ξ	87.24	κ	0.5	ψ	0.04	$\bar{\zeta}_0$	0.025	σ_B^2	0	0
χ	10.54	$\bar{r}$	0.02	σ_Q	0.2	ζ_1	0.25	0	σ_Q^2	0
ϑ	0.001	ϕ_r	0			σ_y	0.1	0	0	σ_y^2
φ	0.03	σ_r	0.02							
γ	4									

Table 1: Parameters and Calibration. This table specifies the parameters of the utility function, the hazard rate, interest model, stock, and income process.

In order to incorporate life-cycle variations in labor income, we consider an age-dependent income growth rate. The expected growth rate $\bar{\zeta}_0$ is 2.5 percent per year as long as the investor works. However, if the level of the short rate is equal to 10 percent, then the income rate doubles to 5 percent because we set ζ_1 acting as a business cycle indicator to 0.25. When the investor enters retirement, the income growth rate $\bar{\zeta}_0$ drops to 0 percent. Along these lines, we assume that the income volatility is equal to 10 percent during worklife while the volatility becomes zero at the beginning of the retirement period.

$$\zeta_0(t) = \begin{cases} \bar{\zeta}_0(t) & \text{if } 20 \leq t \leq 65 \\ 0 & \text{if } t > 65 \end{cases} \tag{38}$$

Our benchmark parameters for the stock market are in line with Cocco, Gomes, and Maenhout (2005), Gomes and Michealides (2005) as well as Munk and Sørensen (2007) and many others. The risk premium of $\psi = 4$ percent is frequently quoted in the literature and reflects the forward looking equity premium over the risk free rate. In our model the risk free rate is equal to the short rate model given by the Vasicek term structure model. The standard deviation for the stock price is set at 20 percent. This falls into the range of historical volatilities found for major US stock market indexes such as the S&P500.

The parameters for the short rate process are taken from Munk and Sørensen (2007) as there is no sufficient data on real bonds available. The speed κ to the long term mean is exactly 50 percent. The standard deviation of the short rate process and the long term short rate are set to 2 percent respectively. We assume a market price of risk $\phi_r = 0$. For our numerical analysis, we also suppose that the investor can only access a 10 year government bond paying real interest.[5]

We set the correlations of the innovations between the financial and non-financial growth rates ρ_{By}=$\rho_{Qy} = 0$ to zero in order to avoid covariance effects in our analysis. The same is true for the correlation between the innovations of stock and bond returns $\rho_{QB} = 0$.

4.3 Numerical Illustration of Insurance Demand

In this section, we illustrate the demand for life insurance and pension annuities over the life cycle. Graph (A) of figure (1) depicts the long and short positions in life insurance as fractions of η. Not surprisingly do life insurances play an important role for small levels of cash on hand. The higher the level of cash on hand, the more is the investor induced to switch to short positions because she has sufficient financial wealth levels in order to bequeath her heirs. The overall insurance demand changes considerably over the life cycle. The total fraction used for life insurance purchases and pension annuities early in life is still limited. This is because premiums of life insurances are extremely cheap as we have seen in the section on insurance markets. On the flip side, short positions are very expensive because they limit the bequest potential which is not sufficiently backed by financial savings. When the investor becomes older, she purchases substantially more pension annuities by shortening life insurance. At the late stage of her life cycle, the investor buys more life contingent products with a higher fraction of η than before because she is more likely to die the next period. Graph (B) of figure (1) shows the insurance demand for an 80 year old female investor along the dimensions normalized cash on hand and short rate level. We find only a marginal influence of the short rate level on the overall demand of life insurance under the current parameterizations of our asset and income model. Probably one reason can be found in the way we model the business cycle effects on income. In our model, the short rate also influences the growth rate of our income process during the retirement period. Here we have offsetting effects because the short rate enters human capital as well as the value function. The numerical analysis

[5] This is a common assumption in the literature. Confer to Munk and Sørensen (2007) for more details.

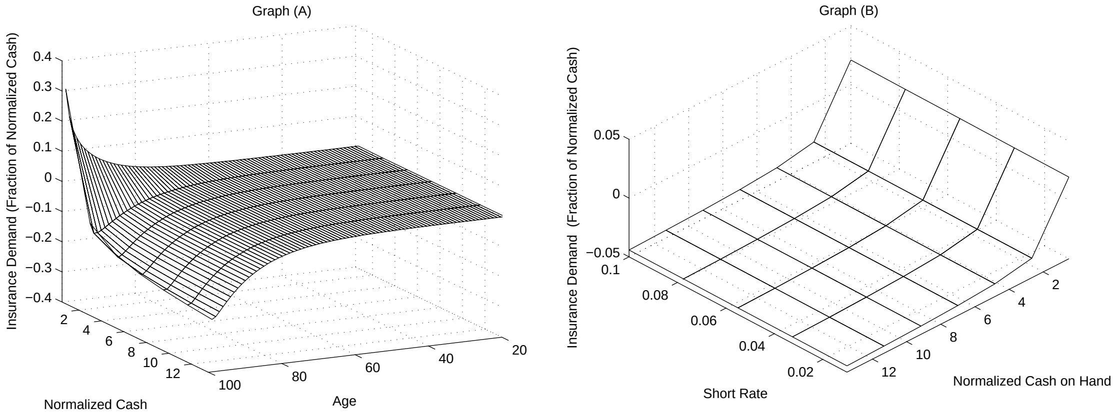

Figure 1: Demand for Pension Annuities and Life Insurance. Graph (A) and (B) assume the parameters from table (1) and uses the optimization given in (37). Graph (A) displays the insurance demand as a fraction of η over the life cycle. The short rate is assumed to be 3.25 percent. Graph (B) shows the insurance demand as a fraction of η for a female investor of age 80.
Source: Author's computations

deviates from the analytical analysis because we assume zero correlation between the innovations of financial and non-financial growth rates.

5 Conclusion

In this paper, we solve a life cycle model with life insurance and pension annuities analytically for the complete market case. Our model assumes a stochastic wage earner with CRRA preferences whose lifetime is random. The investor has to decide among short and long positions in life insurance, stocks, bonds, and money market investment when facing a risky stock market and interest rate risk. We also derive some numerical insight into a realistically calibrated case when income is unspanned.

The optimal life annuity demand depends on all state variables (wealth, income, short rate, and age) as well as all parameters under consideration. The insurance demand is particularly dependent on age. The older the individual, the higher the hazard rate, the greater the absolute demand for life insurance products. The lower the investor's human capital and the higher her financial wealth is, the more likely the investor shifts from life insurance into pension annuities. We find a substantial impact of normalized cash on hand with respect to the insurance rules but we discover a considerably small influence of the short rate on the demand of life insurance if we reasonably calibrate our asset and income model.

6 Appendix (A): Value Function

Using the fact that the utility function is homogenous to the degree $(1-\gamma)$ in augmented wealth, the PDE can be simplified to:

$$\begin{aligned} 0 = k(t) - \left[\left(-r_t - \tfrac{\|\Phi\|^2}{2\gamma}\right)\tfrac{1-\gamma}{\gamma} + \tfrac{1-\gamma}{\gamma}\left(\tfrac{\lambda_t}{1-\gamma} - \mu_t\right) + \tfrac{\varphi}{\gamma}\right] g(r_t, t) \\ + g_t + \left[\kappa[\bar{r} - r_t] - \tfrac{1-\gamma}{\gamma}\phi_r\sigma_r\right] g_r(r_t, t) + \tfrac{1}{2} g_{rr}(r_t, t)\sigma_r^2, \end{aligned} \tag{39}$$

where k(t)is represented by:

$$k(t) = \left[\left(\frac{1}{\mu_t}\right)^{\frac{1-\gamma}{\gamma}} \lambda_t^{\frac{1}{\gamma}} + 1\right]. \tag{40}$$

First, we define the following integral:

$$\bar{D}(\hat{\tau}) = \int_0^{\hat{\tau}} \left(\mu_u - \frac{\lambda_u}{1-\gamma}\right) du, \tag{41}$$

where $\hat{\tau}$ is the time horizon. Then, we use as an educated guess of the following function:

$$\begin{aligned} & g(r,t) = \\ & \textstyle\int_t^{\omega} k(s) \exp\left[-\tfrac{\varphi}{\gamma}(s-t) + \tfrac{1-\gamma}{\gamma}\bar{D}(s-t) + \tfrac{1-\gamma}{\gamma}A_1(s-t) + \tfrac{1-\gamma}{\gamma}A_2(s-t)r_t\right] ds \\ & + \exp\left[-\tfrac{\varphi}{\gamma}(\omega-t) + \tfrac{1-\gamma}{\gamma}\bar{D}(\omega-t) + \tfrac{1-\gamma}{\gamma}A_1(\omega-t) + \tfrac{1-\gamma}{\gamma}A_2(\omega-t)r_t\right], \\ & j(r_t, \hat{\tau}) = \left[-\tfrac{\varphi}{\gamma}(\hat{\tau}) + \tfrac{1-\gamma}{\gamma}\bar{D}(\hat{\tau}) + \tfrac{1-\gamma}{\gamma}A_1(\hat{\tau}) + \tfrac{1-\gamma}{\gamma}A_2(\hat{\tau})r_t\right]. \end{aligned} \tag{42}$$

Now, we have to compute the partial derivatives with respect to r, rr, and t:

$$\begin{aligned} & g_r(r_t, t) = \tfrac{1-\gamma}{\gamma}\textstyle\int_t^{\omega} k(s)(A_2(s-t)e^{j(r_t, s-t)} ds + \tfrac{1-\gamma}{\gamma}A_2(\omega-t)e^{j(r_t,\omega-t)} \\ & g_{rr}(r_t, t) = \left(\tfrac{1-\gamma}{\gamma}\right)^2 \textstyle\int_t^{\omega} k(s)A_2^2(s-t)e^{j(r_t,s-t)} ds + \left(\tfrac{1-\gamma}{\gamma}\right)^2 A_2^2(\omega-t)e^{j(r_t,\omega-t)} \\ & \tfrac{\partial g}{\partial t}(r_t, t) = \textstyle\int_t^{\omega} k(s)e^{j(r_t,s-t)}\left[\tfrac{\varphi}{\gamma} + \tfrac{1-\gamma}{\gamma}(-A_1' - A_2' r_t) + \tfrac{(1-\gamma)}{\gamma}\left(\tfrac{\lambda_t}{1-\gamma} - \mu_t\right)\right] ds - k(t) \\ & + e^{j(r_t,\omega-t)}\left[\tfrac{\varphi}{\gamma} + \tfrac{1-\gamma}{\gamma}(-A_1' - A_2' r_t) + \tfrac{(1-\gamma)}{\gamma}\left(\tfrac{\lambda_t}{1-\gamma} - \mu_t\right)\right]. \end{aligned} \tag{43}$$

Plugging the derivatives into equation (39), we arrive at a new PDE. The new PDE is consistent if A_1, A_2 and the corresponding derivatives fullfil the following PDE themselves:

$$0 = r_t + \tfrac{1}{2\gamma}\|\Phi\|^2 - A_1' - A_2' r_t + \left(\kappa(\bar{r} - r_t) - \tfrac{1-\gamma}{\gamma}\phi_r\sigma_r\right) A_2 + \tfrac{1-\gamma}{2\gamma}\sigma_r^2 A_2^2. \tag{44}$$

The PDE (44) has to hold for all combinations of r and t. We can obtain a system of two ordinary equations with the initial conditions $A_1(0) = A_2(0) = 0$ (cf. to Munk, 2005). The first ordinary differential equation is given by:

$$A_2' = 1 - \kappa A_2(\hat{\tau}). \tag{45}$$

Considering the initial condition $A_2(0) = 0$, we can rewrite A_2:

$$A_2 = \tfrac{1}{\kappa}\left(1 - e^{-\kappa(\hat{\tau})}\right) = b(\hat{\tau}). \tag{46}$$

Including the initial condition $A_1(0) = 0$, the second ordinary differential equation can be immediately put as:

$$A_1 = \tfrac{1}{2\gamma}\|\Phi\|^2(\hat{\tau}) + (\kappa\bar{r} - \tfrac{1-\gamma}{\gamma}\sigma_r\phi_r)\int_0^{\hat{\tau}} b(s)ds + \tfrac{1-\gamma}{2\gamma}\sigma_r^2\int_0^{\hat{\tau}} b(s)^2 ds. \tag{47}$$

The remaining integrals can be solved analytically:

$$\begin{aligned} A_1 &= \tfrac{1}{2\gamma}\|\Phi\|^2(\hat{\tau}) + (\bar{r} + \tfrac{1-\gamma}{2\kappa^2\gamma}[\sigma_r^2 - 2\kappa\sigma_r\phi_r])((\hat{\tau}) - b(\hat{\tau})) \\ &-\tfrac{1-\gamma}{4\kappa\gamma}\sigma_r^2 b(\hat{\tau})^2. \end{aligned} \tag{48}$$

Plugging A_1 and A_2 into (44) and rearranging terms, we find the following solution for the function $g(r,t)$:

$$\begin{aligned} g(r,t) &= \int_t^{\omega} k(s) f(s-t) B^s(r_t,t)^{\frac{\gamma-1}{\gamma}} ds + f(\omega - t) B^{\omega}(r_t,t)^{\frac{\gamma-1}{\gamma}} \\ f(\hat{\tau}) &= \exp\Big(-\tfrac{\varphi}{\gamma}(\hat{\tau}) + \tfrac{1-\gamma}{\gamma}\bar{D}(\hat{\tau}) + \tfrac{1-\gamma}{2\gamma^2}\|\Phi\|^2\hat{\tau} \\ &+ \tfrac{1-\gamma}{\gamma^2}\left((\bar{r} - R_\infty)(\hat{\tau} - b(\hat{\tau})) - \tfrac{\sigma_r^2}{4\kappa} b(\hat{\tau})^2\right)\Big) \\ k(s) &= \left[\left(\tfrac{1}{\mu_s}\right)^{\frac{1-\gamma}{\gamma}} \lambda_s^{\frac{1}{\gamma}} + 1\right]. \end{aligned} \tag{49}$$

.

7 Appendix (B): Human Capital

Human capital can be computed as the present value of the income stream under the Q-measure

$$H(y,r,t) = E^Q \left[\int_t^\omega \frac{\bar{S}(s)}{\bar{S}(t)} y_s e^{-\int_t^s r_u du} ds \right], \tag{50}$$

where the survivor function $\bar{S}$ includes the hazard rate under the risk neutral measure. We consider the dynamics of the function $e^{\kappa t} r_t$ under the Q measure:

$$de^{\kappa t} r_t = e^{\kappa t}(\kappa r_t dt + [\kappa(\bar{r} - r_t) + \sigma_r \phi_r]dt) - \sigma_r e^{\kappa t} d\hat{z}_{rt}. \tag{51}$$

Integrating and transforming, we find that:

$$r_u = e^{-\kappa[u-t]} r_t + \frac{\kappa\bar{r} + \sigma_r\phi_r}{\kappa}\left(1 - e^{-\kappa[u-t]}\right) - \int_t^u \sigma_r e^{-\kappa[u-v]} d\hat{z}_{rv}. \tag{52}$$

Integrating once more, we get:

$$\int_t^s r_u du = \left(r_t - \frac{\kappa\bar{r} + \sigma_r\phi_r}{\kappa}\right) b(s-t) + \frac{\kappa\bar{r} + \sigma_r\phi_r}{\kappa}(s-t) - \int_t^s \sigma_r b(s-u) d\hat{z}_{ru}. \tag{53}$$

Labor dynamics under the Q measure are given by:

$$dy_t = y_t[(\zeta_0(t) - \sigma_y(t)\rho_{yP}^T \Phi + \zeta_1 r_t)dt + \sigma_y(\rho_{yB} d\hat{z}_{rt} + \hat{\rho}_{yQ} d\hat{z}_{Qt})]. \tag{54}$$

Integrating, we find the following expression:

$$\begin{aligned} y_s = y_t \exp[&\textstyle\int_t^s \left(\zeta_0(u) - \sigma_y(u)\rho_{yP}^T \Phi + \zeta_1 r_u - \frac{1}{2}\sigma_y(u)^2\right) du \\ &+ \textstyle\int_t^s \sigma_y(u)\rho_{yB} d\hat{z}_{ru} + \int_t^s \sigma_y(u)\hat{\rho}_{yQ} d\hat{z}_{Qu}]. \end{aligned} \tag{55}$$

Combining the previous equations (53) and (55):

$$\begin{aligned} y_s \exp^{-\int_t^s r_u du} = y_t \exp(&\textstyle\int_t^s \left(\zeta_0(u) - \sigma_y(u)\rho_{yP}^T \Phi - \frac{1}{2}\sigma_y(u)^2\right) du \\ &+ (\zeta_1 - 1)(r_t - \frac{\kappa\bar{r} + \sigma_r\phi_r}{\kappa}) b(s-t) + \frac{\kappa\bar{r} + \sigma_r\phi_r}{\kappa}(\zeta_1 - 1)(s-t) \\ &+ \textstyle\int_t^s (\sigma_y(u)\rho_{yB} - (\zeta_1 - 1)\sigma_r b(s-u)) d\hat{z}_{ru} + \int_t^s \sigma_y(u)\hat{\rho}_{yQ} d\hat{z}_{Qu}. \end{aligned} \tag{56}$$

Taking the expected value and rearranging, we get:

$$\begin{aligned}
&H(y,r,t) = E^Q\left[\int_t^\omega \tfrac{\bar{S}(s)}{\bar{S}(t)} y_s e^{-\int_t^s r_u du} ds\right]\\
&H(y,r,t) = \int_t^\omega \tfrac{\bar{S}(s)}{\bar{S}(t)} y_t h(t,s)(B^s(r,t))^{1-\zeta_1} ds\\
&\ln h(t,s) = \int_t^s \left(\zeta_0(u) - \sigma_y(u)\rho_{yP}^T\Phi - (\zeta_1 - 1)\sigma_y(u)\rho_{yB}\sigma_r b(s-u)\right) du\\
&+\zeta_1(\zeta_1 - 1)\tfrac{\sigma_r^2}{2\kappa^2}((s-t) - b(s-t) - \tfrac{\kappa}{2} b(s-t)^2).
\end{aligned} \tag{57}$$

.

8 Appendix (C): Numerical Methods

We solve the nonlinear PDE (37) by applying an explicit finite difference scheme for equally spaced arrays defining the (η, r, t)-state-space. The grid points are given as: $(\eta_i,r_j,t_n \mid i = 1,...,I;\ j = 1,...,I;\ n = 1,...,N)$. For each interval, we use some fixed positive spacing values $\Delta\eta$, $\Delta\ r$, and $\Delta\ t$ determining how coarse the grid is in each dimension. We denote the approximated value function by $F_{i,j,n}$ at the grid point (η_i, r_j, t_n). We approximate the derivatives by finite differences. In

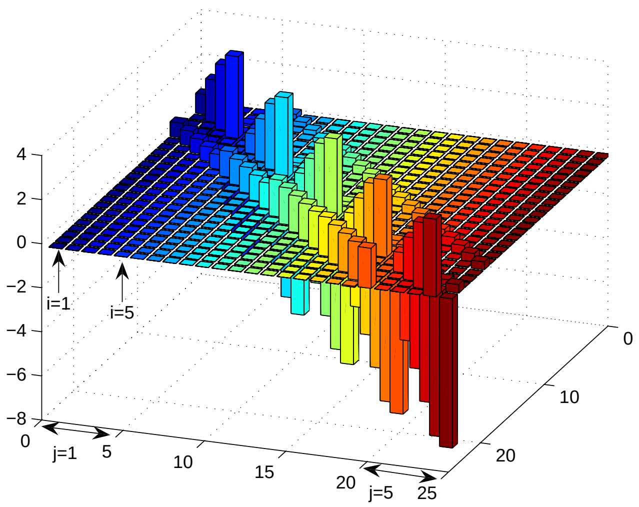

Figure 2: Illustrative Coefficient Array. This illustration assumes an 5 by 5 discretization of the (η, r)-space at time $N-1$. In the remainder, we will refer to this array as matrix M_{N-1}
Source: Author's computations

our computational efforts, we use an upwind-scheme to stabilize the finite difference approach. Derivatives of the first order and mixed derivatives change according to the sign of their coefficients (+/−) in the state-space we span. Indeed this is necessary because of the complexity we have to deal with when computing the value function. The definitions in (58) show how we approximate the various derivatives involved in solving the PDE in (37). For each combination (i,j) we fill out an array we denote as matrix M_n, where n stands for the point in time. In total we obtain a matrix M_n which is of size (IxJ,IxJ). The matrix M has several borders where we assume the state space to start or to end respectively. At each border, we adjust the coefficient matrix by linearly extrapolating the function (37) to obtain the transition probabilities of exceeding the borderlines marked by the inner arrays of matrix M_n.

$$\begin{aligned}
&F_t = \tfrac{F_{i,j,n+1}-F_{i,j,n}}{\Delta t} \\
&F_{\eta\eta} = \tfrac{F_{i+1,j,n}-2F_{i,j,n}+F_{i-1,j,n}}{(\Delta\eta)^2} \quad F_{rr} = \tfrac{F_{i,j+1,n}-2F_{i,j,n}+F_{i,j-1,n}}{(\Delta r)^2} \\
&F^+_{\eta} = \tfrac{F_{i+1,j,n}-F_{i,j,n}}{(\Delta\eta)} \quad F^-_{\eta} = \tfrac{F_{i,j,n}-F_{i-1,j,n}}{(\Delta\eta)} \\
&F^+_{r} = \tfrac{F_{i,j+1,n}-F_{i,j,n}}{(\Delta r)} \quad F^-_{r} = \tfrac{F_{i,j,n}-F_{i,j-1,n}}{(\Delta r)} \\
&F^+_{\eta r} = \frac{1}{2}\frac{\frac{F_{i+1,j+1,n}-F_{i,j+1,n}}{(\Delta\eta)}-\frac{F_{i+1,j,n}-F_{i,j,n}}{(\Delta\eta)}}{(\Delta r)} + \frac{1}{2}\frac{\frac{F_{i,j,n}-F_{i-1,j,n}}{(\Delta\eta)}-\frac{F_{i,j-1,n}-F_{i-1,j-1,n}}{(\Delta\eta)}}{(\Delta r)} \\
&F^-_{\eta r} = \frac{1}{2}\frac{\frac{F_{i,j+1,n}-F_{i-1,j+1,n}}{(\Delta\eta)}-\frac{F_{i,j,n}-F_{i-1,j,n}}{(\Delta\eta)}}{(\Delta r)} + \frac{1}{2}\frac{\frac{F_{i+1,j,n}-F_{i,j,n}}{(\Delta\eta)}-\frac{F_{i+1,j-1,n}-F_{i,j-1,n}}{(\Delta\eta)}}{(\Delta r)}
\end{aligned} \tag{58}$$

Figure (2) depicts the matrix M of size (5x5,5x5) at time $N-1$. As one can infer from the graph, the matrix is only of limited bandwidth and can be easily inverted since we want to solve the set of equations given by $M_nF_n = d_n$ at time n, where the vector d_n of size (IxJ,1) includes constants and future values of F_{n+1}. For the optimization procedure, we use the approach described by Brennan et al. (1997) which is similar to the controlled markov chain methods discussed in Kushner and Dupuis (2001). In order to optimize (37), we first plug an ad-hoc guess of the optimal controls into the PDE we want to solve for. After obtaining the value function by means of the finite difference method, we compute the first order conditions to find the optimal controls given the previous value function at time point n. After obtaining the first estimate of the optimal controls, we take the new values of the optimal controls and plough them back into the value function at same point in time n as before. Then we solve again for the value function at time n. This iterated process is continued until the value function has converged to its optimum. Usually, we only need two iterative steps to find both the optimal controls and the optimized value function. Then we can move forward to the next point in time $n-1$ and do the whole procedure over again. For our problem under consideration, we used a state space (η, r, t) of (15,5,81) on a equidistant grid. The lower value of η is set to 0.5 and the upper value to 20. The interval for the short rate lies between 0.01 and 0.1.

References

Brennan, M., E. Schwartz, and R. Lagnado, 1997, "Strategic Asset Allocation," *Journal of Economic Dynamics and Control*, 21, 1377–1403.

Brown, J., O. Mitchell, J. Poterba, and M. Warshawsky, 2001, *The Role of Annuity Markets in Financing Retirement*, MIT Press, Cambridge, M.A.

Brugiavini, A., 1993, "Uncertainty Resolution and the Timing of Annuity Purchases," *Journal of Public Economics*, 50 (1), 31–62.

Campbell, J., and L. Viceira, 2001, "Who Should Buy Long-Term Bonds," *American Economic Review*, 91, 99–127.

Campbell, R., 1980, "The Demand for Life Insurance: An application of the Economics of Uncertainty," *Journal of Finance*, 35 (5), 1155–1172.

Cocco, J., 2005, "Portfolio Choice in the Presence of Housing," *The Review of Financial Studies*, 112, 61–156.

Cocco, J., F. Gomes, and P. Maenhout, 2005, "Consumption and Portfolio Choice over the Life Cycle," *The Review of Financial Studies*, 18, 491–533.

Deelstra, G., M. Grasselli, and P.-F. Koehl, 2000, "Optimal Investment Strategies in a CIR Framework," *Journal of Applied Probability*, 37, 1–12.

Duffie, D., W. Flemming, H. Soner, and T. Zariphopoulou, 1997, "Hedging in Incomplete Markets with HARA Utility," *Journal of Economic Dynamics and Control*, 21 (4-5), 753–782.

Dybvig, P., and H. Liu, 2004, "Lifetime Consumption and Investment: Retirement and Constrained Borrowing," Discussion paper, Washington University.

Fischer, S., 1973, "A Life Cycle Model of Life Insurance Purchases," *International Economic Review*, 14 (1), 132–152.

Gomes, F., and A. Michaelides, 2005, "Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence," *Journal of Finance*, 60, 869–904.

Hakansson, N. H., 1969, "Consumption over the Life Cycle under Risk, an Uncertain Lifetime, and Insurance," *International Economic Review*, 10 (3), 443–466.

Heaton, J., and D. Lucas, 1997, "Market Frictions, Savings and Portfolio Choice," *Macroeconomic Dynamics*, 1, 76–101.

Horneff, W., R. Maurer, and M. Stamos, 2008a, "Life-Cycle Asset Allocation with Annuity Markets," *Journal of Economic Dynamics and Control*, accepted, 2008.

Horneff, W., R. Maurer, and M. Stamos, 2008b, "Optimal Gradual Annuitization: Quantifying the Costs of Switching to Annuities," *The Journal of Risk and Insurance*, in press, 2008.

Huang, H., and M. Milevsky, 2008, "Portfolio Choice and mortality contingent claims: The general HARA case," *Journal of Banking and Finance*, forthcoming.

Huang, H., M. Milevsky, and J. Wang, 2005, "Portfolio Choice and Life Insurance," Discussion paper, Ifid Centre, York University.

Hurd, M., 1989, "Mortality Risk and Bequests," *Econometrica*, 57, 779–813.

Koo, H., 1998, "Consumption and Portfolio Selection with Labor Income: A Continuous Time Approach," *Mathematical Finance*, 8 (1), 49–65.

Kraft, H., and M. Steffensen, 2008, "Optimal consumption and insurance: a continuous-time Markov chain approach," *ASTIN Bulletin*, forthcoming.

Kushner, H., and P. Dupuis, 2001, *Numerical Methods for Stochastic Control Problems in Continuous Time*, Springer-Verlag, New York.

Lewis, F. S., 1989, "Dependents and the Demand for Life Insurance," *American Economic Review*, 79 (3), 452–467.

Liu, J., 2007, "Portfolio Selection in Stochastic Environments," *The Review of Financial Studies*, 20 (1), 1–39.

Milevsky, M., and V. Young, 2007, "Annuitization and Asset Allocation," *Journal of Economic Dynamics and Control*, 31, 3138–3177.

Munk, C., 2005, “Dynamic Asset Allocation,” Discussion paper, Lecture Notes, University of Southern Denmark.

Munk, C., and C. Sørensen, 2007, “Dynamic Asset Allocation with Stochastic Income and Interest Rates,” Discussion paper, University of Southern Denmark.

Pliska, S. R., and J. Ye, 2007, “Optimal Life Insurance Purchase and Consumption/Investment under Uncertain Lifetime,” *Journal of Banking and Finance*, 31, 1307–1319.

Purcal, S., 2003, “A Stochastic Control Model for the Individual Asset Liability Management,” Discussion paper, University of New South Wales.

Richard, S., 1975, “Optimal Consumption, Portfolio and Life Insurance Rules for an Uncertain Lived Individual in a Continuous Time Model,” *Journal of Financial Economics*, 2, 187–203.

Sangvinatsos, A., and J. Wachter, 2005, “Does the Failure of the Expectation Hypothesis Matter for Long-Term Investor?,” *Journal of Finance*, 60(1), 179–230.

Vasicek, O., 1977, “An Equilibrium Characterization of the Term Structure,” *Journal of Financial Economics*, 5, 177–188.

Viceira, L., 2001, “Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income.,” *Journal of Finance*, 56 (2), 433–470.

Yaari, M., 1965, “Uncertain Lifetime, Life Insurance, and the Theory of the Consumer,” *Review of Economic Studies*, 32, 137–150.

Ye, J., 2006, “Optimal Life Insurance Purchase, Consumption and Portfolio under An Uncertain Life,” Discussion paper, PhD Thesis, University of Chicago.