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ANALYSIS OF THE ROLLING-BALL VISCOMETER

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I. INTRODUCTION

Analysis of the motion in the rolling-ball viscometer provides an additional method of determining absolute viscosity. In practice, the instrument is used comparatively in that a set of balls with various clearances is calibrated using standard liquids of known viscosity. Because the diameters of the spheres are only slightly smaller than that of the cylinder, the primary resistance to flow is concentrated in the vicinity of the region of smallest clearance. Consequently, the forces and the flow can be sufficiently simplified to make possible a theoretical analysis similar to that used for the theory of lubrication. For small clearances, the viscosity can be directly determined if sufficient precision is maintained in the manufacture of the cylinder and the balls and in the determinations of their diameters.

For the slow motions specified for these viscometers, the flow through the space between the ball and the cylinder is laminar. The liquid moves primarily in the upward direction, the net upward flow compensating for the downward motion of the ball. The tube is inclined from the vertical. Consequently, the ball rolls or simultaneously slides and rolls along the lowermost element of the cylinder, and the opening at the minimum section is crescent shaped. The gradually-varied flow is assumed to take place in meridional planes. Because the changes in piezometric head between points below and above the ball are the same regardless of the path followed by the fluid, the rate of flow per unit of circumference varies from zero at the point of contact to a maximum at the opposite side. The longitudinal force resulting from the difference in piezometric head counterbalances the component of immersed weight. Other effects such as shear can be shown to be negligible if the clearance is small.

The simpler case of a sphere falling along the axis of a vertical cylinder has already been treated [1].* Other aspects of the performance of these viscometers have been discussed extensively, e.g., [2]. Comparisons of the results from the analysis of the unsymmetrical case with calibration data for commercial instruments indicate the reliability of the analysis for comparatively small clearances.

†At Iowa Institute of Hydraulic Research when investigation was performed.
*Numbers in brackets refer to the bibliography at the end of the paper.

II. ANALYSIS

Basic to the analysis of the flow is the relationship for the gradient of piezometric head. In addition, relationships for continuity, geometry, and the aforementioned balance of longitudinal forces are required. Because of the involved geometry of the flow passage, definition of the variables is complicated. With reference to Fig. 1, x is the spacing between the ball and the cylinder and y the distance along the axis of the cylinder. At the equator (i.e., the section of minimum clearance), y is zero and the variable spacing is denoted as x_0. The angle θ designates position around the sphere. The fluid velocity v varies with y and θ and with position across the flow passage. The maximum ordinate of the parabolic curve of velocity distribution for any value of y (and θ) is designated as v_m and that for y = 0 as V_m. The velocity of the center of the sphere is V_s.

The familiar expression for flow between parallel plates [3] can be written in the form

$$\frac{dh}{dy} = -\frac{8\mu v_m}{\gamma x^2}, \tag{1}$$

in which $[h = (p/\gamma) + z]$ denotes piezometric head, and μ and γ are the dynamic viscosity and unit weight of the fluid, respectively. Both v_m and x are considered in this instance to be variable with y but only gradually so. From Fig. 1, to the second order of small quantities,

$$x = x_0 + \frac{y^2}{d}, \tag{2}$$

and

$$x_0 = \frac{(D - d)}{2}(1 + \cos\theta). \tag{3}$$

If the quantity of flow between two closely-spaced meridians is a constant,

$$v_m = \frac{x_0}{x} V_m .$$

If these values are substituted into (1) and the expression is integrated, a relationship is obtained between Δh, the change in piezometric head across the sphere, and V_m:

$$\Delta h = \frac{16\mu}{\gamma}\int_0^\infty \frac{v_m dy}{x^2} = \frac{3\pi\mu d^{1/2} V_m}{\gamma x_0^{3/2}} . \tag{4}$$

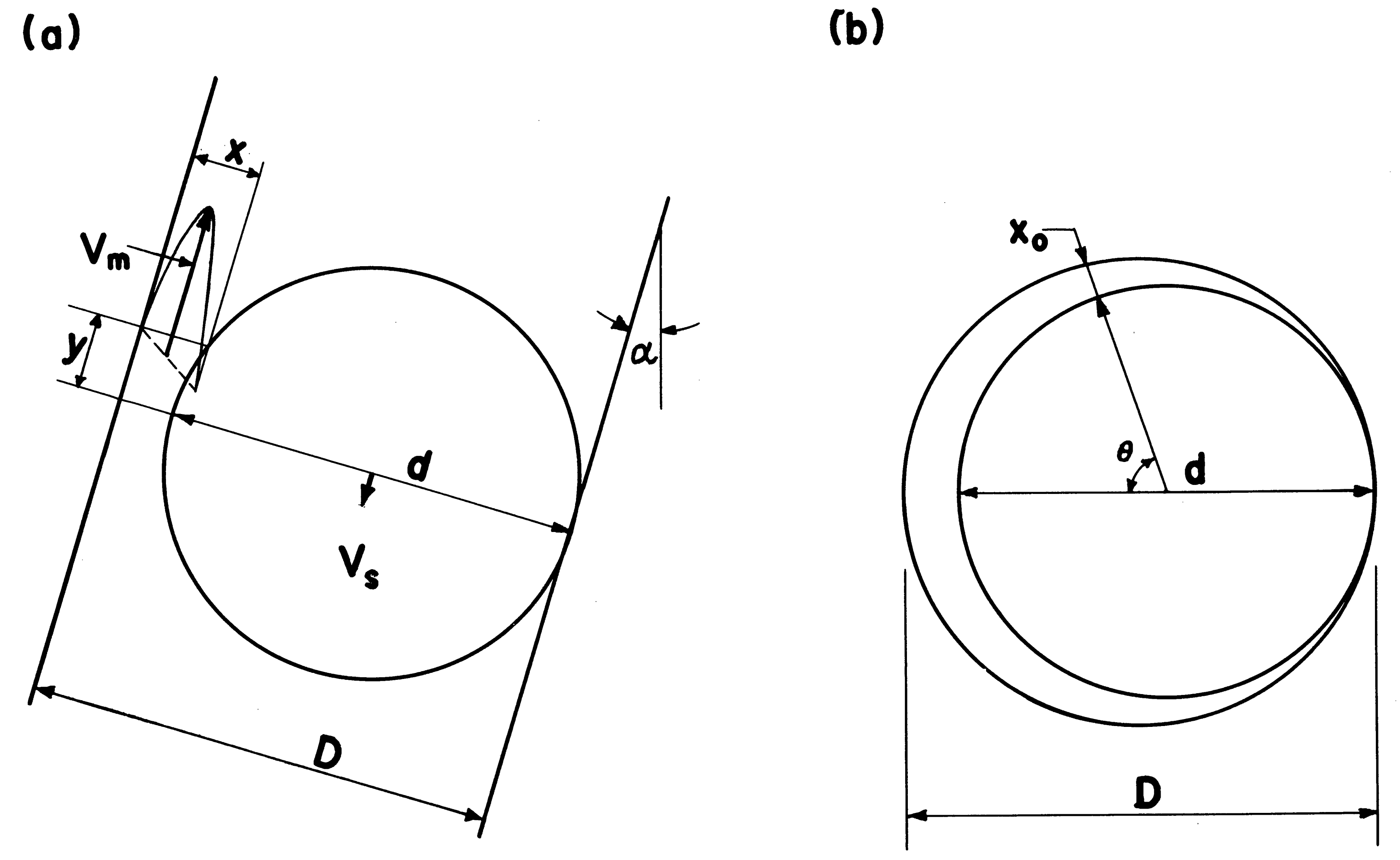

Fig. I Definition Sketch

The upper limit of ∞ is used for convenience. Any moderately large upper limit would produce the same result for the integration of this rapidly converging function.

The appropriate equation of continuity relates the net upward flow of the fluid (at the equator) to the downward motion of the ball:

$$\frac{\pi d^2 V_s}{4} = \frac{2d}{3} \int_0^{\pi} V_m x_0 d\theta \ .$$

The integration if performed after V_m has been expressed in terms of x_0 from (4) and x_0 in terms of θ from (3). Thus,

$$V_s = \frac{4A\gamma\Delta h d \epsilon^{5/2}}{\pi\mu} , \qquad (5)$$

in which ϵ is the average relative spacing,

$$\epsilon = \frac{D - d}{2D} ,$$

and A is a constant,

$$A = \frac{128\sqrt{2}}{135\pi} = 0.427 \ .$$

If the concept of equilibrium of forces is to be utilized, an expression for the integrated shear force S must be examined:

$$S = 2\int_0^{\infty}\int_0^{\pi} (\tau d) \, d\theta dy ,$$

in which the local shear stress τ can be closely approximated by means of the relationship,

$$\tau = \frac{4V_m}{x} \ .$$

Hence,

$$S = \frac{8\gamma\Delta h d^2 \ \epsilon^{5/2}}{3\pi\mu} \int_0^{\pi} \int_0^{\infty} \frac{(1 + \cos\theta)^{5/2}}{(x_0 + \frac{y^2}{d})^2} \, dy d\theta = \gamma\Delta h \, \frac{\pi d^2}{4} \, \frac{8\epsilon}{3} \ .$$

Finally, from the equilibrium of forces in the longitudinal direction,

$$\gamma\Delta h \, \frac{\pi d^2}{4} \, (1 + \frac{8}{3}\epsilon) = \frac{\pi d^3}{6} \Delta\gamma \cos\alpha \ . \qquad (6)$$

The order of magnitude of the force due to shear is less than the primary terms. This and several other second-order effects are omitted in the following relationships. These are considered briefly at the end of this section.

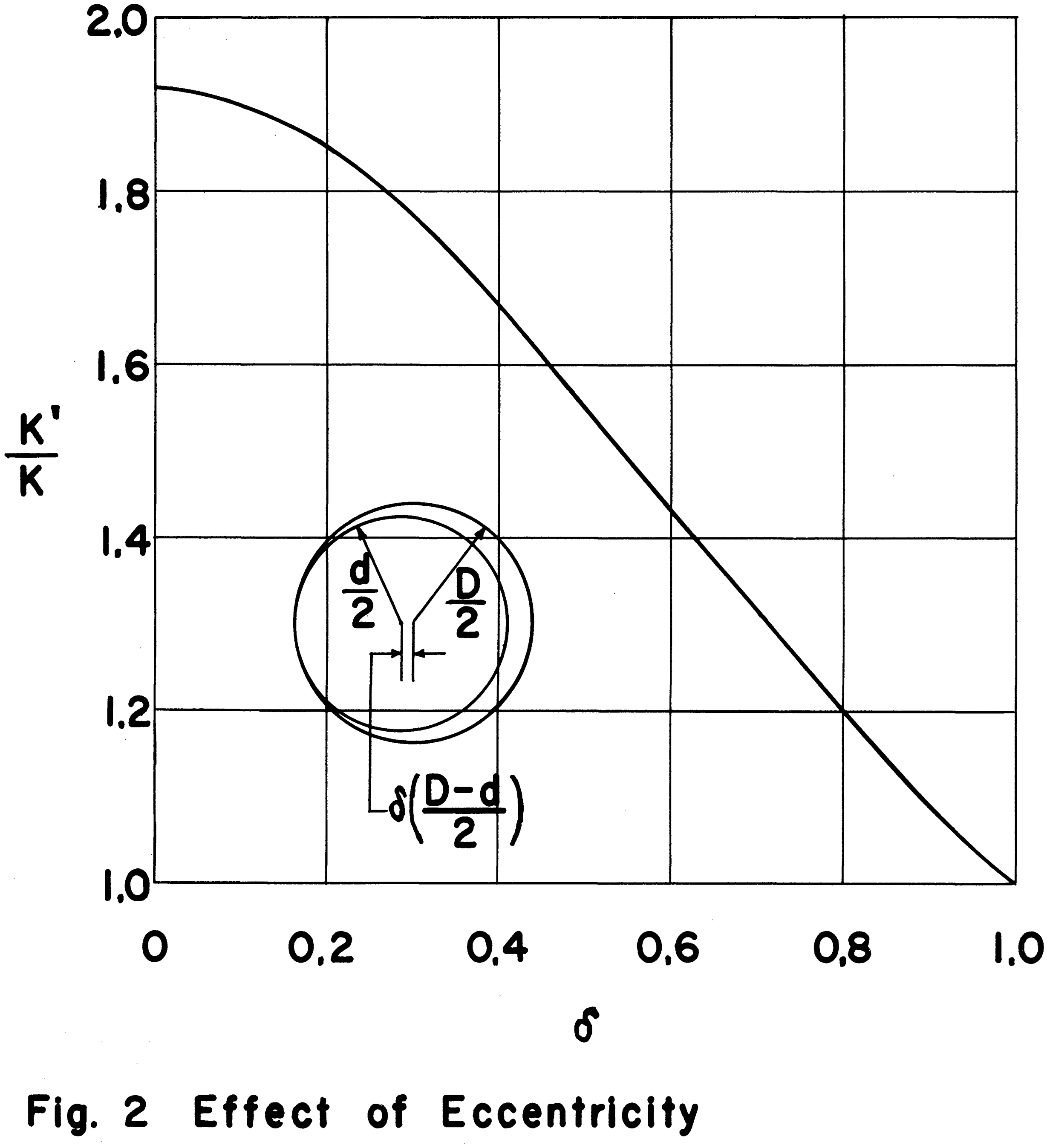

Fig. 2 Effect of Eccentricity

The last step in the analysis is the evaluation of a resistance coefficient K as a function of ϵ. The quantity K is a coefficient to be applied to the Stokes law:

$$K = \frac{F}{3\pi\mu V_s\, d} = \frac{d^2\, \Delta\gamma \cos\alpha}{18\mu V_s} . \qquad (7)$$

From (5), (6), and (7),

$$K = 0.153\, \epsilon^{-5/2} = 0.868 \left(\frac{D - d}{D}\right)^{-5/2} . \qquad (8)$$

Several contributions of second order were considered in addition to the term for shear. For these, V_s is not negligible with respect to V_m, and d is not approximately equal to D, but rather,

$$d = D\,(1 - 2\epsilon) .$$

Consequently, the continuity equation must be modified. Also, a reduction must be included for variation of pressure on the curved surfaces, a contact force must be considered, and the question of sliding versus rolling must be resolved. Thus (4), (5), and (6) are modified by the inclusion of additional terms. The combined effect for a rolling contact is a reduction in the value of K by a factor of $1 - 3.3\epsilon$. However, the motion is too complex for one to be certain that all second order effects have been included.

If the ball were to move in such a way as to be near the wall but not in contact with it, a different type of second-order effect would occur. The spatial relationship in (3) for x_0 would have to be modified:

$$x_0 = \frac{(D - d)}{2}\,(1 + \delta \cos\theta) ,$$

in which δ varies from unity for the case already treated to zero for the symmetrical one. Use of this modified expression in the continuity relationship leads to a more complex form of (5) involving elliptic integrals. The results of the integration are shown in Fig. 2 in which the modified coefficient K' is compared to the value obtained from (8).

Although the gradually varied Poiseuille flow is considered to be locally independent of the geometrical variations, the integrated result is directly affected by the complex geometry of the space between the moving and fixed boundaries. Various aspects of the problem contribute to the final result in quite different ways. The shape of the longitudinal passage, as viewed in Fig. 1a, determines the exponent of the clearance. For example, a long cylindrical object used in place of the spheres would require the exponent of -3 rather than -2.5. The position of the sphere in the cylinder (Fig. 1a) determines the value of the coefficient as shown in Fig. 2. The rolling or sliding motion, the force of friction at the wall, the integrated shear, and the more complicated aspects of continuity affect the result only to the second order (i.e., to $\epsilon^{-3/2}$).

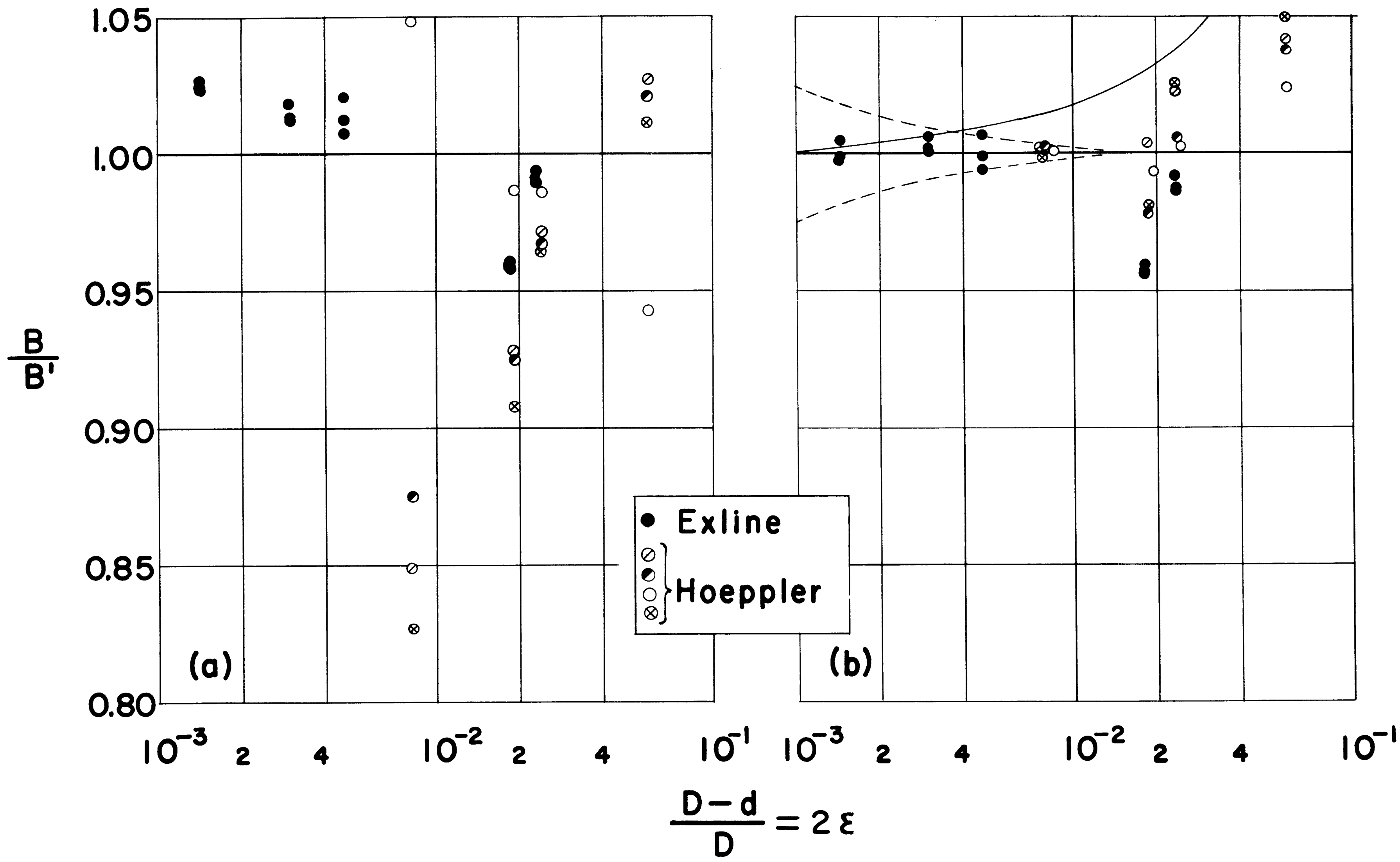

Fig. 3 Comparison of Observed and Computed Ball Constants:
(a) Original Data: (b) Adjusted Data

III. COMPARISON WITH RESULTS OF VISCOMETER CALIBRATIONS

Upon consideration of the results of the foregoing analysis, one is struck by the practical difficulties inherent in its experimental verification. The theory holds only for small clearances, and small clearances require extreme precision in the manufacturing of the instrument. The best data for comparison are those obtained during the calibrations of commercial viscometers. Sets of data for Hoeppler viscometers were obtained from the laboratories at the Iowa Institute of Hydraulic Research, the University of Illinois, the University of Texas, and the California Institute of Technology. Still more valuable were data from the Exline Engineering Company of Tulsa, Oklahoma, which were obtained during the recent development of a more precise instrument.

In the original calibrations of the various instruments, standard liquids were used, and ball constants were determined for each sphere. These constants, designated as B, are conventionally defined as

$$B = \frac{\mu'}{T\,(s_b - s_f)} \quad ,$$

in which μ' is the viscosity in centipoises, T is the time required for the ball to travel a specified distance L along the tube, and s_b and s_f are the specific gravities of the ball and fluid, respectively. It follows directly that

$$B' = \frac{981 \times 100}{18} \, \frac{d^2 \cos \alpha}{LK} \quad ,$$

in which B' represents the theoretical value of B; the numerical coefficients are included to provide consistent units, d^2 and L are in centimeters, and K is dimensionless. The tubes for the Hoeppler viscometer are inclined 10^o from the vertical so that $\cos \alpha = 0.985$; various positions are used in the Exline instrument.

From the tube and ball diameters furnished with each calibrated viscometer, values of B were computed and are compared in Fig. 3a with those obtained by calibration.

$$\frac{B}{B'} = \frac{BL}{6190\, d^2 \cos \alpha} \left(\frac{D - d}{D}\right)^{-5/2} \quad .$$

The solid points are the several almost equal values for various inclinations of the Exline tube. The trend of the results as indicated in Fig. 3a is poorly defined, and is particularly unsatisfactory in that the theoretical value is not approached as (D-d)/D approaches zero. It is, however, closely approached for the results of the more precise Exline instrument. Even for this instrument the values begin to diverge from the ideal value of unity for very small clearances.

As (D-d)/D approaches zero each of the assumptions made in the theoretical derivation becomes more precise, but the calculation also becomes more critically dependent upon the accurate determination of D - d. In fact, an error in the determination of D would cause the computed and measured results to diverge as $d \rightarrow D$ as shown in Fig. 3(a). Consequently, the effect of an error in the measurements of the diameters was examined in an attempt to explain why the predicted trend was not found for very small clearances. Because of the greater difficulty of measuring precisely an inside diameter, an arbitrary correction to each value of D was assigned such that the most nearly consistent correspondence between theory and experiment was obtained for all the balls used with that tube.

The tube diameters for all instruments were slightly less than 1.6 cm, and were given to the nearest 10^{-5} cm. The required corrections varied from 3 x 10^{-4} to 9 x 10^{-4} cm for the four Hoeppler instruments, but was only 2 x 10^{-5} cm for the Exline instrument. Approximate data on maximum and minimum diameters of the tube used for the latter were made available. Because the overall variation was more than ten times the arbitrary adjustment required to produce excellent agreement, the original discrepancy can very likely be ascribed to a very slight error in evaluating the effective mean diameter of the tube.

The small adjustments in the tube diameter brought about the marked improvement in the results shown in Fig. 2b. For large clearances the points were changed very little. For small clearances the corresponding values of B/B' were considerably altered; even the values of (D - d)/D were changed appreciably in some cases. Two factors make these arbitrary adjustments rather convincing evidence of the reliability of the theory for small clearances. For each set of data only the value of D was changed, computations for all balls in the set being based on the same value of D. Also, the change for the most precise set of data was well within the limits of accuracy to which the diameter could be manufactured or measured. For relative clearances of 0.01 or less, the theory is evidently more reliable than the available data. For relative clearances between 0.01 and 0.1, second-order effects become appreciable, and values up to 1.05 occur for the Hoeppler data and down to 0.96 for the Exline instrument.

The solid line in the Fig. 3b is the value of B/B' to be expected if the second-order calculation was complete and no slipping occurred. The corresponding correction is too large. Also included in Fig. 2b are the dotted lines which indicate the effect of changes of $\pm$ 10^{-5} in the relative clearance ($\pm$ 1.6 x 10^{-5} cm). These lines show the increase in the effect of an error in measurement as the relative clearance decreases.

The facts that the ball rolled and remained in contact were examined carefully because a question had been raised on this point in other work (4). The motion of a marked ball was examined both as the mark was in contact and over the entire distance of travel. No slippage was evident and the mark on the ball returned to exactly the same spot after a double traverse of the tube. The remarkably close correspondence between the results of theory and the most precise data is further confirmation that a contact was maintained. Visual evidence is sufficient to say that the ball is very nearly in contact. The calculation of the effect of a variation in space indicates that a thin film between the tube and the sphere would not alter the results significantly. Finally, the results for various inclinations of the Exline tube would differ if contact were not maintained. Such a difference was found only for one set of tests at rather large Reynolds numbers.

The several possible discrepancies mentioned in the foregoing do not necessarily reflect adversely on the accuracy of this type of viscometer. Errors in measurement of diameter would have no effect whatever on the calibration. Conversely, differences such as that for the largest value of (D - d)/D can only result from a different type of error such as one in some aspect of the calibration. The nature of the theoretical prediction is such that a smooth curve should result from the reduction of the data, in the absence of errors of measurement, even if the coefficients of higher-order terms are unknown.

IV. CONCLUSION

From a combination of theory and calibration-type experiments, the available results for rolling-ball viscometers have been systematized. The use of this instrument for the absolute determination of viscosity is suggested, subject to the attainment of extremely precise measurement and manufacture of both the tube and the spheres. Also implicit in the foregoing is a method for the measurement of the bore of precision tubing by means of comparison with precisely-measured spheres. Because of known limitations to the accuracy of both theory and measurement, the results are perhaps not entirely conclusive; but they indicate strongly that a reliable means of predicting the occurrence has been found.

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