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Technical Note No. 3

ON THERMO-ELASTIC STRESS-STRAIN RELATIONS FOR THIN ISOTROPIC SHELLS

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In the absence of thermal effects, the formulation of suitable stress-strain relations in the linear theory of thin elastic shells (where the effects of transverse shear deformation and transverse normal stress are accounted for) has been recently carried out by E. Reissner[1] for axisymmetric deformation of shells of revolution, and by the writer[2] for the general shell where the deformation is referred to lines of curvature. It is also relevant to mention here that the usual formulation of problems of thermo-elastic shells of isotropic materials (e.g., as in Melan and Parkus[3]) in the spirit of Love's first approximation is defective in the sense that it does not conform to the requirement that the initially stress-free isotropic shell (in the absence of suitable edge constraints), when subjected to a uniform temperature field, should remain stress free and undergo only a uniform dilatation.

It is the purpose of the present note to extend the previous results[2] and to modify the derived stress-strain relations so as to include the thermal effects in a manner consistent with the assumptions made in Ref. 2. While there is no difficulty in carrying out the extension mentioned, in view of the considerable recent interest in thermo-elastic problems of shells, it is desirable to record for future reference the necessary and relatively minor modifications in the stress-strain relations for isotropic materials which (on account of the assumed functional form of the transverse normal stress and the normal component of the displacement[2]) will be free from the defect mentioned above.

For brevity we employ the same notation as in Ref. 2, and also recall the assumed functional form of the normal displacement W and transverse normal stress σ_ζ given by Eqs. $(2.7)_2$* and $(2.9)_2$, i.e.,

$$W = w + \zeta w' + \frac{1}{2}\zeta^2 w'' \tag{1}$$

$$\begin{aligned}\sigma_\zeta = \left(1+\frac{\zeta}{R_1}\right)^{-1}\left(1+\frac{\zeta}{R_2}\right)^{-1}\Bigg\{&\left[\frac{3}{2}\frac{S}{h}+\frac{T}{4h}\left(\frac{\zeta}{h/2}\right)\right]\left[1-\left(\frac{\zeta}{h/2}\right)^2\right]\\ &+\frac{1}{2}\,q^+H^+\left[1+\frac{3}{2}\left(\frac{\zeta}{h/2}\right)-\frac{1}{2}\left(\frac{\zeta}{h/2}\right)^3\right]\\ &+\frac{1}{2}\,q^-H^-\left[1-\frac{3}{2}\left(\frac{\zeta}{h/2}\right)+\frac{1}{2}\left(\frac{\zeta}{h/2}\right)^3\right]\Bigg\}\end{aligned} \tag{2}$$

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*The subscript 2 after an equation number refers to that equation in Ref. 2.

where the functions w', w", S, and T are specified by Eq. $(3.5)_2$. To include the thermal effects, it is necessary to modify Eq. $(3.1b)_2$ to read

$$\Lambda = \alpha\theta\left[\sigma_1 + \sigma_2 + \sigma_\zeta\right] + \cdots \tag{3}$$

where the dots in (3) refer to the right-hand side of $(3.1b)_2$, α is the coefficient of linear thermal expansion (which may depend on temperature and coordinates), and $\theta = \theta(\xi_1, \xi_2, \zeta)$ denotes the change in temperature from the initially stress free temperature state.

With the notation

$$\begin{Bmatrix} \theta_{1N} \\ \theta_{2N} \end{Bmatrix} = \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha\theta \begin{Bmatrix} (1+\frac{\zeta}{R_1}) \\ (1+\frac{\zeta}{R_2}) \end{Bmatrix} d\zeta \tag{4a}$$

$$\begin{Bmatrix} \theta_{1M} \\ \theta_{2M} \end{Bmatrix} = \frac{12}{h^3}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha\theta \begin{Bmatrix} (1+\frac{\zeta}{R_1}) \\ (1+\frac{\zeta}{R_2}) \end{Bmatrix} \zeta\, d\zeta \tag{4b}$$

$$\begin{Bmatrix} \theta_{1\zeta} \\ \theta_{2\zeta} \end{Bmatrix} = \frac{1}{2h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \alpha\theta \begin{Bmatrix} 3 \\ \frac{1}{2}(\frac{\zeta}{h/2}) \end{Bmatrix} \left[1-(\frac{\zeta}{h/2})^2\right] d\zeta \tag{4c}$$

and with Λ as given by (3), some of the various coefficients in Eq. $(3.4)_2$ are modified as indicated below

$$\begin{Bmatrix} \delta N_1 : -\theta_{1N} & ; & \delta N_2 : -\theta_{2N} \\ \delta M_1 : -\theta_{1M} & ; & \delta M_2 : -\theta_{2M} \\ \delta S : -\theta_{1\zeta} & ; & \delta T : -\theta_{2\zeta} \end{Bmatrix} \tag{5}$$

and the first two of $(3.5)_2$ will contain additional terms, i.e.,

$$w' = \theta_{1\zeta} + \cdots$$
$$w'' = \theta_{2\zeta}\left(\frac{60}{h}\right) + \cdots \tag{6}$$

but the last two of $(3.5)_2$, as to be expected, remain unaltered. It follows that, in the presence of a temperature field, the modified stress-strain relations (for normal components of stress resultants and stress couples as well as shear-stress resultants) read

$$\begin{aligned}
\varepsilon_1^\circ &= \left(\theta_{1N} - \frac{5h}{2R_1}\theta_{2\zeta}\right) + \cdots \\
\varepsilon_2^\circ &= \left(\theta_{2N} - \frac{5h}{2R_2}\theta_{2\zeta}\right) + \cdots \\
\varkappa_1 &= \left(\theta_{1M} - \frac{\theta_{1\zeta}}{R_1}\right) + \cdots \\
\varkappa_2 &= \left(\theta_{2M} - \frac{\theta_{1\zeta}}{R_2}\right) + \cdots
\end{aligned} \tag{7}$$

$$\begin{Bmatrix} \gamma_{1\zeta}^\circ \\ \gamma_{2\zeta}^\circ \end{Bmatrix} = -\left(\frac{3h}{2}\right)\begin{Bmatrix} \dfrac{1}{\alpha_1}\dfrac{\partial}{\partial \xi_1} \\ \dfrac{1}{\alpha_2}\dfrac{\partial}{\partial \xi_2} \end{Bmatrix}\theta_{2\zeta} + \cdots$$

$$\varepsilon_{\zeta\zeta} = \left(\theta_{1\zeta} + 30\left(\frac{\zeta}{h/2}\right)\theta_{2\zeta}\right) + \cdots$$

An examination of (4) and (7) reveals that when Θ = const. (say Θ_0), then

$$\theta_{1N} = \theta_{2N} = \theta_{1\zeta} = \alpha\theta_0 \;, \quad \theta_{2\zeta} = 0$$
$$\theta_{1M} = \frac{\alpha\theta_0}{R_1} \;, \quad \theta_{2M} = \frac{\alpha\theta_0}{R_2} \tag{8a}$$

and

$$\varepsilon_1^\circ = \varepsilon_2^\circ = \varepsilon_\zeta = \alpha\theta_0 \;, \quad \varkappa_1 = \varkappa_2 = 0 \tag{8b}$$

It is also worth noting that in order for the theory of Love's first approximation (in the presence of thermal effects) to be free of the defect mentioned earlier, it will suffice to assume for the normal displacement the form

$$W = w + \zeta \theta_{1\zeta} \tag{9}$$

in which case, since $\theta_{2\zeta} = 0$, (7) simplifies considerably.

REFERENCES

1. Reissner, E., "Stress Strain Relations in the Theory of Thin Elastic Shells," J. Math. Phys., 31, 109-119 (1952).

2. Naghdi, P. M., "On the Theory of Thin Elastic Shells," Quart. Appl. Math., 14, 369-380 (1957).

3. Melan, E., and Parkus, H., Warmespannungen (Springer Verlag, Wien, 1953), pp. 89-98.