Shih-Ying Sheu
Deneb Robotics, Inc.
P.O. Box 214687
Auburn Hills, Michigan

Michael W. Walker
Artificial Intelligence Laboratory
Department of Electrical Engineering and Computer Science
The University of Michigan
Ann Arbor, Michigan 48109

Identifying the Independent Inertial Parameter Space of Robot Manipulators

Abstract

This paper presents a new approach to the problem of finding the minimum number of inertial parameters of robot manipulator dynamic equations of motion. Based upon the energy difference equation, it is equally applicable to serial link manipulators as well as graph structured manipulators. The method is conceptually simple, computationally efficient, and easy to implement. In particular, the manipulator kinematics and the joint positions and velocities are the only inputs to the algorithm. Applications to a serial link and a graph structured manipulator are illustrated.

1. Introduction

Many advanced manipulator control schemes are based on an inverse dynamics calculation of the nonlinear manipulator dynamics. These inverse dynamics algorithms use the knowledge of the current joint positions and velocities, the desired joint accelerations, and the inertial parameters of each link to calculate the required actuator forces. Each link of a manipulator has 10 different inertial parameters. They are the mass, the three components of the center of mass vector times the mass, and the six unique components of the moment of inertia matrix. However, not all of these parameters are actually present in the manipulator dynamic equations of motion, and the complexity of these equations makes it very difficult to determine which are present and which are not. The main objective of this article is to develop a method for finding a set of independent inertial parameters required for a complete model of the manipulator dynamics.

Several authors have addressed the problem of determining which inertial parameters are identifiable (Neuman and Khosla 1985; Khosla and Kanade 1985; Atkeson et al. 1986; Khalil et al. 1986; Mayeda et al. 1988, 1989; Gautier and Khalil 1988a,b, 1989). A popular method proposed by Khalil and co-workers (1986) is to expand the Lagrangian formulation symbolically and use physical properties of the equations to obtain a solution. For serial-link manipulators, the method has been used to determine the number of independent parameters of a particular manipulator.

Gautier and Khalil (1988a, 1989) have developed a direct method using recursive symbolic expressions of the inertial parameters for both serial-link and tree-structured manipulators. The method is superior to the previous method (Khalil et al. 1986), since a closed-form solution and an upper bound for most of the parameters are obtained.

Mayeda et al. (1988, 1989) use symbolic Lagrangiar equations to obtain the minimum number of inertial parameters. The results apply to manipulators with rotational and translational joints and whose consecutive rotational joints are parallel and perpendicular. Again, their method is applied only for serial-link robots.

Gautier and Khalil (1988b) use an energy difference equation to identify the dynamic parameters of a serial-link manipulator. In the energy difference equation, joint positions, joint velocities, and applied motor torques are the only inputs needed for estimation. The inertial parameters are regrouped by symbolic analysis. The grouped parameters are linearly independent and completely identifiable. However,

The International Journal of Robotics Research,
Vol. 10, No. 6, December 1991,

an off-line symbolic analysis for the particular robot used is required. The symbolic analysis limits their applications when the link number is increased or the manipulator structure is complex (e..g, in graph-structured manipulators).

In this article, a numeric method that is also based on the energy difference equation is used to analyze the inertial parameters of robot manipulators. In the first part of the article, the inertial parameters affecting the dynamic equations of motion are collected as the essential parameter vector. A set of basis vectors that span the vector space are identified. The advantages over existing methods are its ease of implementation and its applicability to serial-link, tree-structured, and graph-structured manipulators. Our method also provides an efficient way to selectively remove the basis vectors of the inertial parameters space that have less effect on the dynamics. Thus the complexity of the equations of motion is reduced.

In the second part of this article, the basis vectors of the essential parameter space are transformed into a set of fundamental vectors, $\mathbf{e}_i$, which are vectors of all zeros except for one at the ith component. The subspace spanned by the fundamental vectors is called an *independent inertial parameter space*. The projection of the essential parameter vector into this space is called an *independent inertial parameter vector*, whose nonzero elements are *independent inertial parameters*.

The article is organized as follows: Section 2 introduces the essential parameter space and proves that the components of the inertial parameters contained in the manipulator equations of motion are the same as the components contained in the energy difference equation. Section 3 presents the numeric method of determining a basis that spans the essential parameter space. Section 4 formulates the transformation from the essential parameter space to the independent inertial parameter space. Section 5 provides example applications to a serial-link and a graph-structured manipulator, which enlighten our analyses of both the essential and the independent parameter space. The final section concludes the article.

2. Essential Parameter Space

The *essential parameter space* of the manipulator dynamic equations of motion is the smallest subspace that contains a solution of the dynamic equations. The solution will not contain any null space component and therefore has the least vector length.

2.1. Construction of the Energy Function

It is very important to express the energy of a manipulator as a function of the positions and velocities of the primary joints of the manipulator. This allows one to directly obtain the equations of motion of any manipulator using Lagrange's equation.

Suppose a manipulator has n joints, l loops, and N $(= n - l)$ rigid links, we can always choose the joint positions, $(q_1 \cdots q_n)$, as a set of joint variables. If there are r independent joint variables, we can choose them as primary variables and assign them the symbols $(p_1 \cdots p_r)$. The primary variable vector is defined as $\mathbf{p} = [p_1 p_2 \cdots p_r]^T$. As in Walker (1988), the joint variables can be written as functions of the primary variables through the constraint equations

$$\mathbf{q} = \mathbf{U}(\mathbf{p}), \tag{1}$$

and the joint velocities are calculated by

$$\dot{\mathbf{q}} = \frac{\partial \mathbf{U}(\mathbf{p})}{\partial \mathbf{p}} \dot{\mathbf{p}} \tag{2}$$

As in Paul (1975), the kinetic energy can be formulated as

$$K(\mathbf{p}, \dot{\mathbf{p}}) = \frac{1}{2} \sum_{i,j=1}^{r} I_{ij} \dot{p}_i \dot{p}_j = \tfrac{1}{2} \dot{\mathbf{p}}^T \mathbf{I}(\mathbf{p}) \dot{\mathbf{p}}, \tag{3}$$

where I_{ij} (*generalized inertia coefficients*) are the ijth element of the matrix $\mathbf{I}$.

The potential energy is obtained by adding the potential energies of all real links and is also a function of primary variables

$$V(\mathbf{p}) = - \sum_{i=1}^{N} m_i \mathbf{g}^T \mathbf{r}_{ci}(\mathbf{p}), \tag{4}$$

where $\mathbf{g}$ is the acceleration vector of gravity, m_i is the total mass of link i, and $\mathbf{r}_{ci}$ is the position vector of the center of mass of link i with respect to the base.

2.2. The Lagrangian Dynamic Equations of Motion

The Lagrangian equations describe the dynamic behavior of robot systems in terms of work and energy stored in the system. When the equations are formulated using the primary (or generalized) coordinates, the constraint forces involved in the system are automatically eliminated (Asada and Slotine 1986). In this section, we write the components of Lagrange's equation as linear functions of the inertial parameters. Later we directly use this form of the equation to prove that both the energy and the dynamic equations contain the same components of the inertial parameters.

It is well known that energy can be written as a linear function of the inertial parameters (Yang and Tzeng 1986; Gautier and Khalil 1988b). The energies are formulated as functions of the primary variables as:

$$T(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{D}) = \mathbf{E}^T(\mathbf{p}, \dot{\mathbf{p}})\mathbf{D}$$
$$V(\mathbf{p}, \mathbf{D}) = T(\mathbf{p}, \mathbf{0}, \mathbf{D}) = \mathbf{E}^T(\mathbf{p}, \mathbf{0})\mathbf{D},$$

where the state variables $\mathbf{p}, \dot{\mathbf{p}} \in R^r$ are the positions and velocities of the independent joints of an N-link manipulator system, and $T(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{D})$ is the sum of the kinetic and potential energy. Thus the manipulator has r degrees of freedom (DOF). $\mathbf{E}(\mathbf{p}, \dot{\mathbf{p}})$ is a nonlinear function of the state variables. $\mathbf{D} \in R^{10N}$ is a vector, called the *inertial parameter vector* (IPV), that contains all the inertial parameters of the manipulator,

$$\mathbf{D} = [\mathbf{D}_1^T, \mathbf{D}_2^T, \ldots, \mathbf{D}_N^T]^T,$$

and the vector $\mathbf{D}_i \in R^{10}$ contains the 10 constant inertial parameters relative to link i coordinates. The *Lagrangian L* is defined as:

$$\begin{aligned} L(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{D}) &= K(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{D}) - V(\mathbf{p}, \mathbf{D}) \\ &= T(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{D}) - 2V(\mathbf{p}, \mathbf{D}) \\ &= [\mathbf{E}^T(\mathbf{p}, \dot{\mathbf{p}}) - 2\mathbf{E}^T(\mathbf{p}, \mathbf{0})]\mathbf{D} \\ &\triangleq \mathbf{W}^T(\mathbf{p}, \dot{\mathbf{p}})\mathbf{D}, \end{aligned} \tag{5}$$

The Lagrangian form of the dynamic equation of motion is:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{p}}} - \frac{\partial L}{\partial \mathbf{p}} = \mathbf{Q}, \tag{6}$$

where $\mathbf{Q}$ is the nonconservative active forces at the r independent joints. Note that we have selected the independent joints $\mathbf{p}$ to correspond to those joints where the actuators are located. Because the inertial parameters are assumed to be constant, this equation can be written in the form:

$$\mathbf{Q}(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{D}) = \left(\frac{d}{dt}\frac{\partial \mathbf{W}}{\partial \dot{\mathbf{p}}} - \frac{\partial \mathbf{W}}{\partial \mathbf{p}}\right)^T \mathbf{D} = \mathbf{\Psi}(\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}})\mathbf{D}. \tag{7}$$

Therefore the Lagrangian dynamic equations are linear in the inertial parameters. Because of this linearity, existing vector space theories are applicable to analyzing manipulator dynamics such that a more fundamental understanding of the manipulator dynamics is achieved.

2.3. *The Essential Parameter Space*

The essential parameter space is the orthogonal complement of the null space for the dynamic equations of motion and is to be used in deriving independent inertial parameters in section 4. The null space of the dynamic equations of motion, $\mathcal{N}(\mathbf{Q})$, is defined as the set of inertial parameter vectors such that $\mathbf{Q}(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{D}) \equiv \mathbf{0}$, $\forall \mathbf{p}, \dot{\mathbf{p}}$. The orthogonal complement of the null space $\mathcal{N}(\mathbf{Q})^\perp$ plays an important role in manipulator dynamics. We define $\mathbf{E}_D \triangleq \mathcal{N}(\mathbf{Q})^\perp$ and call it the *essential parameter space* (EPS) of the manipulator dynamic equations of motion.

The *essential parameter vector* $\mathbf{D}_E$ (EPV) is the projection of the real inertia vector, IPV, onto the essential parameter space. Its components are called the *essential parameters*. The EPV is identifiable. On the other hand, Atkeson et al. (1986) and Khosla (1986) have shown that the parameters that are identified as linear combinations can be reduced by consistently setting certain parameters in these sets to zero so that only independent and completely identifiable parameters are left in the dynamic equations. The number of independent inertial parameters left in the dynamic equations is equal to the dimension of our EPS. However, the number of nonzero components of the essential parameter vector is not necessarily equal to the number of independent inertial parameters. In the next section, we will transform the essential parameter vector into a set of independent inertial parameters, which will reduce the computational cost of the dynamic equations.

2.4. *The Identifiable Parameter Space*

In this article the following energy difference equation is used in the analysis. It is simply the difference in energy content at two different states of the manipulator, $\mathbf{p}_1, \dot{\mathbf{p}}_1$ and $\mathbf{p}_2, \dot{\mathbf{p}}_2$:

$$\begin{aligned} \hbar(\mathbf{p}_1, \dot{\mathbf{p}}_1, \mathbf{p}_2, \dot{\mathbf{p}}_2, \mathbf{D}) &\triangleq T(\mathbf{p}_2, \dot{\mathbf{p}}_2, \mathbf{D}) - T(\mathbf{p}_1, \dot{\mathbf{p}}_1, \mathbf{D}) \\ &= [\mathbf{E}(\mathbf{p}_2, \dot{\mathbf{p}}_2) - \mathbf{E}(\mathbf{p}_1, \dot{\mathbf{p}}_1)]^T\mathbf{D} \\ &= \mathbf{\Phi}(\mathbf{p}_1, \dot{\mathbf{p}}_1, \mathbf{p}_2, \dot{\mathbf{p}}_2)^T\mathbf{D}. \end{aligned} \tag{8}$$

The null space of the energy difference equation, $\mathcal{N}(\hbar)$, is defined as the set of $\mathbf{D}$ such that $\hbar(\mathbf{p}_1, \dot{\mathbf{p}}_1, \mathbf{p}_2, \dot{\mathbf{p}}_2, \mathbf{D}) \equiv \mathbf{0}$, $\forall \mathbf{p}_1, \dot{\mathbf{p}}_1, \mathbf{p}_2, \dot{\mathbf{p}}_2$.

Equation (8) is used for determining the component of the inertial parameters in the equations of motion. The $\mathbf{D} \in \mathcal{N}(\hbar)^\perp$ are not identifiable, because they do not affect the above equation; only the components belonging to $\mathcal{N}(\hbar)^\perp$ are identifiable. Therefore we can define $\mathbf{I}_D \triangleq \mathcal{N}(\hbar)^\perp$, which will be called the *identifiable parameter space* (IDPS). The projection of $\mathbf{D}$ into this space, $\mathbf{D}_I$, is called the *identifiable parameter vector* (IDPV), and its components are called *identifiable parameters*.

2.5. *On the Equivalence of the Essential Parameter Space and the Identifiable Parameter Space*

Although the energy difference equation is formulated much more easily than the dynamic equations, the latter is used for control problems. This section proves that both the essential parameter space, obtained from the equations of motion, and the identifiable parameter space, obtained from the energy difference equation, contain the same components of the inertial parameters. Therefore these components can be identified through the use of the energy difference equation and then utilized in the dynamic equations of motion.

From sections 2.3 and 2.4, we conclude that both the dynamic equations and the energy difference equation are linear in the inertial parameter vectors; **i.e.,**

$$\mathbf{Q}(\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}}, \mathbf{D}) = \mathbf{\Psi}(\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}})\mathbf{D} \quad (9)$$

and

$$\hbar(\mathbf{p}_1, \dot{\mathbf{p}}_1, \mathbf{p}_2, \dot{\mathbf{p}}_2, \mathbf{D}) = \mathbf{\Phi}(\mathbf{p}_1, \dot{\mathbf{p}}_1, \mathbf{p}_2, \dot{\mathbf{p}}_2)^T\mathbf{D}. \quad (10)$$

The objective of this section is to show that both equations use the same components of **D** (i.e., that only the identifiable inertial parameters affect the dynamic equations).

The null space of the dynamic and energy difference equations are defined as the set of inertial parameter vectors, **D**, such that

$$\mathbf{\Psi}(\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}})\mathbf{D} \equiv \mathbf{0} \qquad \forall \mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}},$$

$$\mathbf{\Phi}(\mathbf{p}_1, \dot{\mathbf{p}}_1, \mathbf{p}_2, \dot{\mathbf{p}}_2)^T\mathbf{D} \equiv 0 \qquad \forall \mathbf{p}_1, \dot{\mathbf{p}}_1, \mathbf{p}_2, \dot{\mathbf{p}}_2,$$

or

$$\left(\frac{d}{dt}\frac{\partial \mathbf{W}}{\partial \dot{\mathbf{p}}} - \frac{\partial \mathbf{W}}{\partial \mathbf{p}}\right)^T \mathbf{D} \equiv \mathbf{0} \quad (11)$$

$$[\mathbf{E}(\mathbf{p}_2, \dot{\mathbf{p}}_2) - \mathbf{E}(\mathbf{p}_1, \dot{\mathbf{p}}_1)]^T\mathbf{D} \equiv 0. \quad (12)$$

As mentioned earlier, the identifiable parameter space $\mathbf{I_D}$ and essential parameter space $\mathbf{E_D}$ are the orthogonal complements of the null spaces $\mathcal{N}(\hbar)$ and $\mathcal{N}(\mathbf{Q})$; i.e.,

$$\mathbf{I_D} \triangleq \mathcal{N}(\hbar)^\perp \quad \text{and} \quad \mathbf{E_D} \triangleq \mathcal{N}(\mathbf{Q})^\perp,$$

respectively. Thus the identifiable parameter space is equivalent to the essential parameter space if and only if the $\mathcal{N}(\hbar)$ is equivalent to the $\mathcal{N}(\mathbf{Q})$.

THEOREM 2.1.

$$\mathcal{N}(\hbar) = \mathcal{N}(\mathbf{Q}).$$

□

Proof: The rate of change of the total energy is:

$$\frac{dT(\mathbf{p}, \dot{\mathbf{p}}, \mathbf{D})}{dt} = \dot{\mathbf{p}}^T\mathbf{Q}(\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}}, \mathbf{D})$$

Therefore the change in energy over a time interval t_1 to t_2 is:

$$\hbar(\mathbf{p}_1, \dot{\mathbf{p}}_1, \mathbf{p}_2, \dot{\mathbf{p}}_2, \mathbf{D}) = \int_{t_1}^{t_2} \dot{\mathbf{p}}^T\mathbf{Q}(\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}}, \mathbf{D})dt$$

Thus if $\mathbf{D} \in \mathcal{N}(\mathbf{Q})$, then $\mathbf{D} \in \mathcal{N}(\hbar)$. Therefore $\mathcal{N}(\mathbf{Q}) \subset \mathcal{N}(\hbar)$.

Conversely, from equations (5) and (8),

$$L(\mathbf{p}_2, \dot{\mathbf{p}}_2, \mathbf{D}) - L(\mathbf{p}_2, \dot{\mathbf{p}}_2, \mathbf{D})$$
$$= \hbar(\mathbf{p}_1, \dot{\mathbf{p}}_1, \mathbf{p}_2, \dot{\mathbf{p}}_2, \mathbf{D}) - 2\hbar(\mathbf{p}_1, 0, \mathbf{p}_2, 0, \mathbf{D})$$

Thus if $\mathbf{D} \in \mathcal{N}(\hbar)$, then $\mathbf{D} \in \mathcal{N}(\mathbf{Q})$. Therefore $\mathcal{N}(\hbar) \subset \mathcal{N}(\mathbf{Q})$.

Hence $\mathcal{N}(\hbar) = \mathcal{N}(\mathbf{Q})$.

Q.E.D.

Theorem 2.1 implies that $\mathcal{N}(\hbar)^\perp = \mathcal{N}(\mathbf{Q})^\perp$. As a consequence, the identifiable parameter space, $\mathbf{I_D} = \mathcal{N}(\hbar)^\perp$, is the same as the essential parameter space $\mathbf{E_D} = \mathcal{N}(\mathbf{Q})^\perp$.

3. Basis Set for the Essential Parameter Space

The numeric method for finding a basis of the essential parameter space is introduced in this section. In addition, the contributions of the basis vectors along with their corresponding singular values to the dynamic equations of motion are also analyzed.

3.1. *Obtaining a Basis Set for the Essential Parameter Space*

This section presents a method of finding a set of basis vectors for the essential parameter space using the singular value decomposition (SVD) method. This method is similar to the method used by Atkeson and co-workers (1986) for the estimation of the inertial parameters. The difference in our approach is:

- The energy equation is used rather than the dynamic equations of motion, which eliminates the need of acceleration inputs and reduces the computational complexity of the problem.
- It is equally applicable to manipulators with closed kinematic loops and to serial-link manipulators.
- The solution is a set of basis vectors that span the essential parameter space, rather than a list of the independent and linearly dependent parameters.

The last point is important in that it removes the idea of independence and linear dependence of the initial set of inertial parameters and allows one to focus on simply determining a set of basis vectors that span the essential parameter space. Therefore existing linear vector space theory applies to the dynamics analysis and gives a better understanding of the dynamics.

Assuming the joint trajectories are persistently exciting and sampling the energy difference equation (8) at $m + 1$ points in time, we obtain the following equation:

$$\mathbf{HD} = \mathbf{e}, \tag{13}$$

where $\mathbf{H} \in \mathbf{R}^{m \times 10N}$ and $\mathbf{e} \in \mathbf{R}^{m}$ are computed, and $\mathbf{D} \in \mathbf{R}^{10N}$ is the inertial parameter vector. Note that (1) we choose $m > 10N$, since dim(EPS) $\leq 10N$; and (2) $\mathbf{H}$ is a function of the manipulator kinematic parameters and the positions and velocities of the independent joints of the manipulator.

The matrix $\mathbf{H}$ can be decomposed into the following form using singular value decomposition (Klema and Laub 1980):

$$\mathbf{H} = \mathbf{U\Sigma V}^{T}, \tag{14}$$

where

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{S} = diag(\sigma_1, \sigma_2, \cdots, \sigma_k)$$
$$\text{with} \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0.$$

$\mathbf{U} = [\mathbf{U}_1\, \mathbf{U}_2]$ and $\mathbf{V} = [\mathbf{V}_1\, \mathbf{V}_2]$ are orthonormal matrices, and

$$\mathbf{U}_1 = [\mathbf{u}_1\mathbf{u}_2 \cdots \mathbf{u}_k], \qquad \mathbf{U}_2 = [\mathbf{u}_{k+1}\mathbf{u}_{k+2} \cdots \mathbf{u}_m],$$

$$\mathbf{V}_1 = [\mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_k], \qquad \mathbf{V}_2 = [\mathbf{v}_{k+1}\mathbf{v}_{k+2} \cdots \mathbf{v}_{10N}].$$

Using equation (14), the energy equation (13) becomes:

$$\mathbf{e} = \mathbf{U\Sigma V}^{T}\mathbf{D} = \sum_{i=1}^{k} \sigma_i(\mathbf{v}_i^T\mathbf{D})\mathbf{u}_i = \sum_{i=1}^{k} \sigma_i\alpha_i\mathbf{u}_i, \tag{15}$$

where

$$\alpha_i = \mathbf{v}_i^T\mathbf{D}. \tag{16}$$

From (15) and (16), we note that only the projections of $\mathbf{D}$ onto the basis vectors $\{\mathbf{v}_i, i = 1, \ldots, k\}$ are contained in the energy equation. Thus $\{\mathbf{v}_i, i = 1, \ldots, k\}$ form a set of basis vectors for the essential parameter space. Our objective is to determine k, which is the dimension of the essential parameter space.

Note that because of numeric errors, many of the singular values that should be zero are only very close to zero. The standard method of determining which singular values are effectively zero is to compare their values to the machine precision. Suppose that the coefficient matrix is scaled so that $\sigma_1 = 1$; if the value of σ_{k+1} is less than the square root of the machine precision, then, $\sigma_{k+1}, \sigma_{k+2}, \ldots, \sigma_{10N}$ are considered to be zero (Klema and Laub 1980). Alternatively, $\mathbf{H}$ is considered to have rank k if $(\sigma_{k+1}^2 + \sigma_{k+2}^2 + \cdots \sigma_{10N}^2)^{1/2}$ is less than the square root of the machine precision (Stewart 1973).

3.2. *Model Reduction*

As described earlier, only the projection of the components of the inertial parameters onto the essential parameter space is contained in the manipulator equations of motion. However, for a given manipulator, some of these components may be very small. In this section we quantify the size of these components for a specified trajectory in terms of their effect on the energy content of the manipulator.

The vectors $\{\mathbf{v}_i, i = 1, \ldots, k\}$ span the essential parameter space. Hence any $\mathbf{D}_E$ contained in that space can be written as:

$$\mathbf{D}_E = \sum_{i=1}^{k} \alpha_i\mathbf{v}_i, \tag{17}$$

where, from (16), $\alpha_i = \mathbf{v}_i^T\mathbf{D}$, for $i = 1, \ldots, k$, and $\mathbf{D}$ is the IPV.

Recall that the energy is:

$$\mathbf{e} = \sum_{i=1}^{k} \sigma_i\alpha_i\mathbf{u}_i. \tag{18}$$

Thus the component of the energy resulting from the α_i component of $\mathbf{D}_E$ is proportional to the product of the corresponding singular value σ_i and α_i. Thus although α_i may be relatively small, its effect on the manipulator dynamics can be very large if the corresponding singular value is large.

Let $\beta_j = \sigma_{\eta(j)}\,\alpha_{\eta(j)}$, where $\eta(j)$ is a permutation function that orders the β_j in descending order (that is, $|\beta_j| \geq |\beta_{j+1}|$). Then the energy equation can be written as:

$$\mathbf{e} = \sum_{j=1}^{k} \beta_j\mathbf{u}_{\eta(j)}. \tag{19}$$

An approximation, $\hat{\mathbf{D}}_E$, of the inertial parameters can be made by using only the components corresponding to the first s ($< k$) values of the β_j. That is,

$$\hat{\mathbf{D}}_E = \sum_{j=1}^{s} \alpha_{\eta(j)}\mathbf{v}_{\eta(j)}.$$

Using this for the inertial parameters gives an energy value of

$$\hat{\mathbf{e}} = \mathbf{U}_1\mathbf{S}\mathbf{V}_1^T\hat{\mathbf{D}}_E = \sum_{j=1}^{s} \beta_j \mathbf{u}_{\eta(j)}. \tag{20}$$

The error in energy is then

$$\mathbf{e} - \hat{\mathbf{e}} = \sum_{j=s+1}^{k} \beta_j \mathbf{u}_{\eta(j)},$$

and the squared error in energy is:

$$\| \mathbf{e} - \hat{\mathbf{e}} \|_2^2 = (\mathbf{e} - \hat{\mathbf{e}})^T(\mathbf{e} - \hat{\mathbf{e}}) = \sum_{j=s+1}^{k} \beta_j^2. \tag{21}$$

Defining the *%error* as:

$$\%error \triangleq \frac{\| \mathbf{e} - \hat{\mathbf{e}} \|_2}{\| \mathbf{e} \|_2} \cdot 100 \tag{22}$$

gives

$$\%error = \left(\frac{\sum_{j=s+1}^{k} \beta_j^2}{\sum_{j=1}^{k} \beta_j^2} \right)^{1/2} \cdot 100. \tag{23}$$

The value of s is chosen dependent on the acceptable *%error*. If we specify *%error* $\leq 100 \cdot \delta$, then s must satisfy

$$\sum_{j=s+1}^{k} \beta_j^2 \leq \delta^2 \sum_{j=1}^{k} \beta_j^2. \tag{24}$$

Therefore by accepting a specified *%error* of the energy content, we can formulate the manipulator model using a smaller number of basis vectors $\{\mathbf{v}_i, i = 1, \ldots, s\}$.

4. Independent Inertial Parameter Space

Because not all inertial parameters affect the dynamic equations of motion, the true inertial parameters can be replaced by a set of independent inertial parameters to reduce the computational load.

In the previous section, the essential parameter for the dynamic equations of motion was found; however, the number of nonzero elements of the EPV could be greater than k, the dimension of EPS, or the number of independent inertial parameters. Moreover, the computational cost of the inverse dynamics depends on the number of inertial parameters present in the dynamic equations. Therefore the use of the EPV does not necessarily reduce the computation burden of the manipulator dynamics. To reduce these computations, the EPV must be transformed into an independent inertial parameter vector that has at most k nonzero independent components. The transformation between the two vectors is formulated in this section.

4.1. Fundamental Basis of the Independent Inertial Parameter Space

In this section we present a numeric approach for finding a fundamental basis that spans an independent inertial parameter vector of a robot manipulator. The *fundamental basis* is a subset of the *standard basis* of R^{10N} and has dimension $K = \dim$ (EPS). The *independent inertial parameter vector* (IIPV) is an inertial parameter vector whose nonzero components are the independent inertial parameters of the manipulator.

With the fundamental basis, one can easily determine which of the inertial parameters are linearly independent. Note that there may exist more than one set of independent inertial parameters. The minimal subspace that contains the IIPV of a manipulator is called the *independent inertial parameter space* (IIPS), which is spanned by the fundamental basis of the manipulator and also may not be unique. Although it is of the same dimension as the essential parameter space, it may contain some components of the null space of the dynamic equations of motion.

THEOREM 4.1. Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{10N}\}$ be the standard basis of R^{10N}. Then there exists a subset of k of these vectors and a k-dimensional component vector β such that the projection of the vector

$$\mathbf{D}_I = \mathbf{F}_1\beta$$

into the EPS is equal to the essential parameter vector, where $\mathbf{F}_1$ is a $10N \times k$ matrix containing k of the $\mathbf{e}_i$. Thus the components of the vector β represent a set of k independent inertial parameters that are contained in the dynamic equations of motion. All other parameters in the equations of motion can be set to zero.

□

Proof: Let $\mathbf{F}_2$ be a $10N \times (10N - k)$ matrix containing $10N - k$ of the standard basis vectors, $\mathbf{e}_i$ such that $\mathbf{F}_2^T\mathbf{V}_2$ is nonsingular. That is, $\mathbf{F}_2^T\mathbf{V}_2$ contains $(10 - k)$ linearly independent rows of the matrix $\mathbf{V}_2$. Let $\mathbf{F}_1$ be a $10N \times k$ matrix containing the remaining $\mathbf{e}_i$. Let

$$\mathbf{D}_B = (\mathbf{I} - \mathbf{V}_2(\mathbf{F}_2^T\mathbf{V}_2)^{-1}\mathbf{F}_2^T)\mathbf{V}_1\mathbf{V}_1^T\mathbf{D}_R, \tag{25}$$

where $\mathbf{D}_R$ is the real parameter vector. Then

$$\mathbf{F}_2^T \mathbf{D}_B = \mathbf{0}.$$

Thus $\mathbf{D}_B$ contains zeros everywhere except those rows corresponding to the nonzero rows of $\mathbf{F}_1$. Thus

$$\beta = \mathbf{F}_1^T \mathbf{D}_B$$

and

$$\mathbf{D}_I = \mathbf{F}_1 \beta = \mathbf{D}_B. \tag{26}$$

The projection of $\mathbf{D}_I$ onto the essential parameter space is:

$$\mathbf{V}_1 \mathbf{V}_1^T \mathbf{D}_I = \mathbf{V}_1 \mathbf{V}_1^T \mathbf{D}_R = \mathbf{D}_E.$$

Therefore the independent parameter vector $\mathbf{D}_I$ is contained in the independent parameter space spanned by the basis vectors consisting of the columns of $\mathbf{F}_1$, and the projection of this vector onto the essential parameter space is equal to the essential parameter vector, $\mathbf{D}_E$.

Q.E.D.

To obtain the independent parameter vector, $\mathbf{D}_I$, we need only identify the k-dimensional coordinate vector β, which is called the *fundamental parameter vector* (FPV) of the manipulator. In fact, the elements of the fundamental parameter vector are a set of independent inertial parameters of the dynamic equations of motion.

The significance of the fundamental parameter vector is a result of its application in adaptive controls, parameter identifications, and inverse dynamics computations. For example, if a set of independent parameters of the dynamic equations is found, then the rest of the inertial parameters in the equations can be set to zeros. Therefore the unknowns in the dynamic model are reduced, and both the parameter identification and inverse dynamics computation problems are significantly simplified.

4.2. Two-Dimensional Inertial Parameter Space Example

This is a two-dimensional inertial parameter space example that shows the transformations among three two-dimensional inertial parameter vectors: the real, the essential, and the independent inertial parameter vectors.

The Essential Parameter Space

Suppose that the energy difference equation (8) is equal to

$$\hbar(t_1, t_2, \mathbf{D}) = \mathbf{\Phi}^T \mathbf{D}, \tag{27}$$

where

$$\mathbf{\Phi} = [\phi_1 \phi_2]^T, \quad \mathbf{D} = [m_1 m_2]^T,$$

$$\text{and} \quad \phi_2 = c\phi_1, \quad c \in R.$$

If $\mathbf{D}$ is in the null space of the energy difference equation, then

$$\mathbf{\Phi}^T \mathbf{D} = 0 \quad \text{or} \quad (m_1 + cm_2)\phi_1 = 0.$$

Because ϕ_1 is not always zero, the null space is equal to the line

$$m_1 + cm_2 = 0. \tag{28}$$

The essential parameter space for the two-dimensional inertial parameter vector is the line orthogonal to the null space. Both lines, the EPS and the null space, are shown in Figure 1. Let $\mathbf{e}_1$ and $\mathbf{n}_1$ be basis vectors for the EPS and the null space, respectively, which are also shown in the figure and are equal to

$$\mathbf{e}_1 = \frac{1}{\sqrt{1 + c^2}} \begin{bmatrix} 1 \\ c \end{bmatrix}, \quad \text{and} \quad \mathbf{n}_1 = \frac{1}{\sqrt{1 + c^2}} \begin{bmatrix} -c \\ 1 \end{bmatrix}.$$

The Essential Parameter Vector

The vector $\mathbf{D}_R \in R^2$ shown in Figure 1 is defined as

$$\mathbf{D}_R = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix},$$

where M_1 and M_2 are the true values of inertial parameters. The essential parameter vector $\mathbf{D}_E$,

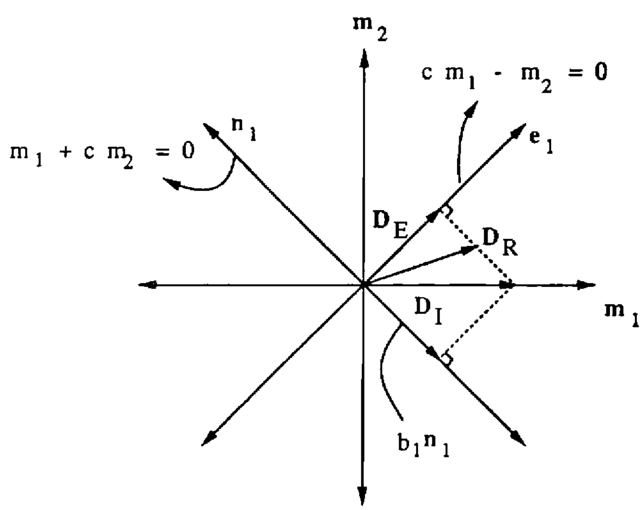

Fig. 1. The transformations among three kinds of IPVs.

which is the projection of $\mathbf{D}_R$ into the EPS, is equal to

$$\mathbf{D}_E = (\mathbf{D}_R^T\mathbf{e}_1)\mathbf{e}_1 = \frac{M_1 + cM_2}{1 + c^2}\begin{bmatrix}1\\c\end{bmatrix}. \quad (29)$$

This shows that the essential parameter vector belongs to the one-dimensional essential parameter space, which is spanned by the basis vector $\mathbf{e}_1$.

The Independent Parameter Vector

Shown in Figure 1, $b_1\mathbf{n}_1$ is a vector contained in null space. By adding it to the essential parameter vector, the EPV is transformed to an independent parameter vector $\mathbf{D}_I$, which contains only one component, m_1. The transformation of $\mathbf{D}_I$ is given as follows:

$$\begin{aligned}\mathbf{D}_I &= \mathbf{D}_E + b_1\mathbf{n}_1\\ &= \frac{M_1 + cM_2}{1 + c^2}\begin{bmatrix}1\\c\end{bmatrix} + b_1\frac{1}{\sqrt{1 + c^2}}\begin{bmatrix}-c\\1\end{bmatrix} \quad (30)\\ &= (M_1 + cM_2)\begin{bmatrix}1\\0\end{bmatrix},\end{aligned}$$

where b_1 is chosen to be

$$b_1 = -\frac{(M_1 + cM_2)c}{\sqrt{1 + c^2}}.$$

Equation (30) shows that the independent IPV, $\mathbf{D}_I$, contains only one fundamental parameter.

Summary

From previous calculations, the EPV and independent IPV are transformed from the real inertial parameters. Although the three vectors are different in the two-space, they are all effective solutions of the energy difference equation. This is easy to verify:

$$\mathbf{\Phi}^T\mathbf{D}_R = (M_1 + cM_2)\phi_1$$

$$\begin{aligned}\mathbf{\Phi}^T\mathbf{D}_E &= \begin{bmatrix}\phi_1\\c\phi_1\end{bmatrix}^T\left(\frac{M_1 + cM_2}{1 + c^2}\begin{bmatrix}1\\c\end{bmatrix}\right)\\ &= (\phi_1 + c^2\phi_1)\frac{M_1 + cM_2}{1 + c^2}\\ &= (M_1 + cM_2)\phi_1\end{aligned}$$

$$\begin{aligned}\mathbf{\Phi}^T\mathbf{D}_I &= \begin{bmatrix}\phi_1\\c\phi_1\end{bmatrix}^T\left((M_1 + cM_2)\begin{bmatrix}1\\0\end{bmatrix}\right)\\ &= (M_1 + cM_2)\phi_1\end{aligned}$$

5. Applications

This section presents application examples that enlighten our analyses of both the essential and the independent parameter space. The examples include a serial-link manipulator (PUMA 560) and a graph-structured manipulator (semi-direct-drive manipulator). In each application, the essential parameter space is found and is then transformed into an independent parameter space. For both robot manipulators, we also investigate the effect of model reduction on the computed torque values. Finally, the geometric structure and the dimension of the essential parameter space of these two manipulators are compared.

5.1. A Serial-Link Manipulator

The PUMA 560 manipulator is chosen as an example of a serial-link manipulator. As shown in Figure 2, the PUMA 560 is a 6-DOF serial-link manipulator. Hence each joint of the manipulator is a primary joint.

A Basis for the Essential Parameter Space

A total of 101 samples of the joint positions $\mathbf{p}$ and velocities $\dot{\mathbf{p}}$ were used to compute the $\mathbf{H}$ coefficient matrix in equation (13). The joint positions and velocities were generated ramdomly within the ranges described in Table 1.

The matrix $\mathbf{H}$ was then decomposed by the singular value decomposition method to obtain the nonzero singular values σ_i ($i = 1, \ldots, k$) and basis vectors $\{\mathbf{v}_i, i = 1, \ldots, k\}$ for the essential parameter space. The singular values are listed in Table 2.

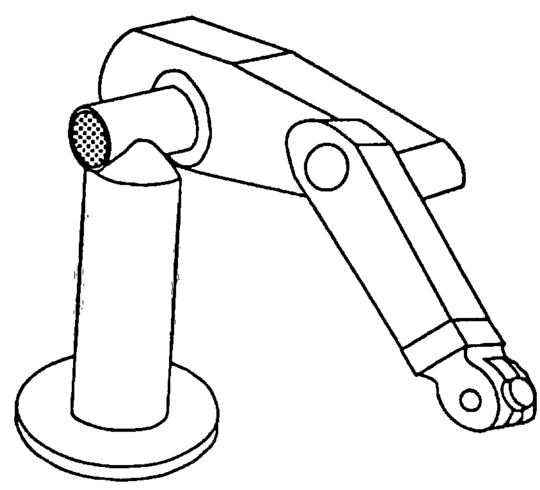

Fig. 2. Puma 560 manipulator.

Table 1. The Range of PUMA 560 Joint Positions and Velocities

Joints	1	2	3	4	5	6
Position (radians)	−2.79 | +2.79	−3.93 | +0.79	−0.79 | +3.93	−1.92 | +2.97	−1.75 | +1.75	−4.64 | +4.64
Velocity (rad/s)	−25.13 | +25.13	−35.34 | +35.34	−35.34 | +35.34	−26.70 | +26.70	−15.71 | +15.71	−41.78 | +41.78

From Table 2, we find that $\sigma_{37} = 5.7E - 12$, which is less than the square root of the machine precision. Therefore there are 36 nonzero singular values for the PUMA 560 manipulator. Thus the essential parameter space, spanned by the basis vectors $\{\mathbf{v}_i, i = 1, \ldots, 36\}$, is of dimension 36.

Model Reduction

We now investigate the manipulator model obtained by using less than all of the 36 basis vectors contained in the essential parameter space. This is quantified in two ways: first, we investigate the effect on the measure of the energy content of the manipulator; second, we investigate the effect on the computed torque value.

The Effect on Energy Content. A plot of the *%error* in energy is given in Figure 3 as a function of s. As can be seen from this figure, the *%error* is 5.3% when only 30 components of the inertial parameters are used, compared with 0.3% when 34 components of the inertial parameters are used. From Figure 3, we see that the first 34 basis vectors have almost equally significant effect on energy content.

The Effect on Computed Torque. Again we approximate the inertial parameter vector by only using the α_i corresponding to the first s β_i. With these parameter vectors, we computed the actuator torques resulting from a sinusoidal joint trajectory. Figure 4 is a plot of the computed torque for the first joint of the manipulator. There are four plots in the figure. The first is the torque computed using the actual parameters, and the three others are the torques computed using 10, 25, and 36 of the largest components of the inertial parameters. The mean square errors between the actual and the approximated torques are listed in the figure. Table 3 provides more mean square errors of the computed torques of all six joints, which were calculated by using different numbers of inertial components from the essential parameter space. As can be seen, the computed torque is very close to the actual value only if the number of basis vectors used in the essential param-

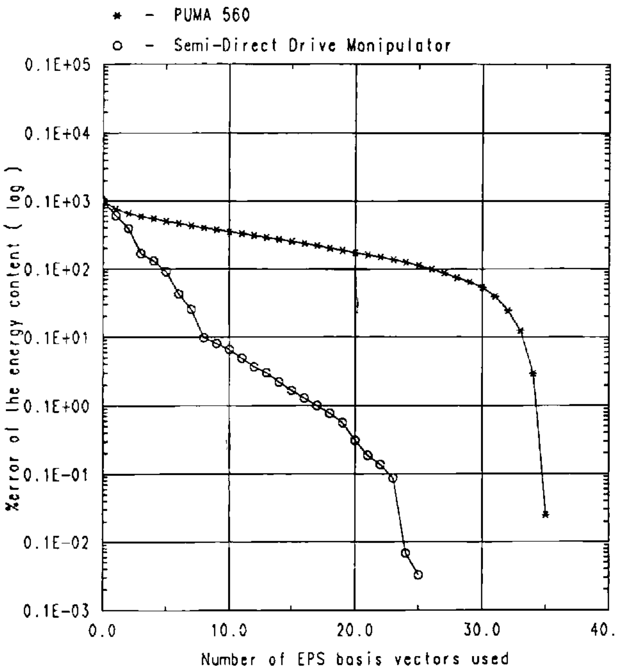

*Fig. 3. The relations between %*error *and the number of inertial components used from the essential parameter space of the PUMA 560 and the semi-direct-drive manipulator.*

Table 2. A Set of Singular Values for the Serial PUMA 560

4.4E + 04	4.1E + 04	3.8E + 04	3.7E + 04	2.8E + 04	2.4E + 04	2.3E + 04	2.0E + 04	2.0E + 04	1.7E + 04
1.6E + 04	1.6E + 04	1.4E + 04	1.2E + 04	1.1E + 04	1.1E + 04	1.0E + 04	9.3E + 03	8.8E + 03	8.0E + 03
7.5E + 03	7.1E + 03	6.6E + 03	5.9E + 03	5.4E + 03	5.0E + 03	4.9E + 03	4.0E + 03	3.8E + 03	3.5E + 03
3.3E + 03	2.6E + 03	2.1E + 03	1.6E + 03	5.5E + 01	3.6E + 01	5.7E − 12	4.7E − 12	4.5E − 12	4.5E − 12
4.3E − 12	3.9E − 12	3.6E − 12	3.3E − 12	3.3E − 12	3.2E − 12	2.7E − 12	2.7E − 12	2.5E − 12	2.4E − 12
2.3E − 12	2.2E − 12	2.1E − 12	1.9E − 12	1.8E − 12	1.7E − 12	1.6E − 12	1.6E − 12	1.3E − 12	0.0E + 00

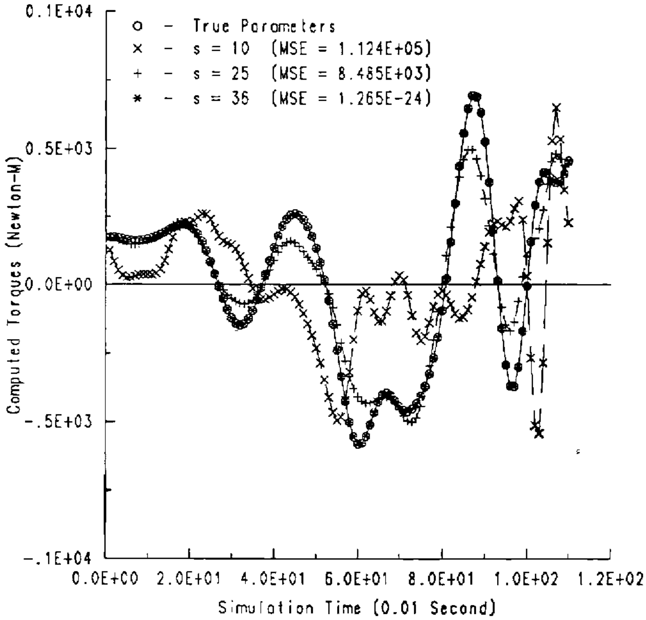

Fig. 4. The relations between joint 1 computed torques and the number of inertial components used from the essential parameter space of the PUMA 560.

eter space is greater than 34. Therefore the PUMA 560 requires at least 34 essential parameter space basis vectors to formulate an accurate dynamic model.

Independent Inertial Parameters for the Puma 560

This section shows the transformations among the real, essential, and independent inertial parameter vectors for the PUMA 560 manipulator.

We first introduce a systematic method for selecting a set of independent inertial parameters. As mentioned in section 4.1, if the rank of the essential parameter space is less than the inertial parameter space, then there exists more than one set of independent inertial parameters of the manipulator. The importance of selecting independent inertial parameters is a result of:

1. *Computational efficiency.* The coefficients of the independent inertial parameters are directly related to the number of computations for the dynamic equations. Therefore we prefer to select the independent inertial parameters whose coefficients require less computational load than other parameters.
2. *Model simplification.* For a complex manipulator system, we want the independent inertial parameters to belong to as few links as possible.

However, the independency is determined by the geometric structure of the manipulator. Theorem 4.1 presents a systematic method for finding a set of independent inertial parameters by deleting the parameters corresponding to the independent rows of the $\mathbf{V}_2$ matrix, whose column vectors are the null space basis vectors of the manipulator inertial parameters. If the independent rows of the matrix are chosen to correspond to the unwanted inertial parameters, the independent parameters will be our preference. Therefore we set up a priority table, Table 4, which specifies the priority of the inertial parameters to be excluded from the PUMA 560 dynamic model. According to the priority table and

Table 3. Mean-Squares Error of Computed Torque vs. Number of Inertial Components Used from the Essential Parameter Space of the PUMA 560

	No. of EPS Components Used					
	10	25	30	34	35	36
Joints	35.2*	11.3*	5.3*	0.3*	0.002*	0*
1	$1.1 \cdot 10^5$	$8.5 \cdot 10^3$	$4.9 \cdot 10^2$	$1.4 \cdot 10^1$	$4.2 \cdot 10^{-4}$	$1.3 \cdot 10^{-24}$
2	$2.1 \cdot 10^4$	$5.3 \cdot 10^3$	$1.1 \cdot 10^3$	$2.7 \cdot 10^2$	$9.9 \cdot 10^{-3}$	$1.5 \cdot 10^{-24}$
3	$2.3 \cdot 10^4$	$5.2 \cdot 10^3$	$6.8 \cdot 10^2$	1.5	$8.0 \cdot 10^{-5}$	$4.4 \cdot 10^{-25}$
4	$1.1 \cdot 10^4$	$3.3 \cdot 10^3$	$4.6 \cdot 10^2$	1.2	$8.7 \cdot 10^{-5}$	$2.8 \cdot 10^{-25}$
5	$1.5 \cdot 10^4$	$2.1 \cdot 10^3$	$4.8 \cdot 10^2$	$8.9 \cdot 10^{-1}$	$7.3 \cdot 10^{-5}$	$3.2 \cdot 10^{-25}$
6	$2.2 \cdot 10^3$	$1.9 \cdot 10^2$	$3.5 \cdot 10^1$	$5.7 \cdot 10^{-2}$	$1.4 \cdot 10^{-5}$	$3.0 \cdot 10^{-26}$

* *%error* of the energy content.

Table 4. Priority Table for Removing Inertial Parameters from PUMA 560 Dynamic Model

Links	1	2	3	4	5	6
m	1	2	3	4	5	6
mc_x	7	8	9	10	11	12
mc_y	13	14	15	16	17	18
mc_z	19	20	21	22	23	24
I_x	43	44	45	46	47	48
I_y	49	50	51	52	53	54
I_z	55	56	57	58	59	60
I_{xy}	25	26	27	28	29	30
I_{xz}	31	32	33	34	35	36
I_{yz}	37	38	39	40	41	42

Table 5. A Set of Independent Inertial Parameters of PUMA 560

Links	1	2	3	4	5	6
m						
mc_x		x	x	x	x	x
mc_y		x				x
mc_z			x	x		x
I_x			x	x		x
I_y	x	x	x	x	x	x
I_z		x	x	x	x	x
I_{xy}		x	x	x	x	x
I_{xz}		x	x	x	x	x
I_{yz}		x				x

applying the *modified Gram-Schmidt* (MGS) method (Noble and Daniel 1977; Golub and Van Loan 1983), we pick out the first $10N - k$ independent rows whose corresponding inertial parameters will be excluded from the dynamic model. In the table, the number i means that the corresponding inertial parameter is the ith preferred candidate to be an independent row of the matrix $\mathbf{V}_2$ and is the ith unwanted inertial parameter of the dynamic model.

Table 5 lists a set of independent inertial parameters that are found using theorem 4.1. As shown in the table, the blanks are the inertial parameters that are not included in the dynamic equations and therefore can be ignored, whereas the x values indicate that the corresponding inertial parameters are the independent inertial parameters of the dynamic equations. There are 36 independent inertial parameters for the PUMA 560; this number is the same as the dimension of its EPS. The independent parameters are the only parameters that can be identified for the PUMA dynamic model. Note that because the number of independent parameters is always significantly less than the number of all inertial parameters, the computational load of the dynamics is reduced. The study of reducing the computational load can be found in Khalil et al. (1986).

Table 6 shows the "true" values for the real inertial parameter vector $\mathbf{D}_R$ that are assigned to use in this example. The real inertial parameter vector is transformed to the essential inertial parameter vector $\mathbf{D}_E = \mathbf{V}_I\mathbf{V}_I^T\mathbf{D}_R$. The essential inertial parameters are shown in Table 7.

The essential inertial parameter vector $\mathbf{D}_E$ is then transformed to the independent inertial parameter vector $\mathbf{D}_I$ using equations (25) and (26). The result is

Table 6. The True* Inertial Parameters of PUMA 560

Links	1	2	3	4	5	6
m	4.7	14.3	3.3	1.25	0.33	0.12
mc_x	0	−3.232	0	0	0	0
mc_y	0	0	0	0	0	0
mc_z	−0.324	0.980	0.436	0	0.023	0.012
I_x	0.216	0.175	0.171	0.004	0.052	0.157
I_y	0.216	1.859	0.174	0.003	0.052	0.157
I_z	0.529	2.032	1.590	0.190	0.170	0.178
I_{xy}	0	0	0	0	0	0
I_{xz}	0	−0.221	0	0	0	0
I_{yz}	0	0	0	0	0	0

* The true values are assigned values, which may not equal to the real parameter values.

Table 7. The Essential Inertial Parameters of PUMA 560

Links	1	2	3	4	5	6
m	0.000	1.220	1.220	1.150	1.150	1.149
mc_x	0.000	2.559	−0.007	0.000	0.000	0.000
mc_y	0.000	0.000	0.211	0.206	0.000	0.000
mc_z	0.000	0.211	−0.117	−0.000	−0.020	−0.004
I_x	0.000	0.748	−0.339	−0.0805	0.008	0.072
I_y	0.680	−0.069	0.236	0.131	0.136	0.072
I_z	0.000	−0.469	1.018	−0.022	0.042	0.178
I_{xy}	0.000	0.000	0.004	0.000	0.000	0.000
I_{xz}	0.000	0.019	0.000	0.000	0.000	0.000
I_{yz}	0.000	0.000	−0.000	0.000	−0.000	0.000

shown in Table 8, where we find that the number of the nonzero components of $\mathbf{D}_I$ is less than both the real and the essential inertial parameter vectors. Its components are a set of independent inertial parameters for the PUMA 560.

5.2. A Graph-Structured Manipulator

In this second example, we focus on the semi-direct-drive manipulator (Asada and Youcef-Toumi 1983) shown in Figure 5. This manipulator has three actuators on its primary joints with a five-bar linkage structure forming a closed kinematic chain.

A Basis for the Essential Parameter Space

As for the serial-link manipulator, 101 random samples of the primary joint positions $\mathbf{p}$ and velocities $\dot{\mathbf{p}}$ were used to compute the $\mathbf{H}$ coefficient matrix in equation (13). The ranges of the primary joint positions and velocities are listed in Table 9.

The matrix $\mathbf{H}$ was then decomposed using the SVD method to obtain the nonzero singular values σ_i ($i = 1, \ldots, k$) and basis vectors $\{\mathbf{v}_i, i = 1, \ldots, k\}$ for the essential parameter space. For the semi-direct-drive manipulator, we found there were 26 nonzero singular values. Thus the basis vectors $\{\mathbf{v}_i, i = 1, \ldots, 26\}$ span the essential parameter space, which has 26 dimensions.

Model Reduction

We now investigate the manipulator model obtained by using the IPVs that are generated using less than the 26 basis vectors contained in the essential parameter space.

Table 8. The Independent Inertial Parameters of PUMA 560

Links	1	2	3	4	5	6
m						
mc_x		5.102	−0.102	0.000	0.000	0.000
mc_y		0.000				0.000
mc_z			1.172	0.000		0.041
I_x			0.490	0.004		0.205
I_y	1.112	−1.915	0.491	0.003	0.000	0.205
I_z		−1.567	1.588	0.190	0.170	0.178
I_{xy}		0.000	0.000	0.000	0.000	0.000
I_{xz}		0.202	0.000	0.000	0.000	0.000
I_{yz}		0.000				0.000

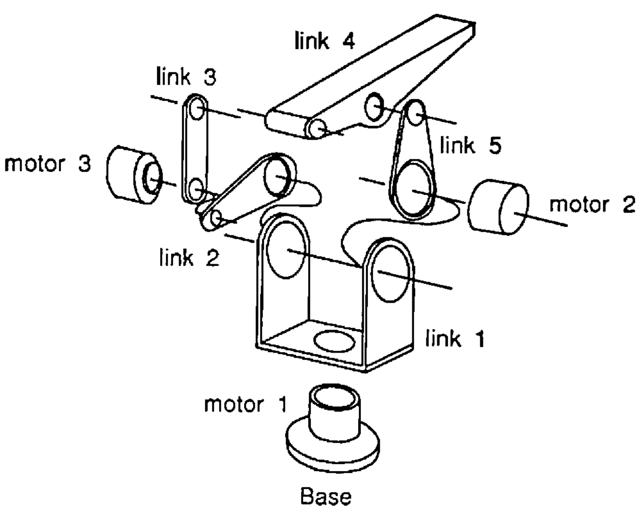

Fig. 5. Semi-direct-drive manipulator.

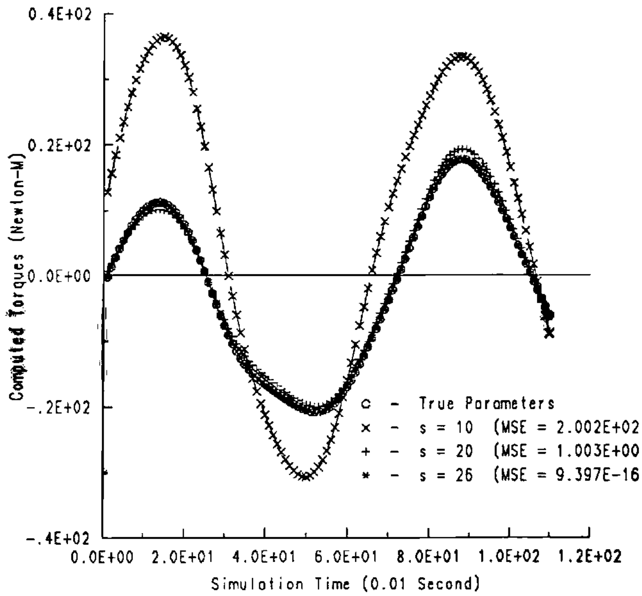

Fig. 6. The relations between joint 2 computed torques and the number of inertial components used from the essential parameter space of the semi-direct-drive manipulator.

The Effect on Energy Content. A plot of the *%error* in energy was given in Figure 3 as a function of s. It can be seen from this figure that the *%error* is less than 0.5% by only using 11 components of the inertial parameters and less than 0.1% by using 17 components. From Figure 3, we see that some basis vectors affect the semi-direct-drive manipulator dynamics more significantly than other basis vectors.

The Effect on Computed Torque. We approximate the IPV by only using the α_i corresponding to the first s β_i. Using the parameter vector we computed the actuator torques resulting from sinusoidal joint trajectories. Figure 6 is a plot of the computed torque for joint 2 resulting from a sinusoidal joint trajectory. There are four plots in the figure. The first is the torque computed using the actual parameters, and the three others are the torques computed using 10, 20, and 26 of the largest components of the inertial parameters. The mean square errors between the actual and the approximated torques are listed in the figure.

Table 10 provides more mean square errors of the computed torques of all three primary joints that were calculated by using different numbers of inertial components from the essential parameter space. As can be seen, the computed torque is very close to the actual value only if the number of basis vectors used in the essential parameter space is greater than 20. Therefore the semi-direct-drive manipulator requires at least 20 essential parameter space basis vectors to formulate an accurate dynamic model.

Table 9. The Range of Semi-Direct-Drive Manipulator Joint* Positions and Velocities

	Joint 1	Joint 2	Joint 3†
Position (radians)	−2.793 | +2.793	−3.927 | +0.785	p_3
Velocity (rad/sec.)	−25.133 | +25.133	−35.343 | +35.343	−35.343 | +35.343

* The primary joint variables only.

† To avoid singular positions, joint 3 positions were forced to satisfy:

$$0.5236 < p_2 - p_3 < 2.618.$$

Independent Inertial Parameters for the Semi-Direct-Drive Manipulator

This section provides a set of independent inertial parameters for the semi-direct-drive manipulator, which contains one closed kinematic loop. This example is intended to demonstrate that our method is applicable to the analyses of graph-structured mechanisms.

As in the previous example, we first set up a priority table, Table 11, which states the preference of the semi-direct-drive manipulator inertial parameters to be excluded from the dynamic model.

Table 10. Mean-Squares Error of Computed Torque vs. Number of Inertial Components Used from the Essential Parameter Space of the Semi-Direct-Drive Arm

	No. of EPS Components Used				
	10	20	23	25	26
Joints	0.7*	0.03*	0.009*	0.0003*	0*
1	$7.7 \cdot 10^{-1}$	$6.7 \cdot 10^{-3}$	$7.3 \cdot 10^{-4}$	$3.8 \cdot 10^{-7}$	$5.5 \cdot 10^{-9}$
2	$2.0 \cdot 10^{2}$	1.0	$1.5 \cdot 10^{-1}$	$4.9 \cdot 10^{-4}$	$9.4 \cdot 10^{-16}$
3	$2.4 \cdot 10^{2}$	1.2	$1.5 \cdot 10^{-1}$	$4.3 \cdot 10^{-4}$	$5.8 \cdot 10^{-16}$

* *%error* of the energy content.

Table 12 lists a set of independent inertial parameters that are found using theorem 4.1. There are 26 independent inertial parameters for the semi-direct-drive manipulator; this number is the same as the dimension of the EPS. With the independent parameters, the computational load of the dynamics is reduced.

5.3. *Comparisons Between the Two Types of Manipulators*

The semi-direct-drive arm is designed to overcome some drawbacks of direct-drive arms (Asada and Youcef-Toumi 1983). However, it contains a closed kinematic loop and has more links than a serial-link manipulator. For the same mobility (3-DOF motion of its end point), it is very interesting to know how much the computational load would be increased by using the semi-direct-drive arm instead of using a serial-link manipulator. Because the computation cost of the dynamic equations of motion is directly related to the number of independent inertial parameters, it is evaluated by comparing the dim(EPS) of the two types of manipulators. The 3-DOF serial-link manipulator is simulated here by fixing the positions of the last three joints of the PUMA manipulator.

The results are shown in Table 13. As can be seen, the serial-link manipulator has only 16 independent inertial parameters, whereas the semi-direct-drive manipulator has 26 independent inertial parameters.

Table 13 implies that the serial-link manipulator would require fewer computations than the semi-direct drive arm. As in Youcef-Toumi and Asada (1985), the dynamic equations of the semi-direct-drive arm can be simplified by decoupling the inertial tensor and making it configuration invariant to improve the high-speed dynamic performance for

Table 11. Priority Table for Removing Inertial Parameters from Semi-Direct-Drive Manipulator Dynamic Model

Links	1	2	3	4	5
m	1	2	3	4	5
mc_x	6	7	8	9	10
mc_y	11	12	13	14	15
mc_z	16	17	18	19	20
I_x	36	37	38	39	40
I_y	41	42	43	44	45
I_z	46	47	48	49	50
I_{xy}	21	22	23	24	25
I_{xz}	26	27	28	29	30
I_{yz}	31	32	33	34	35

Table 12. A Set of Independent Inertial Parameters of the Semi-Direct-Drive Arm

Links	1	2	3	4	5
m					
mc_x			x	x	x
mc_y		x	x	x	x
mc_z					
I_x					
I_y	x	x	x	x	x
I_z		x	x	x	x
I_{xy}		x	x	x	x
I_{xz}			x	x	x
I_{yz}			x	x	x

Table 13. Comparisons of the Semi-Direct-Drive Arm and PUMA 560

	PUMA 560*	Semi-Direct-Drive
No. of links	3	5
No. of joints	3	6
No. of primary joints	3	3
No. of actuators	3	3
Dimension of essential space	16	26
Total no. of inertial parameters	30	50

* Only the first 3 links are included in model.

manipulator control. The simplification is done by introducing mass redistribution constraints. Because the constraints are present, the number of independent inertial parameters is reduced, and therefore the computational cost is reduced. For instance, to make all three off-diagonal terms in the inertia tensor zero, three constraint equations are required. Therefore in this particular case, the number of independent inertial parameters is reduced to 23. However, compared with the dim(EPS) of the 3-DOF PUMA, it still required more computations.

6. Conclusions

A simple numeric method has been given to determine a set of basis vectors for the essential inertial parameters and to transform the basis to a set of fundamental vectors to specify the independent inertial parameters of robot manipulators. The advantages of this method over existing methods are that it is easy to implement, has less computational load, and is applicable to large and/or complex systems such as graph-structured mechanisms, task dynamics, and multirobot systems. Applications to two manipulators, the PUMA 560 manipulator and a semi-direct-drive manipulator, have been given to demonstrate this applicability.

When applied to the PUMA 560 manipulator, it was found there were only 36 independent inertial parameters, compared with 60 total parameters. This result is consistent with previous works by Gautier and Khalil (1988a) and Mayeda et al. (1988). The semi-direct-drive manipulator, on the other hand, has 26 independent inertial parameters.

In addition to determining the independent inertial parameters, the method provides a method of simplifying the manipulator model by only using those parameters that have the greatest effect on the manipulator dynamics. Thus when applied to the computed torque problem, it was found that 34 components are needed to obtain a very good approximation of the actual torque for the PUMA 560 manipulator. In comparison, the semi-direct-drive manipulator needs only 20 components to obtain a very good approximation of the actual torque.

A comparison of a 3-DOF serial-link manipulator and a 3-DOF graph-structured manipulator was presented. It was found that the number of independent inertial parameters was only eight for the serial-link manipulator, compared with 26 for the graph-structured manipulator. Therefore to overcome the drawbacks of direct drive arms, the semi-direct-drive arm requires a more complex mechanism at the expense of its computational efficiency.

Future work will be directed toward the application of the results. One application is in manipulator design. Once the kinematics are chosen for a particular manipulator, the effect of adding mass to any link could easily be determined by finding the projection of its associated inertial parameters onto the essential parameter space. An optimum design would be one that minimizes these components. The second application is in inverse dynamics. Because the number of independent inertial parameters is significantly less than the total number of the inertial parameters, the dynamic equations could be formulated using the independent parameters to minimize the computational load. A third application is in parameter identification. By formulating the manipulator system energy as a function of primary joint variables, the energy difference equation can be easily applied to identify serial-link manipulators, graph-structured manipulators, and the interacting tasks of the manipulators. The independent inertial parameters are the only unknowns of the dynamic model to be identified and are completely identifiable.

References

Asada, H., and Slotine, J. J. E. 1986. *Robot Analysis and Control*. New York: John Wiley & Sons.

Asada, H., and Youcef-Toumi, K. 1983 (San Francisco). Analysis and design of semi-direct-drive robot arms. *Proc. of 1983 American Control Conference*, pp. 757–764.

Atkeson, C. G., An, C. H., and Hollerbach, J. M. 1986. Estimation of inertial parameters of manipulator loads and links. *Int. J. Robot. Res.* 5(3):101–119.

Gautier, M., and Khalil, W. 1988a. A direct determination of minimum inertial parameters of robots. *Proceedings of IEEE Int. Conf. on Robotics and Automation*, pp. 1682–1687.

Gautier, M., and Khalil, W. 1988b. On the identification of the inertial parameters of robots. *Proceedings of the 27th IEEE Conf. on Decision and Control*, pp. 2264–2269.

Gautier, M., and Khalil, W. 1989. Identification of the minimum inertial parameters of robots. *Proceedings of IEEE Int. Conf. on Robotics and Automation*, pp. 1529–1534.

Golub, G. H., and Van Loan, C. F. 1983. *Matrix Computations*. Baltimore, MD: Johns Hopkins University Press.

Khalil, W., Kleinfinger, J. F., and Gautier, M. 1986. Reducing the computational burden of the dynamic model of robots. *Proceedings of IEEE Conf. on Robotics and Automation*, pp. 525–531.

Khosla, P. K. 1986. *Real-Time Control and Identification of Direct-Drive Manipulators*. Ph.D. thesis. Department of Electrical and Computer Engineering, Carnegie-Mellon University.

Khosla, P. K., and Kanade, T. 1985. Parameter identification of robot dynamics. *Proceedings of 24th IEEE Conf. on Decision and Control*, pp. 1754–1760.

Klema, V. C., and Laub, A. J. 1980. The singular value decomposition: Its computation and some applications. *IEEE Trans. Auto. Control* AC-25(2):164–176.

Mayeda, H., Yoshida, K., and Osuka, K. 1988. Base parameters of manipulator dynamic models. *Proceedings of IEEE Int. Conf. on Robotics and Automation*, pp. 1367–1372.

Mayeda, H., Yoshida, K., and Ohashi, K. 1989. Base parameters of dynamic models for manipulators with rotational and translational joints. *Proceedings of IEEE Int. Conf. on Robotics and Automation*, pp. 1523–1528.

Neuman, C. P., and Khosla, P. K. 1985. Identification of robot dynamics: An application of recursive estimation. *Proc. 4th Yale Workshop on Applications of Adaptive Sys. Theory*, pp. 42–49.

Noble, B., and Daniel, J. W. 1977. *Applied Linear Algebra*. Englewood Cliffs, NJ: Prentice-Hall.

Paul, B. 1975. Analytical dynamics of mechanisms—a computer oriented overview. *Mech. Machine Theory* 10:481–507.

Sheu, S. Y., and Walker, M. W. 1989. Estimating the essential parameter space of the robot manipulator dynamics. *Proceedings of the 28th IEEE CDC*, pp. 1517–1522.

Stewart, G. W. 1973. *Introduction to Matrix Computations*. New York: Academic Press.

Walker, M. W. 1988. An efficient algorithm for the adaptive control of a manipulator. *Proceedings of IEEE Int. Conf. on Robotics and Automation*, pp. 682–690.

Yang, D. C. H. and Tzeng, S. W. 1986. Simplification and linearization of manipulator dynamics by the design of inertia distribution. *Int. J. Robot. Res.* 5(3):120–128.

Youcef-Toumi, K., and Asada, H. 1985 (Miami). The design of arm linkages with decoupled and configuration-invariant inertia tensors: Part II: Actuator relocation and mass redistribution. *Proc. of 1985 ASME, Winter Annual Meeting*, pp. 153–161.