Structure of the combinatorial generalization of hypergeometric functions for $SU(n)$ states. II

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In the construction of the general $SU(5)$ states, the action of *each* individual lowering operators (raised to a power) operating on the semimaximal state leads to an operator-valued polynomial which is shown to belong to the class of generalized hypergeometric functions in the sense of Gel'fand (namely, they are Radon transform of linear forms). Three new functions are found at the $SU(5)$ level and their content in terms of known lower-hierarchy functions are explicitly exhibited. The structure of the general $SU(n)$ states due to the combined action of all lowering operators is quite complicated, but the action of each individual lowering operator taken one at a time may still be manageable for higher n, and, in the spirit of boson operator formalism, this may be one systematical way of producing high-hierarchy generalized hypergeometric functions.

I. INTRODUCTION

Previous work[1–4] shows that the combinatorics of the boson operator formalism in the construction of the $SU(n)$ states provides a natural scheme for the appearance of certain generalized hypergeometric functions. We recall that a general state is obtained by operating an appropriate string of lowering operators L_j^i (raised to a power) on the so-called semimaximal state, the latter being expressed as products of certain (antisymmetrized) creation operators acting on the vacuum state. As a result of pushing the lowering operators through the creation operators, the nonvanishing commutators thus yield an operator-valued polynomial (operating on the vacuum). For the $SU(3)$ state, this operator-valued polynomial is simply expressed as the Gauss hypergeometric function ${}_2F_1(a,b;c;x)$, as pointed out by Baird and Biedenharn,[1] namely,

$$|\text{general } SU(3) \text{ state}\rangle = \text{const (product of antisymmetrized creation operators)} \times {}_2F_1(a,b;c;x)|0\rangle. \tag{1}$$

Or, symbolically, the relevant ingredient reads

$$SU(3)\colon (L_2^1)^n[aa] \to \text{Gauss } {}_2F_1, \tag{2}$$

where each factor of a in the bracket stands for an antisymmetrized $(a_{i_1 i_2 \cdots i_s})^\beta$ that the lowering operator has to negotiate with.

What is the generalization of the statement (1)? It was found[3,4] that a general $SU(4)$ state which is obtained via a product of three lowering operators $(L_3^1)^n$, $(L_3^2)^n$, $(L_2^1)^n$ does not have a simple form, but may be regarded as folded products of known functions. In other words, at the $SU(4)$ level the action of each individual lowering operator still yields a recognizable function, namely

$SU(4)$: operator	resulting N-fold sum	content	Gel'fand criterion: Radon transform of linear forms
$(L_3^1)^n[aaaa]$	2	Appell function F_2	Yes
$(L_3^2)^n[aa]$	1	Gauss function ${}_2F_1$	Yes
$(L_2^1)^n[aaaa]$	3	Lauricella function $F_D^{(3)}$	Yes

(3)

For higher-rank $SU(n)$ states ($n \geqslant 5$), it turns out that our present repertory of generalized hypergeometric functions clearly is not adequate to accommodate even the action of each individual lowering operator. One has to either invent new names for these generalized hypergeometric functions if one adopts the viewpoint that the boson operator formalism is a good way of generating (hopefully systematically) such functions, or alternatively one may try to exhibit the inner structure thereof in terms of known functions.

In this paper, we examine the structure of the general $SU(5)$ states, obtained by pushing through a set of six lowering operators, $L_4^1, L_4^2, L_4^3, L_3^1, L_3^2$, and L_2^1 (each raised to a power). Their individual action can be summarized as follows (the details are given in Sec. III):

$SU(5)$: operator	resulting N-fold sum	content	Gel'fand criterion: Radon transform of linear forms
$(L_4^1)^n[aaaaaa]$	3	Appell $F_2 \times {}_3F_2$	Yes
$(L_4^2)^n[aaaaa]$	3	Appell $F_2 \times {}_3F_2$	Yes
$(L_4^3)^n[aa]$	1	Gauss ${}_2F_1$	Yes
$(L_3^1)^n[aaaaaaa]$	8	Appell F_2 $\times$ Lauricella $F_D^{(3)}$ $\times$ Lauricella $F_B^{(3)}$	Yes
$(L_3^2)^n[aaa]$	2	Appell F_1	Yes
$(L_2^1)^n[aaaaaaa]$	6	Lauricella $F_D^{(6)}$	Yes

(4)

The following remarks are obvious at the $SU(5)$ level:

(a) The operator $(L_4^3)^n[aa]$ yields the Gauss ${}_2F_1$ function. This result is analogous to the action of $(L_2^1)^n[aa]$ at the $SU(3)$ level, or that of $(L_3^2)^n[aa]$ at the $SU(4)$ level.

(b) The operator $(L_3^2)^n[aaa]$ yields F_1, the Appell function of the first kind (in 2-variables).

(c) The operator $(L_2^1)^n[aaaaaaa]$ yields $F_D^{(6)}$, the Lauricella function of the fourth kind in 6-variables. Basically this is rather similar to the case (b) above, except that $(L_2^1)^n$ here has to push through seven factors of a's. Evidently, the action of $(L_p^{p-1})^n$ [$(s+1)$ factors of a] would yield $F_D^{(s)}$, and Lauricella function of the fourth kind in s-variables. Note that $F_D^{(1)} \equiv$ Gauss ${}_2F_1$, $F_D^{(2)} \equiv$ Appell F_1.

(d) The operators $(L_4^1)^n$, $(L_4^2)^n$, $(L_3^1)^n$ yield three essentially new functions of several variables. Two of them involve tripple sums and the other an eightfold sum. Instead of giving new names to these functions, we have exhibited their content as folded products of known functions. They are shown, however, to belong to the class of generalized hypergeometric functions in the sense of Gel'fand[5] as being the Radon transform of linear forms.

II. GENERAL $SU(5)$ STATES

As is well known, a general $SU(5)$ state may be constructed by applying a set of appropriate lowering operators to the semimaximal state.

|general $SU(5)$ state$\rangle \equiv$

$$\left|\begin{pmatrix} m_{15} & m_{25} & m_{35} & m_{45} & 0 \\ & m_{14} & m_{24} & m_{34} & m_{44} \\ & & m_{13} & m_{23} & m_{33} \\ & & & m_{12} & m_{22} \\ & & & & m_{11} \end{pmatrix}\right\rangle$$

$$= \text{const } (L_2^1)^{n_{12}} (L_3^2)^{n_{23}} (L_3^1)^{n_{13}} (L_4^3)^{n_{34}} (L_4^2)^{n_{24}} (L_4^1)^{n_{14}}$$

$$\times \left|\begin{pmatrix} m_{15} & m_{25} & m_{35} & m_{45} & 0 \\ & m_{14} & m_{24} & m_{34} & m_{44} \\ & & m_{14} & m_{24} & m_{34} \\ & & & m_{14} & m_{24} \\ & & & & m_{14} \end{pmatrix}\right\rangle$$

$$= \text{const } (L_2^1)^{n_{12}} (L_3^2)^{n_{23}} (L_3^1)^{n_{13}} (L_4^3)^{n_{34}} (L_4^2)^{n_{24}} (L_4^1)^{n_{14}} \times (a_{1234})^{\nu_{44}} (a_{1235})^{n_{45}} (a_{123})^{\nu_{34}} (a_{125})^{n_{35}} (a_{12})^{\nu_{24}} \times (a_{15})^{n_{25}} (a_1)^{\nu_{14}} (a_5)^{n_{15}} |0\rangle. \quad (5)$$

The set of lowering operators L_j^i are defined in Ref. 6. Those with $i < j \leqslant 3$ appeared in the discussion of $SU(4)$ case.[3,4] L_4^1 reads explicitly

$$L_4^1 \equiv \mathcal{E}_{12}\mathcal{E}_{13}E_{41} + \mathcal{E}_{13}E_{42}E_{21} + \mathcal{E}_{12}E_{43}E_{31} + E_{43}E_{32}E_2 \quad (6)$$

The exponents n_{ij}, ν_{ij} in Eq. (5) are shorthand notations as before,[4] namely

$$n_{ij} \equiv m_{ij} - m_{ij-1}, \quad \nu_{ij} \equiv m_{ij} - m_{i+1,\,j+1}. \quad (7)$$

III. ACTION OF EACH INDIVIDUAL LOWERING OPERATOR

By a straightforward calculation, the action of each $(L_j^i)^n$ operator on the relevant set of creation operators turns out to be as follows:

Step 1, L_4^1:

$$A \equiv (a_{1235})^{n_{45}} (a_{123})^{\nu_{34}} (a_{125})^{n_{35}} (a_{12})^{\nu_{24}} (a_{15})^{n_{25}} (a_1)^{\nu_{14}}, \quad (8)$$

$$(L_4^1)^{n_{14}} A|0\rangle = \text{const}(w_0)^{n_{14}} A \times \sum_{k_1,k_2,k_3} \frac{(-n_{14})_{k_1+k_2+k_3}(-n_{25})_{k_1}(-n_{35})_{k_2}(-n_{45})_{k_3}(-s_1-1)_{k_3}}{(1+\nu_{14}-n_{14})_{k_1}(-s_2-1)_{k_2+k_3}(-s_3-2)_{k_3}} \times \frac{w_1^{k_1}}{k_1!}\frac{w_2^{k_2}}{k_2!}\frac{w_3^{k_3}}{k_3!}|0\rangle \quad (9a)$$

$$= \text{const}(w_0)^{n_{14}} A \times \sum_{k_1,k_2} \frac{(-n_{14})_{k_1+k_2}(-n_{25})_{k_1}(-n_{35})_{k_2}}{(1+\nu_{14}-n_{14})_{k_1}(-s_2-1)_{k_2}} \frac{w_1^{k_1}}{k_1!}\frac{w_2^{k_2}}{k_2!} \times \sum_{k_3} \frac{(-n_{14}+k_1+k_2)_{k_3}(-n_{45})_{k_3}(-s_1-1)_{k_3}}{(-s_3-2)_{k_3}(-s_2-1+k_2)_{k_3}} \times \frac{w_3^{k_3}}{k_3!}|0\rangle, \quad (9b)$$

where

$$\text{const} \equiv [\nu_{14}!/(\nu_{14}-n_{14})!][(s_2+1)!/(s_2+1-n_{14})!] \times [(s_3+2)!/(s_3+2-n_{14})!], \quad (10)$$

$$s_1 \equiv \nu_{14} + \nu_{24} + n_{25}, \quad s_2 \equiv \sum_{i=1}^{2} (\nu_{i4} + n_{i+1,5}), \quad s_3 \equiv \sum_{i=1}^{3} (\nu_{i4} + n_{i+1,5}), \quad (11)$$

$$w_0 \equiv a_4/a_1, \quad w_1 \equiv a_1 a_{45}/a_4 a_{15}, \quad w_2 \equiv a_{124} a_5/a_{125} a_4, \quad w_3 \equiv a_{1234} a_5/a_{1235} a_4. \quad (12)$$

As a generalized hypergeometric series in three variables, the expression (9a) does not seem to be a known function. Alternatively, Eq. (9b) shows that it may be written as a folded product of an Appell F_2 function (in two variables) with a ${}_3F_2$ function (in one variable).

Step 2, L_4^2:

$$B \equiv (a_{1235})^{n_{45}-k_3} (a_{123})^{\nu_{34}} (a_{125})^{n_{35}-k_2} (a_{124})^{k_2} (a_{12})^{\nu_{24}}, \quad (13)$$

$$(L_4^2)^{n_{24}} B|0\rangle = \text{const}(u_0)^{n_{24}} B \times \sum_{l_1 l_2 l_3} \frac{(-n_{24})_{l_1+l_2+l_3}(-n_{45}+k_3)_{l_2+l_3}}{(1+\nu_{24}-n_{24})_{l_1+l_3}} \times \frac{(-n_{35}+k_2)_{l_1}(-k_2)_{l_3}}{(-s_4-1)_{l_2+l_3}} \frac{u_1^{l_1}}{l_1!}\frac{u_2^{l_2}}{l_2!}\frac{u_3^{l_3}}{l_3!}|0\rangle \quad (14a)$$

$$= \text{const}(u_0)^{n_{24}} B \times \sum_{l_3} \frac{(-n_{24})_{l_3}(-n_{45}+k_3)_{l_3}(-k_2)_{l_3}}{(1+\nu_{24}-n_{24})_{l_3}(-s_4-1)_{l_3}} \frac{u_3^{l_3}}{l_3!} \times \sum_{l_1 l_2} \frac{(-n_{24}+l_3)_{l_1+l_2}(-n_{35}+k_2)_{l_1}(-n_{45}+k_3+l_3)_{l_2}}{(1+\nu_{24}-n_{24}+l_3)_{l_1}(-s_4-1+l_3)_{l_2}} \times \frac{u_1^{l_1}}{l_1!}\frac{u_2^{l_2}}{l_2!}|0\rangle, \quad (14b)$$

where

$$\text{const} \equiv [\nu_{24}!/(\nu_{24}-n_{24})!][(s_4+1)!/(s_4+1-n_{24})!], \quad (15)$$

$$s_4 \equiv \nu_{24} + \nu_{34} + n_{35} + n_{45} - k_3, \quad (16)$$

$$u_0 \equiv a_{14}/a_{12}, \quad u_1 \equiv a_{12}a_{145}/a_{14}a_{125}, \quad u_2 \equiv a_{15}a_{1234}/a_{14}a_{1235}, \quad u_3 \equiv a_{12}a_{145}a_{1234}/a_{14}a_{124}a_{1235}. \quad (17)$$

The expression (14a) in three variables does not seem to be a known function, but Eq. (14b) shows that it has the structure of a folded product of Appell F_2 function with a ${}_3F_2$ function.

Step 3, L_4^3:

$$C \equiv (a_{1235})^{n_{45}-k_3-l_2-l_3}(a_{123})^{\nu_{34}}, \tag{18}$$

$$(L_4^3)^{n_{34}}C\,|\,0\rangle = \text{const}\left(\frac{a_{124}}{a_{123}}\right)^{n_{34}} C\sum_{k_4} \times \frac{(-n_{34})_{k_4}(-n_{45}+k_3+l_2+l_3)_{k_4}}{(1+\nu_{34}-n_{34})_{k_4}} \frac{w_4^{k_4}}{k_4!}\,|\,0\rangle \tag{19a}$$

$$= \text{const}\left(\frac{a_{124}}{a_{123}}\right)^{n_{34}} C \times {}_2F_1(-n_{34}, -n_{45}+k_3+l_2+l_3; 1+\nu_{34}-n_{34}; w_4)\,|\,0\rangle, \tag{19b}$$

where

$$\text{const} \equiv \nu_{34}/(\nu_{34}-n_{34})!, \tag{20}$$

$$w_4 \equiv a_{123}a_{1245}/a_{124}a_{1235}. \tag{21}$$

For the purpose of the subsequent steps, it will be convenient to rewrite (21) with the aid of the identity $a_{123}a_{1245} = a_{124}a_{1235} - a_{125}a_{1234}$ as

$$w_4^{k_4} = \sum_{k_5} \frac{(-k_4)_{k_5}}{k_5!}\left(\frac{a_{125}a_{1234}}{a_{124}a_{1235}}\right)^{k_5}. \tag{22}$$

This has the effect of simplifying the expressions (23), (29), and (33) in not having to include the factor $(a_{1245})^{k_4}$ (which does not commute with L_3^1, L_3^2, nor a_{1345} with L_2^1).

Step 4, L_3^1:

$$D \equiv (a_{124})^{n_{34}+k_2-l_3-k_5}(a_{125})^{n_{35}-k_2-l_1+k_5}(a_{145})^{l_1+l_3}(a_{12})^{\nu_{24}-n_{24}+l_1+l_3}(a_{14})^{n_{24}-l_1-l_2-l_3}(a_{15})^{n_{25}-k_1+l_2}(a_1)^{\nu_{14}-n_{14}+k_1}, \tag{23}$$

$$(L_3^1)^{n_{13}}D\,|\,0\rangle = \text{const}(v_0)^{n_{13}}D\sum_{\sigma}\frac{(-1)^{\sigma_6+\sigma_8}(-n_{13})_{\sigma_1+\sigma_2+\cdots+\sigma_8}(-n_{25}+k_1-l_2)_{\sigma_1+\sigma_6}}{(1+\nu_{14}-n_{14}-n_{13}+k_1)_{\sigma_1+\sigma_2+\sigma_3+\sigma_6+\sigma_7+\sigma_8}}$$
$$\times\frac{(-n_{24}+l_1+l_2+l_3)_{\sigma_2+\sigma_7}(-l_1-l_3)_{\sigma_3+\sigma_8}(-n_{34}-k_2+l_3+k_5)_{\sigma_4+\sigma_6}}{(-s_5-1)_{\sigma_4+\sigma_5+\sigma_6+\sigma_7+\sigma_8}}$$
$$\times(-n_{35}+k_2+l_1+k_5)_{\sigma_5+\sigma_7}(-\nu_{24}+n_{24}-l_1-l_3)_{\sigma_8}\prod_{j=1}^{8}\frac{v_j^{\sigma_j}}{\sigma_j!}\,|\,0\rangle \tag{24a}$$

$$= \text{const}(v_0)^{n_{13}}D\sum_{\sigma_4,\sigma_5}\frac{(-n_{13})_{\sigma_4+\sigma_5}(-n_{34}+k_2+l_3-k_5)_{\sigma_4}(-n_{35}+k_2+l_1+k_5)_{\sigma_5}}{(-s_5-1)_{\sigma_4+\sigma_5}}\frac{v_4^{\sigma_4}}{\sigma_4!}\frac{v_5^{\sigma_5}}{\sigma_5!}$$
$$\times\sum_{\hat\sigma_1\hat\sigma_2\hat\sigma_3}\frac{(-n_{13}+\sigma_4+\sigma_5)_{\hat\sigma_1+\hat\sigma_2+\hat\sigma_3}(-n_{25}+k_1-l_2)_{\hat\sigma_1}(-n_{24}+l_1+l_2+l_3)_{\hat\sigma_2}(-l_1-l_3)_{\hat\sigma_3}}{(1+\nu_{14}-n_{14}-n_{13}+k_1)_{\hat\sigma_1+\hat\sigma_2+\hat\sigma_3}}\frac{v_1^{\hat\sigma_1}v_2^{\hat\sigma_2}v_3^{\hat\sigma_3}}{\hat\sigma_1!\hat\sigma_2!\hat\sigma_3!}$$
$$\times\sum_{\sigma_6,\sigma_7,\sigma_8}\frac{(-1)^{\sigma_7}(-n_{34}-k_2+l_1+k_5+\sigma_4)_{\sigma_6}(-n_{35}+k_2+l_1+k_5+\sigma_5)_{\sigma_7}(-\nu_{24}+n_{24}-l_1-l_3)_{\sigma_8}}{(-s_5-1+\sigma_4+\sigma_5)_{\sigma_6+\sigma_7+\sigma_8}}$$
$$\times(-\hat\sigma_1)_{\sigma_6}(-\hat\sigma_2)_{\sigma_7}(-\hat\sigma_3)_{\sigma_8}\frac{\hat v_6^{\sigma_6}\hat v_7^{\sigma_7}\hat v_8^{\sigma_8}}{\sigma_6!\sigma_7!\sigma_8!}\,|\,0\rangle, \tag{24b}$$

where

$$\text{const} \equiv [(\nu_{14}-n_{14}+k_1)!/(\nu_{14}-n_{14}-n_{13}+k_1)!] \times [(s_5+1)!/(s_5+1-n_{13})!], \tag{25}$$

$$s_5 \equiv n_{25}+n_{34}+n_{35}+\nu_{14}+\nu_{24}-n_{14}, \tag{26}$$

$$v_0 \equiv a_3/a_1, \quad v_1 \equiv a_1a_{35}/a_3a_{15}, \quad v_2 \equiv a_1a_{34}/a_3a_{14},$$
$$v_3 \equiv a_1a_{345}/a_3a_{145}, \quad v_4 \equiv a_{123}a_4/a_{124}a_3,$$
$$v_5 \equiv a_{123}a_5/a_{125}a_3, \quad v_6 \equiv a_1a_{45}a_{123}/a_3a_{15}a_{124},$$
$$v_7 \equiv a_1a_{45}a_{123}/a_3a_{14}a_{125},$$
$$v_8 \equiv a_1a_{45}a_{123}/a_3a_{12}a_{145}. \tag{27}$$

In Eq. (24b), we have for $j = 1, 2, 3$

$$\hat\sigma_j \equiv \sigma_j + \sigma_{j+5}, \quad \hat v_{j+5} \equiv v_{j+5}/v_j. \tag{28}$$

The expression (24a) does not seem to correspond to a known function. On the other hand, Eq. (24b) shows that it has the following structure: Appell function (in v_4, v_5), Lauricella $F_D^{(3)}$ (in v_1, v_2, v_3), and Lauricella $F_B^{(3)}$ (in v_6, v_7, v_8). The last which is a generalization of the Appell F_3 function makes its first appearance at the $SU(5)$ level.

Step 5, L_3^2:

$$E \equiv (a_{124})^{n_{34}+k_2-l_3-k_5-\sigma_4-\sigma_6}(a_{125})^{n_{35}-k_2-l_1+k_5-\sigma_5-\sigma_7} \times (a_{12})^{\nu_{24}-n_{24}+l_1+l_3-\sigma_8} \tag{29}$$

$$(L_3^2)^{n_{23}}E\,|\,0\rangle = \text{const}(\mu_0)^{n_{23}}E \times\sum_{r_1,r_2}\frac{(-n_{23})_{r_1+r_2}(-n_{34}-k_2+l_3+k_5+\sigma_4+\sigma_6)_{r_1}}{(1+\nu_{24}-n_{24}+l_1+l_3-\sigma_8)_{r_1+r_2}}$$

$$\times (-n_{35} + k_2 + l_1 + k_5 + \sigma_5 + \sigma_7)_{r_2} \frac{\mu_1^{r_1}\mu_2^{r_2}}{r_1! r_2!} |0\rangle \tag{30a}$$

$$= \text{const}(\mu_0)^{n_{23}} E \times F_1(-n_{23}; -n_{34} - k_2 + l_3 + k_5 + \sigma_4 + \sigma_6, -n_{35} + k_2 + l_1 + k_5 + \sigma_5 + \sigma_7; \mu_1, \mu_2)|0\rangle. \tag{30b}$$

where

$$\text{const} \equiv (\nu_{24} - n_{24} + l_1 + l_3 - \sigma_8)! / (\nu_{24} - n_{24} - n_{23} + l_1 + l_3 - \sigma_8)!, \tag{31}$$

$$\mu_0 \equiv a_{13}/a_{12}, \quad \mu_1 \equiv a_{12}a_{134}/a_{13}a_{124}, \quad \mu_2 \equiv a_{12}a_{135}/a_{13}a_{125}. \tag{32}$$

Step 6, L_2^1:

$$F \equiv (a_{134})^{r_1}(a_{135})^{r_2}(a_{145})^{l_1+l_3-\hat\sigma_3}(a_{13})^{n_{23}-r_1-r_2} \times (a_{14})^{n_{24}-l_1-l_2-l_3-\hat\sigma_2}(a_{15})^{n_{25}-k_1+l_2-\hat\sigma_1} \times (a_1)^{\nu_{14}-n_{14}-n_{13}+k_1+\hat\sigma_1+\hat\sigma_2+\hat\sigma_3} \tag{33}$$

$$(L_2^1)^{n_{12}}F|0\rangle = \text{const}(a_2/a_1)^{n_{12}}F \times F_D^{(6)}(-n_{12}; -n_{25} + k_1 - l_2 + \hat\sigma_1, -n_{24} + l_1 + l_2 + l_3 + \hat\sigma_2, -l_1 - l_3 + \hat\sigma_3, -n_{23} + r_1 + r_2, -r_1, -r_2; 1 + \nu_{14} - n_{14} - n_{13} - n_{12} + k_1 + \hat\sigma_1 + \hat\sigma_2 + \hat\sigma_3; x_1, x_2, x_3, x_4, x_5, x_6)|0\rangle \tag{34}$$

where

$$\text{const} \equiv (\nu_{14} - n_{14} - n_{13} + k_1 + \hat\sigma_1 + \hat\sigma_2 + \hat\sigma_3)! / (\nu_{14} - n_{14} - n_{13} - n_{12} + k_1 + \hat\sigma_1 + \hat\sigma_2 + \hat\sigma_3)!, \tag{35}$$

$$x_1 \equiv a_1a_{25}/a_2a_{15}, \quad x_2 \equiv a_1a_{24}/a_2a_{14}, \quad x_3 \equiv a_1a_{23}/a_2a_{13}, \quad x_4 \equiv a_1a_{245}/a_2a_{145}, \quad x_5 \equiv a_1a_{235}/a_2a_{135}, \quad x_6 \equiv a_1a_{234}/a_2a_{134}. \tag{36}$$

IV. GEL'FAND CRITERION: RADON TRANSFORM OF LINEAR FORMS

One class of generalized hypergeometric functions has the property that they are Radon transforms of *linear* forms. It so happens that all the known low-hierarchy functions such as Gauss, Appell, and Lauricella functions satisfy this Gel'fand criterion.[5]

From the expression (4), the simple functions associated with the action of each individual operator $(L_4^3)^n$, $(L_3^2)^n$, $(L_2^1)^n$ obviously have this property. For the others, it is not apparent from their contents as folded products of simple functions. In general, the Gel'fand criterion which holds for each constituent may not be preserved under folded multiplication. However, it is rather remarkable that the functions associated with the action of each operator $(L_4^1)^n$, $(L_4^2)^n$, $(L_3^1)^n$, at the $SU(5)$ level *still* satisfy the Gel'fand criterion. The proof of this statement, which consists of using well-known integral representation for each constituent and a simple change of variables, is left for the reader.

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