Asymptotic expansions of Witten–Reshetikhin–Turaev invariants for some simple 3-manifolds

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For any Lie algebra $\mathfrak{g}$ and integral level k, there is defined an invariant $Z_k^*(M, L)$ of embeddings of links L in 3-manifolds M, known as the Witten–Reshetikhin–Turaev invariant. It is known that for links in S^3, $Z_k^*(S^3, L)$ is a polynomial in $q = \exp(2\pi i/(k + c_{\mathfrak{g}}^{v})$, namely, the generalized Jones polynomial of the link L. This paper investigates the invariant $Z_{r-2}^*(M,\varnothing)$ when $\mathfrak{g}=\mathfrak{sl}_2$ for a simple family of rational homology 3-spheres, obtained by integer surgery around $(2, n)$-type torus knots. In particular, we find a closed formula for a formal power series $Z_\infty(M)\in \mathbf{Q}[[h]]$ in $h=q-1$ from which $Z_{r-2}^*(M,\varnothing)$ may be derived for all sufficiently large primes r. We show that this formal power series may be viewed as the asymptotic expansion, around $q=1$, of a multivalued holomorphic function of q with 1 contained on the boundary of its domain of definition. For these particular manifolds, most of which are not $\mathbf{Z}$-homology spheres, this extends work of Ohtsuki and Murakami in which the existence of power series with rational coefficients related to $Z_k^*(M, \varnothing)$ was demonstrated for rational homology spheres. The coefficients in the formal power series $Z_\infty(M)$ are expected to be identical to those obtained from a perturbative expansion of the Witten–Chern–Simons path integral formula for $Z^*(M, \varnothing)$.

I. INTRODUCTION

Suppose that M is a compact oriented 3-manifold without boundary and that $L\subset M$ is a framed link. In Ref. 1, Witten formally defined a topological invariant $Z_k^*(M, L)$ of this pair, dependent on additional pieces of data, namely, a choice of a Lie algebra $\mathfrak{g}$, of a level $k\in\mathbf{Z}$, along with a choice of representation, ρ_i of $\mathfrak{g}$, for each component L_i of L. Witten's formulation of $Z_k^*(M, L)$ was as a functional integral

$$Z_k^*(M, L)=\int_{\mathscr{A}}\prod_i \operatorname{tr}_{\rho_i}\left(P \exp \oint_{L_i} A\, ds\right)e^{ik/4\pi \int_M\langle A,dA+1/3[A,A]\rangle d\tau}\mathscr{D}A \tag{I.1}$$

over a quotient of the space of G-connections on M by an appropriate gauge group. The first term in the integrand is known as the *Wilson loop* associated with L; it is the product of the traces of the holonomies of the connection A around the components of L, the traces being taken in the representations attached to the components. The second term in the integrand is the exponential of a multiple of the Chern–Simons action. The functional integral definition only makes sense when k is an integer so that q is a root of unity, since it is only then that the exponential of the Chern–Simons action is invariant under the action of the gauge group. Although many attempts have been made to directly make sense of this expression, it remains only a formal expression from which valid results can be derived when the functional integral is manipulated according to certain rules; see, for example, Refs. 2–8. The approaches which are closest in spirit to that of (I.1) employ the notion of a topological field theory (see Ref. 9) whose definition is based on

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Segal's mathematical definition of conformal field theory. The situation is much like that which existed for divergent series in the last century and indeed I will indicate some rather close connections between these two stories in this paper.

Many alternative and completely rigorous formulations of $Z_k^*(M, L)$ have been obtained, primarily using a presentation for M as obtained from S^3 by surgery around some link. In particular, in Ref. 10, a construction for the link invariant $Z_k^*(S^3, L)$, which takes values in polynomials in $q = \exp 2\pi i/(k + c_{\mathfrak{g}}^{v})$, is given in terms of the quantum group $U_q\mathfrak{g}$ and representations ρ_i of it, placed on the components of L. In the case when $\mathfrak{g}=\mathfrak{sl}_2$ and all the components of L are endowed with the two-dimensional vector representation, $Z_k^*(S^3, L) = V_L(q)$, the one-variable Jones polynomial of L. More generally, when $\mathfrak{g}=\mathfrak{sl}_m$ and all the components are endowed with the m-dimensional vector representation, $Z_k^*(S^3, L) = X_L(q, q^{m-1})$ is a slice of the two-variable HOMFLY polynomial.

Using the description of a compact connected orientable 3-manifold M, without boundary as obtained by Dehn surgery around a suitable link L_M in S^3, Reshetikhin and Turaev[11] for $\mathfrak{g}=\mathfrak{sl}_2$, and Turaev and Wenzl[12] more generally, obtained $Z_k^*(M, L)$ as a combination of the values of $Z_k^*(S^3, L_M \cup L')$ with all possible choices of irreducible representations attached to the components of L_M. Here $L' \subset S^3 \backslash L_M$ is the image of the link $L \subset M$ under a surgery operation taking M to S^3. This sum will only be finite when q is a root of unity. However, it is still something of a mystery that while $Z_k^*(S^3, L)$ can be defined for all values of q, being a polynomial in q, this is not true in any of the definitions so far known for $Z_k^*(M, L)$ when $M \neq S^3$.

In this paper, we concentrate only on invariants of pairs in which the link is empty and we take $\mathfrak{g}=\mathfrak{sl}_2$. In this case, $Z_k^*(M,\emptyset)$ has a simple combination formulation,[11] see for example Kauffman and Lins[12]; a self-contained summary of this formulation is given in Sec. II B while all the necessary basic notation used throughout this paper is given in Sec. II A. In this formulation it is apparent that $Z_k^*(M, \emptyset)$ can be defined in this way for all roots of unity q, rather than just ones of the form $e^{2\pi i/r}$. Very few concrete computations of $Z_k^*(M, \emptyset)$, as a function of $r=k+2$ (the order of the root of unity q), have been carried out—see Refs. 13–18 for some such computations.

It follows quickly from its definition that, for fixed order r of the root of unity $q, \nabla Z_k^*(M, \emptyset)$ can be written as an algebraic function of q with rational coefficients. In the normalization for which the invariant for S^3 is 1, denote the invariant for the pair $(M,\emptyset)$, as an algebraic function of q at rth roots of unity, by $Z_r(M)$. We now describe the results of Refs. 19–22 on the forms of these functions of $h=q-1$ when r is an odd prime.

Theorem (Murakami/Ohtsuki): *Suppose that r is an odd prime and M is an oriented* **Z**-*homology sphere.*

(a) *(Murakami[19]) As a function of $q, \nabla Z_r(M) \in \mathbf{Z}[h]$, so that for some $a_{m,r}(M) \in \mathbf{Z}$, one has $Z_r(M) = \Sigma_m a_{m,r}(M) h^m$. For $0 \leq m \leq (r-3)/2$, $a_{m,r}(M)$ is uniquely determined by this condition as an element of $\mathbf{Z}/r\mathbf{Z}$ and*

$$a_{0,r}(M) \equiv 1, \quad a_{1,r}(M) \equiv 6\lambda(M),$$

where $\lambda(M)$ denotes the [$SU(2)$-] Casson invariant of M.

(b) *(Ohtsuki[20]) There exist rational numbers $a_{m,\infty}(M)$ such that, for any prime $r \geq 2m+3$, $a_{m,r}(M) \equiv a_{m,\infty}(M)$ as elements of $\mathbf{Z}/r\mathbf{Z}$.*

As a result of part (b) of this Theorem, one may define a formal power series

$$Z_\infty(M) = \sum_{m=0}^{\infty} a_{m,\infty} h^m$$

with rational coefficients, which is an invariant of integral homology 3-spheres M. Some indication that similar results may be obtainable for 3-manifolds which are not necessarily integral homology spheres but only rational homology spheres is given by the following theorem.

Theorem (Murakami[21]): *Suppose that r is an odd prime and that M is a $\mathbf{Z}/r\mathbf{Z}$-homology sphere so that $N=|H_1(M,Z)|$ is coprime to r. Let $\tau_r(M)$ denote the quantum $SO(3)$-invariant of M. Then $\tilde{Z}_r(M)=N\{N\}_r\tau_r(M)$, where $\{N\}_r$ denotes the Legendre symbol of N modulo r, has the following properties:*

(a) $\tilde{Z}_r(M)\in\mathbf{Z}[h]$, *say* $\tilde{Z}_r(M)=\Sigma_m a_{m,r}(M)h^m$ *with* $a_{m,r}(M)\in\mathbf{Z}$;
(b) $a_{0,r}(M)\equiv 1$ *and* $a_{1,r}(M)\equiv 3\lambda'(M)$ *where* $\lambda'(M)$ *denotes the Casson–Walker invariant of* M.

An analog of the second part of the previous theorem for rational homology spheres may be found in Ohtsuki.[22] Note that when M is a $\mathbf{Z}$-homology 3-sphere, $N=1$ and so $\tilde{Z}_r(M)=Z_r(M)$, thus justifying the use above the same notation, $a_{m,r}(M)$, for the coefficients of powers of h in the expansions of $Z_r(M)$ and $\tilde{Z}_r(M)$. Indeed this second theorem is an extension of part (a) of the first theorem above. The difference in the factors (6 and 3) preceding the Casson and Casson–Walker invariants is due to a difference in the normalizations of these invariants in their definitions; see Ref. 23.

In this paper we restrict our attention to a particular two-parameter family of rational homology 3-spheres $\{M_{n,t}\}$, given by integer t-surgery around a $(2,n)$ torus knot. This family contains a subfamily of integral homology spheres, namely, those for which $|n+t|=1$ while the Poincaré homology sphere is included as $M_{-3,2}$. For these manifolds we compute the associated invariants $Z_r(M)$ and derive a closed formula for Ohtsuki's invariant $Z_\infty(M)$; see Theorem IV.9 and Eq. (IV.10). From computations of some coefficients in the power series $Z_\infty(M_{n,t})$ for various n and t, it is suspected that $Z_\infty(M)\in\mathbf{Z}[[h]]$ whenever $M_{n,t}$ is a $\mathbf{Z}$-homology sphere (see Conjecture IV.20). In Theorem V.4 it is shown that the values of $Z_r(M)$, for all sufficiently large primes r, can be reconstructed from $Z_\infty(M)$, by projection of $\mathbf{Z}[[h]]$ onto the quotient by the relation $(q^r-1)/(q-1)=0$. In Theorem VI.3 it is shown that $Z_\infty(M)$ may be regarded as an asymptotic expansion around $q=1$ of a multivalued holomorphic function of q whose domain of definition contains $q=1$ on the boundary.

Theorem (See IV.1, IV.9, IV.19, IV.20, V.4, and VI.3 for details). *Suppose that n is an odd integer and $t\neq -n$ is an integer, while r is an odd prime not dividing $t+n$. Put $h=q-1$.*

(i) *There is a formal power series, $Z_\infty(M_{n,t})$ in h, with coefficients in $\mathbf{Z}[\frac{1}{2}, 1/|t+n|]$, such that the coefficients of h^i in*

$$\tilde{Z}_r(M_{n,t})-|t+n|^{(r-1)/2}Z_\infty(M_{n,t})$$

are divisible by r for $0\leq i\leq (r-3)/2$. [It is not necessary to assume $t+n\not\equiv 0(r)$ for this part of the theorem.]

(ii) $Z_\infty(M_{n,t})=(-1)^tA^{2n}(q^{\Delta_1(\bar{B})}-q^{\Delta_2(\bar{B})})/[2(1-q^{-\delta})]$ *where* $\delta=\mathrm{sgn}(t+n)$ *while* $\Delta_1(x)$ *and* $\Delta_2(x)$ *are two quadratics given by* (IV.8) *and we have used a symbolic notation defined in Sec. II A.*

(iii) $\tilde{Z}_r(M_{n,t})-\{|t+n|\}_rZ_\infty(M_{n,t})$ *is divisible by* $(q^r-1)/(q-1)$ *in the ring of formal power series in* $h=q-1$ *with rational coefficients whose denominators are not divisible by* r.
(iv) *The formal power series for* $Z_\infty(M_{n,t})$ *in* $h=q-1$ *may be obtained as an asymptotic expansion around* $q=1$ *(or* $\ln q=0$*) of an appropriate holomorphic function of* $\ln q$ *defined on* $\mathbf{C}\backslash\mathbf{R}$.
(v) *It is conjectured that* $Z_\infty(M_{n,t})\in\mathbf{Z}[[h]]$ *when* $M_{n,t}$ *is a* $\mathbf{Z}$*-homology sphere.*

Another way to state these results is to say that the formal power series $Z_\infty(M_{n,t})$, although it has zero radius of convergence in the usual complex topology, converges in the r-adic topology at

rth roots of unity to $\{|t+n|\}\bar{Z}_r(M_{n,t})$, so long as r is an odd prime not dividing $|t+n|=|H_1(M_{n,t}, \mathbf{Z})|$.

The restriction to the particular family of manifolds discussed in this paper is necessitated by the fact that it is only for these manifolds that the associated state-sum expressions for $Z_r(M)$ involve only "trivial" quantum $6j$-symbols which cancel, leaving a relatively simple sum. The techniques developed in Sec. III to enable the computation of $Z_\infty(M)$ can currently only be applied in these cases, although it is hoped that they can be extended to deal with all 3-manifolds with only relatively minor modifications.

An outline of the present paper is as follows. In Sec. II, a combinatorial formulation of $Z_r(M)$ is summarized and is used to obtain a presentation for $Z_r(M_{n,t})$ in Sec. II C as a quotient of two sums. This is reformulated in Sec. II D into a shape more amenable to the techniques of the subsequent sections. The denominator is a Gauss sum and the numerator is a two-dimensional variant. In Sec. III, the modulo r properties of sums over the same domain as in the numerator are investigated and these results are applied in Sec. IV to obtain the asymptotic expansion $Z_\infty(M_{n,t})$. In Sec. V a more exact analysis of the sums involved in $Z_r(M_{n,t})$ is carried out and enables these values, for all sufficiently large primes r, to be reconstructed from $Z_\infty(M_{n,t})$. In Sec. VI, a reconstruction of a holomorphic function is carried out from the formal power series for $Z_\infty(M_{n,t})$. Finally, in Sec. VII, some conjectures are made on generalizations of the theorems proved in this paper to more general manifolds.

II. WITTEN–RESHETIKHIN–TURAEV INVARIANTS

In this section a state-sum form is obtained for the Witten–Reshetikhin–Turaev invariants $Z_k(M, \varnothing)$ for the manifolds M used in this paper. This expression is obtained in Sec. II C using the formalism of Ref. 12 (summarized in Sec. II B) for computing $\mathfrak{sl}_2$ invariants derived from the recoupling theory of the Temperley–Lieb algebra. A reformulation of the sum in a form more convenient for the computation of asymptotic expansions in Sec. IV is derived in Sec. II D.

A. Notation

1. Links and manifolds

Any compact connected oriented closed 3-manifold may be obtained from S^3 by surgery around an appropriate framed link. For integers n and t with n odd, let $M_{n,t}$ denote the manifold obtained from S^3 by integer surgery around the $(2,n)$ torus knot with t additional twists. The framing number of the knot is $n+t$ and therefore $M_{n,t}$ will be an integral homology sphere for $t=-n\pm 1$ and a rational homology sphere when $t\neq -n$.

Example II.1: The mirror image of $M_{n,t}$ is $M_{-n,-t}$ for all n and t. The Poincaré homology sphere is realized as $M_{3,-2}$ and the framed knot in S^3 giving rise to this manifold is shown in Fig. 1, where the knot is given the blackboard framing. This diagram also serves to identify positive twists, the two extra curls being negative twists.

2. The q-symbols

Throughout this paper, $r\in\mathbf{N}$ will denote the order of a root of unity q. Set $I=\{0,1,\ldots,r-2\}$. Let $A=q^{1/4}$ and define the q-numbers by

$$[n]_q=\frac{A^{2n}-A^{-2n}}{A^2-A^{-2}}.$$

The q-factorials are defined by $[n]!_q=\Pi_{i=1}^n[i]_q$. A triple of non-negative integers (a, b, c) will be said to be *q-admissible* when $b+c-a$, $c+a-b$, $a+b-c$, and $2r-4-a-b-c$ are all positive and even.

If a is a non-negative integer, set $\Delta_a=(-1)^a[a+1]_q$. If (a, b, c) is a q-admissible triple, set

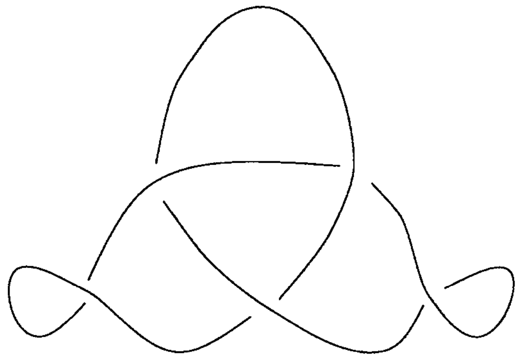

FIG. 1. Knot for Poincaré homology sphere.

$$\lambda_c^{a,b}=(-1)^{(a+b-c)/2}A^{[a(a+2)+b(b+2)-c(c+2)]/2},$$

$$\theta(a,b,c)=(-1)^x\frac{[x+1]!_q[x-a]!_q[x-b]!_q[x-c]!_q}{[a]!_q[b]!_q[c]!_q},$$

where $2x=a+b+c$. The latter expression is known as a *θ-net.*

Suppose that $\{a_i\}_{i=1}^6$ are non-negative integers such that (a_i, a_j, a_k) is a q-admissible triple for each $(i,j,k)\in S$ where $S=\{(1,2,3), (1,4,5), (2,4,6), (3,5,6)\}$. If the edges of a tetrahedron are numbered 1–6 as shown in Fig. 2, then the elements of S are precisely those triples of numbers whose associated edges share a common vertex; that is, the elements of S index the vertices of the tetrahedron. Considering the integer a_i to be placed on the ith edge, it is given that those triples of integers on edges emerging from any vertex form a q-admissible triple. Define the associated tetrahedral net (a variant of the quantum $6j$-symbol) to be

$$\begin{bmatrix} a_4 & a_5 & a_6 \\ a_3 & a_2 & a_1\end{bmatrix}=\frac{\Pi_{v,e}[y_e-x_v]!_q}{\Pi_{i=1}^6[a_i]!_q}\sum_{s=\max(x_v)}^{\min(y_e)}\frac{(-1)^s[s+1]!_q}{\Pi_v[s-x_v]!_q\Pi_e[y_e-s]!_q}.$$

Here $2x_v=\Sigma_{i\in v}a_i$ for each $v\in S$ while if e denotes a pair of opposite edges, of which there are three, then $2y_e=\Sigma_{i\notin e}a_i$.

It is easily verified that when all indices are elements of the set I, the quantity Δ_a along with the values of θ-nets and tetrahedral nets are all nonzero. The θ-net depends in a totally symmetric way on the three indices, while the tetrahedral net exhibits the S_4 symmetry of the tetrahedron. The special values

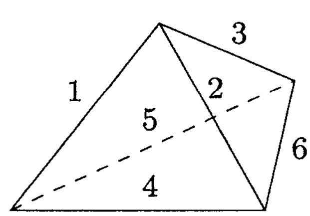

FIG. 2. Tetrahedral net.

$$\lambda_a^{a,0}=1, \quad \theta(a,a,0)=\Delta_a, \quad \begin{bmatrix} 0 & a & a \\ i & a & a \end{bmatrix}=\theta(a,a,i)$$

will be used in the next section.

3. Symbolic notation

The final main piece of notation used in this paper is symbolic. Let B_m denote the mth Bernoulli number, as defined by the generating function

$$\sum_{m=0}^{\infty} \frac{B_m z^m}{m!}=\frac{z}{e^z-1}. \tag{II.2}$$

Following Ref. 24, we will use a symbolic notation employing the symbol B so that B^m refers to B_m. This has the particular property that

$$\sum_{i=a}^{b-1} f(i)=\int_{a+B}^{b+B} f(x)dx \tag{II.3}$$

for integers a and b. Using $f(i)=\binom{a}{p}$, we get

$$\binom{b}{p+1}-\binom{a}{p+1}=\int_{a+B}^{b+B}\binom{x}{p}dx$$

for all $a,b\in\mathbf{N}$. Since both sides are polynomial functions of a and b, this equality also holds for any $a,b\in\mathbf{Q}$.

Put

$$\bar{B}_m=2\int_{B+1/4}^{B+3/4} x^m\, dx.$$

Using the generating function for B_m in (II.2), it follows that

$$\sum_{m=0}^{\infty} \frac{\bar{B}_m z^m}{m!}=2(\mathrm{e}^{z/4}+\mathrm{e}^{-z/4})^{-1}. \tag{II.4}$$

In particular, $\bar{B}_m=0$ for m odd while $\bar{B}_0=1$, $\bar{B}_2=-1/16$, $\bar{B}_4=5/256$, $\bar{B}_6=-61/4096$, $\bar{B}_8=1385/65\,536$, and $\bar{B}_{10}=-50\,521/1\,048\,576$. It follows from (II.4) that

$$\sum_{s=0}^{m}\binom{2m}{2s}2^{4s}\bar{B}_{2s}=\delta_{m=0},$$

from which one deduces that $2^{4m}\bar{B}_{2m}$ is an odd integer for all m. Indeed, $2^{2m}\bar{B}_m$ is the mth Euler number and other expressions for $\bar{B}_m$ are

$$\bar{B}_m=4i.m!\sum_{s=-\infty}^{\infty}(-1)^s(2\pi i(2s+1))^{-m-1}=4\int_{-\infty}^{\infty}\frac{(iz)^m\, dz}{\mathrm{e}^{2\pi z}+\mathrm{e}^{-2\pi z}}; \tag{II.5}$$

the series is absolutely convergent for $m>0$ and convergent, but not absolutely convergent, for $m=0$. For large m, it may be seem that $\bar{B}_{2m}\sim 4/\pi^{-1}(2m)!(-4\pi^2)^{-m}$.

Suppose now that $f(x)$ is a polynomial function of x. Then we may write

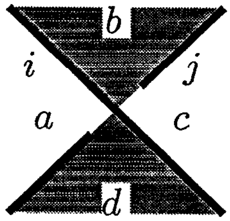

FIG. 3. Local labels at a vertex in $\mathscr{D}$.

$$\int_{B+1/4}^{B+3/4} f(x)dx=\frac{1}{2} f(\bar{B}), \tag{II.6}$$

using a symbolic notation in which $\bar{B}^m$ is replaced by $\bar{B}_m$. In this notation, $f(\bar{B})=f(-\bar{B})$ for all polynomials f. From (II.5), it follows that an integral presentation for $f(\bar{B})$ is given by

$$f(\bar{B})=4\int_{-\infty}^{\infty} \frac{f(iz)dz}{e^{2\pi z}+e^{-2\pi z}}. \tag{II.7}$$

B. A summary of a state-sum formulation of invariants

Suppose that M is a 3-manifold obtained by surgery around the framed link L in S^3. Represent L by a link diagram $\mathscr{D}$ with the blackboard framing and place the checkerboard coloring on the regions into which $\mathscr{D}$ divides the plane where the exterior region is unshaded. The $\mathfrak{sl}_2$ Witten–Reshetikhin–Turaev invariant of the empty link in M, at the r^{th} root of unity q, will be denoted $Z_r(M)$. We here summarize the state-sum formulation of $Z_r(M)$ as described in Ref. 12.

Define a state model in which what is meant by a state is an allowed assignment of an element of I to each of the components of L as well as to each of the regions into which $\mathscr{D}$ divides the plane. Such an assignment is said to be *allowed* so long as the infinite region is labeled 0 and, for each edge of $\mathscr{D}$, the triple of integers assigned to the two adjacent regions and the component containing the edge form a q-admissible triple. For a fixed state σ, define local weights on each vertex, edge, face, and component of $\mathscr{D}$ as follows. If e is an edge of $\mathscr{D}$, set

$$w_e(\sigma)=\theta(a,\ b,\ c)^{-\chi},$$

where a,b,c are the assignments given by σ to the component of L containing e and the two regions adjacent to e and χ is the Euler characteristic of the edge (1 unless the edge contains no vertices, in which case it is 0). If f is a face or component of $\mathscr{D}$ set

$$w_f(\sigma)=\Delta_{\sigma(f)}^{\chi},$$

where $\chi=1$, unless f is a face containing no vertices in which case $\chi=0$. Finally, if v is a vertex of $\mathscr{D}$, set

$$w_v(\sigma)=(\lambda_b^{a,i})^{-\epsilon}(\lambda_c^{d,i})^{\epsilon}\begin{bmatrix} a & b & j \\ c & d & i \end{bmatrix},$$

where i,j and a,b,c,d are the labels assigned to the two components of L and the four regions meeting at v, respectively, while $\epsilon=\pm 1$ according to the orientation of the crossing (over/under) relative to the local shading of regions. The convention on local labels and the sign ϵ is determined by Fig. 3 in which the sign is positive.

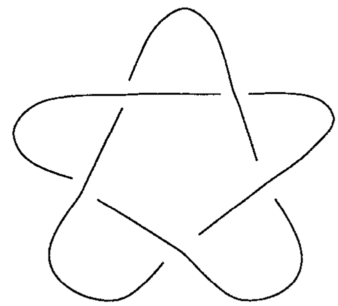

FIG. 4. Link diagram for $M_{5,0}$.

To the state σ we now assign a global weight

$$W_{\mathscr{A}}(\sigma)=\prod_{\substack{\text{vertices}\\ v}} w_v(\sigma)\prod_{\substack{\text{edges}\\ e}} w_e(\sigma)\prod_{\substack{\text{regions}\\ f}} w_f(\sigma)\prod_{\substack{\text{components}\\ c}} w_c(\sigma).$$

The invariant $Z_r(M)$ is now obtained from the partition function of this state model by renormalization, so that

$$Z_r(M)=G_+^{-n_+}G_-^{-n_-}\sum_{\text{states }\sigma} W_{\mathscr{A}}(\sigma),$$

where n_+ and n_- are the numbers of positive and negative eigenvalues, respectively, of the linking matrix defined by the framed link L. Also, G_+ and G_- denote the partition function evaluations on an unknot with framings 1 and -1, respectively.

Finally, to simplify computations, it may be noted that if a link is changed by altering the framing on one of its components, then the global weight associated with a state scales by the term

$$(-1)^{at}A^{a(a+2)t},$$

where n denotes the number assigned by the state to the component and t denotes the number of positive twists added. Applying this fact to compute $G_\pm$ one obtains

$$G_\epsilon=\sum_{a\in I}(-1)^aA^{\epsilon a(a+2)}\Delta_a^2=\frac{A^{-3\epsilon}}{\epsilon(A^{-2}-A^2)}\sum_{a=0}^{2r-1}(-1)^aA^{\epsilon a^2}, \tag{II.8}$$

which is a Gauss sum. For odd r, putting $q=A^4$, one has

$$G_\epsilon=\frac{A^{-3\epsilon}(1-A^{\epsilon r^2})}{\epsilon(A^{-2}-A^2)}\sum_{a=0}^{r-1}q^{\epsilon a^2}. \tag{II.9}$$

One important property of the invariant $Z_r(M)$ is that it transforms according to $q\mapsto q^{-1}$ when the manifold M is replaced by its mirror image.

C. Computing the invariants

In this section we apply the method of Sec. II B to compute $Z_r(M_{n,t})$ for odd n with $n\neq -t$. Let $\mathscr{D}_n$ be the link diagram of the $(2,n)$ torus knot containing n vertices with $n+2$ regions, precisely two of which are unshaded in the associated checkerboard coloring. Figure 4 shows $\mathscr{D}_5$;

the associated blackboard framed link defines $M_{5,0}$.

In the state model for $Z_r(M_{n,0})$ described in Sec. II B, the states are indexed by a pair $a,i \in I$, where a is the label assigned to the single component of L (and thus also to the n shaded regions) and i is the label assigned to the interior unshaded region. The only constraint is that $(a,\ a,\ i)$ be a q-admissible triple; that is, i must be even with $0 \leqslant i \leqslant \min(2a,\ 2r-4-2a)$. The n vertices all have $\epsilon=-1$ and so the global weight contribution to $Z_r(M_{n,0})$ associated with this state for the diagram $\mathscr{D}_n$ is

$$\Delta_a^{n+1}\Delta_i\left(\frac{\lambda_a^{0,a}(\lambda_i^{a,a})^{-1}\begin{bmatrix}0 & a & a\\ i & a & a\end{bmatrix}}{\theta(a,\ a,\ i)\theta(a,\ a,\ 0)}\right)^n = \Delta_a\Delta_i(\lambda_i^{a,a})^{-n}.$$

The invariant for the manifold $M_{n,t}$ is therefore

$$Z_r(M_{n,t}) = \frac{1}{G_\delta}\sum_{a=0}^{r-2}\ \sum_{j=0}^{\min(a,\ r-2-a)} (-1)^{at}A^{a(a+2)t}\cdot\Delta_a\Delta_{2j}((-1)^{a-j}A^{a(a+2)-2j(j+1)})^{-n}, \tag{II.10}$$

where $\delta=\text{sgn}(n+t)$ and we have put $i=2j$.

D. Reformulating the sum

In this section we reformulate the sum in (II.10) into a more manageable form for the purpose of computing asymptotic expansions. Putting $a=b-1$ and $j=k-1$ gives

$$G_\delta Z_r(M_{n,t}) = (-1)^t\sum_{b=1}^{r-1}(-1)^{(t+1-n)b}A^{(t-n)(b^2-1)}[b]_q\sum_{k=1}^{\min(b,\ r-b)}(-1)^{nk}q^{nk(k-1)/2}[2k-1]_q. \tag{II.11}$$

Since $[-a]_q=-[a]_q$, the inner summand scales by $(-1)^{n+1}=1$ under the transformation $k\to 1-k$; indeed, under this transformation, the term $A^{4k-2}/(A^2-A^{-2})$ in $[2k-1]_q$ transforms to the other half of $[2k-1]_q$. Therefore, replacing the double summation by one over a diamond for $(b,\ k)$ defined by $\max(b+1-r,\ 1-b)\leqslant k \leqslant \min(b,\ r-b)$ doubles the result, while if one of the terms of $[2k-1]_q$ is simultaneously removed, then the result remains unchanged. Next, change variables to $x=k+b$ and $y=k-b$; the region of summation now consists of those integer points $(x,\ y)\in[1,\ r]\times[1-r,\ 0]$ for which $x+y$ is even. Thus

$$G_\delta Z_r(M_{n,t}) = \frac{(-1)^t A^{n-t-2}}{(A^2-A^{-2})^2}\sum_{x=1}^{r}\ \sum_{\substack{y=1-r\\ x\equiv y(2)}}^{0}(-1)^{(x(t+1)+y(2n-t-1))/2}A^{1/4(t+n)(x^2+y^2)+1/2(3n-t)xy}$$
$$\times(A^{(3-n)x+(1-n)y}-A^{(1-n)x+(3-n)y}). \tag{II.12}$$

For the rest of this paper we assume that r is odd. Under the transformation $b\to r-b$, the summand in (II.11) scales by a factor $(-1)^{r+(r+b)(t-n)}A^{(t-n)r^2}$, since $[r-b]_q=[b]_q$. Since r is odd, this means that a factor $(1-(-A^{r^2})^{t-n})$ may be extracted from the sum by restricting the region of summation to even b. In (II.12), this results in a restriction of the region of summation to those integer points in $[1,\ r]\times[1-r,\ 0]$ whose coordinates are congruent modulo 4. Thus, putting $q=A^4$,

$$G_\delta Z_r(M_{n,t})=\frac{(-1)^t A^{n-t-2}}{(A^2-A^{-2})^2}\,(1-(-A^{r^2})^{t-n})\sum_{x=1}^{r}\sum_{\substack{y=1-r\\ x\equiv y(4)}}^{0} s_{x,y}(q^{g(x,y)}-q^{g(y,x)}), \tag{II.13}$$

where

$$s_{x,y}=(-1)^{(x(t+1)+y(2n-t-1))/2},$$

$$g(x,\,y)=\tfrac{1}{16}(t+n)(x^2+y^2)+\tfrac{1}{8}(3n-t)xy+\tfrac{1}{4}(3-n)x+\tfrac{1}{4}(1-n)y.$$

Observe that $g(x,\,y)$ takes integer values while $s_{x,y}=(-1)^{ny}$ on the region of summation in (II.13). Since r is odd, we may define a $\mathbf{Z}/r\mathbf{Z}$-valued function $g_1(x,\,y)$ on $\mathbf{Z}\times\mathbf{Z}$ for which $16g(x,\,y)\equiv 16g_1(x,\,y)(r)$. Clearly $g_1(x,\,y)$ is well defined on $\mathbf{Z}/r\mathbf{Z}\times\mathbf{Z}/r\mathbf{Z}$, that is, $g_1(x+r,\,y)=g_1(x,\,y+r)=g(x,\,y)$ as elements of $\mathbf{Z}/r\mathbf{Z}$. The double sum in (II.13) can now be rewritten as

$$\begin{aligned}
&\sum_{x=1}^{r}\sum_{\substack{y=1-r\\ y\equiv x(4)}}^{0}(-1)^y q^{g_1(x,\,y)}+(-1)^{x-r}q^{g_1(y,\,x)}\\
&=\sum_{x=1}^{r}\sum_{\substack{y=1-r\\ y\equiv x(4)}}^{0}(-1)^y q^{g_1(x,\,y)}+\sum_{x=1-r}^{0}\sum_{\substack{y=1\\ y-r\equiv x+r(4)}}^{r}(-1)^x q^{g_1(y-r,x+r)}\\
&=\sum_{x=1}^{r}\sum_{\substack{y=1-r\\ y\equiv x(2)}}^{0}(-1)^y q^{g_1(x,\,y)}=(-1)^{(r-1)/2}\sum_{a=(1-r)/2}^{(r-1)/2}\sum_{\substack{b=(1-r)/2\\ b\equiv a+1(2)}}^{(r-1)/2}(-1)^b\\
&\qquad\times q^{g_1(a+(1+r)/2,b+(1-r)/2)},
\end{aligned}$$

where in the last line we have changed variables according to $x=a+\frac{1}{2}(1+r)$ and $y=b+\frac{1}{2}(1-r)$. The exponent of q in the last line may be rewritten as $\frac{1}{4}(3a+b+2)+G(a,\,b)$ where

$$G(a,\,b)=\frac{n}{8}\,(a+b)^2+\frac{t-n}{16}\,(a-b)^2-\frac{n}{8}\in\mathbf{Z}/r\mathbf{Z}.$$

Combining this with (II.9) in (II.13) and observing that $-A^{r^2}=A^{-r^2}$, we obtain

$$Z_r(M_{n,t})=(-1)^{t+\alpha}A^{n-t+\delta}\,\frac{1+A^{-(n+t)r^2}}{1+A^{-\delta r^2}}\,\frac{\Sigma_{(a,\,b)\in X}(-1)^b q^{G(a,\,b)+(1/4)(3a+b)}}{(q^{-\delta}-1)\Sigma_{a=0}^{r-1}q^{\delta a^2}}, \tag{II.14}$$

where $\alpha=(r-1)/2$ and X denotes the set of integral points in $[-\alpha,\,\alpha]\times[-\alpha,\,\alpha]$ whose coordinates have opposite parity.

It may now be verified that the expression on the right-hand side of (II.14) is invariant under the simultaneous transformation on the variables t, n, and q in which $t\to -t$, $n\to -n$, and $q\to q^{-1}$, so that the manifold $M_{n,t}$ changes to that with the opposite orientation. The only term for which this is not immediate is the sum over X in the numerator. For this term note that G transforms to $-G$ under $t\to -t$, $n\to -n$ so that applying the transformation $(a,\,b)\to(-a,\,-b)$ now verifies invariance.

For the purposes of investigating the behavior of $Z_r(M_{n,t})$ as a function of q when r varies, it is convenient to consider a slight variant of $Z_r(M_{n,t})$ defined as follows. Let $c_r(m)=A^{-m}(1+A^{-mr^2})$. Let $\bar{Z}_r(M_{n,t})$ be given by $(-1)^{t+\alpha}A^{2n}$ times the ratio of the sums involved in (II.14), so that

$$Z_r(M_{n,t})=\frac{c_r(n+t)}{c_r(\delta)}\bar{Z}_r(M_{n,t}).$$

In particular, $\bar{Z}_r(M_{n,t})$ can be written as a polynomial in q.

III. CONGRUENCE PROPERTIES OF SUMS OVER SPECIAL SETS

Throughout this section, it is assumed that $r=2\alpha+1$ is an odd prime. All parameters are integral and all congruences are modulo r, unless otherwise stated. Also, X denotes the subset of $[-\alpha,\alpha]\times[-\alpha,\alpha]$ consisting of integral points whose coordinates have opposite parity.

A. Gauss sums

Before investigating the behavior of the sum in the numerator of the right-hand side of (II.14), we will consider that of the Gauss sum in the denominator. Put $q=e^k$. Then

$$\sum_{a=0}^{r-1} q^{\epsilon a^2}=\sum_{i=0}^{\infty}\frac{(\epsilon k)^i}{i!}\sum_{a=0}^{r-1}a^{2i}.$$

However, $\{1,\ldots,r-1\}$ forms a cyclic group of order $r-1$ under multiplication modulo r and therefore,

$$\sum_{a=1}^{r-1} a^i\equiv\begin{cases}-1, & \text{if } (r-1)|i,\\ 0, & \text{otherwise.}\end{cases}$$

Hence

$$\sum_{a=0}^{r-1} q^{\epsilon a^2}=-\frac{(\epsilon k)^{(r-1)/2}}{((r-1)/2)!}+O(k^{r-1}). \tag{III.1}$$

This equation holds modulo r, in the sense that the coefficient of k^i in the expansion of the left-hand side is a rational whose denominator is not divisible by r, for $0\leq i<r-1$. Furthermore, for each i, this rational is congruent modulo r to the coefficient of k^i in the right-hand side.

B. Sums over X

Next consider sums of the form involved in the numerator of (II.14), that is, of functions $(-1)^y f(x,y)$ over the region X, where f is a polynomial in a and b. Let $\binom{x}{a}$ denote the usual binomial coefficient with $\binom{x}{0}$ defined to be 1 for all x. Now the functions $\{a!\binom{x/2}{a}|a\in\mathbf{N}\cup\{0\}\}$ form a basis over $\mathbf{Z}[1/2]$ for polynomial functions in x with coefficients in $\mathbf{Z}[1/2]$. Indeed, working modulo r, only those basis elements with $a<r$ are relevant. Therefore, it is only necessary to compute sums of the form

$$S_{a,b}=\sum_{(x,y)\in X}(-1)^y\binom{x/2}{a}\binom{y/2}{b},$$

where $0\leq a<b<r$. Note that $S_{a,b}$ is antisymmetric in a and b.

Lemma III.2:

$$S_{a,b}\equiv\begin{cases}0, & \text{if } \ 0\leq a<b<r-1,\\ (-1)^\alpha\left\{\binom{1/4}{a+1}-\binom{3/4}{a+1}\right\}, & \text{if } 0\leq a<b=r-1.\end{cases}$$

Proof: Assume that $0\leq a<b<r$. Let $c_a^i=\sum_{x=-\alpha, x\equiv i(2)}^{\alpha}\binom{x/2}{a}$ for $i=0,1$. Then $S_{a,b}=c_a^1c_b^0-c_a^0c_b^1$. Observe that

$$c_a^0+c_a^1\equiv\sum_{\substack{x=-\alpha\\ x\ \text{even}}}^{\alpha+r}\binom{x/2}{a}=\binom{[(\alpha+r)/2]+1}{a+1}-\binom{-[\alpha/2]}{a+1}\equiv 0.$$

Thus $S_{a,b}\equiv -c_a^0(c_b^0+c_b^1)$ and the first case of the lemma is now immediate.

Next note that $\binom{x}{r-1}\equiv 0$ unless $x\equiv -1$, in which case it is $\equiv 1$. Thus $c_{r-1}^0+c_{r-1}^1\equiv 1$, so that

$$S_{a,r-1}\equiv -c_a^0=\sum_{x=-[\alpha/2]}^{[\alpha/2]}\binom{x}{a}=\binom{[\alpha/2]+1}{a+1}-\binom{-[\alpha/2]}{a+1}.$$

Finally, from the observation that

$$\left[\frac{\alpha}{2}\right]\equiv\begin{cases}-1/4, & \text{if } \alpha \text{ is even},\\ -3/4, & \text{if } \alpha \text{ is odd},\end{cases}$$

the lemma now follows, since $x\equiv y$ implies $\binom{x}{a}\equiv\binom{y}{a}$ so long as $0\leq a<r$. ■

Lemma III.3: Suppose that $f(x, y)$ is a polynomial function over $\mathbf{Z}[1/2]$, of x and y. Let $F(x)$ denote the sum, over all positive integers j, of the coefficient of $y^{j(r-1)}$ in $f(2x, 2y)-f(2y, 2x)$. Then

$$\sum_{(x,y)\in X}(-1)^y f(x,y)\equiv\frac{1}{2}(-1)^{\alpha}F(\bar{B}).$$

Proof: Let $f_{a,b}$ denote the coordinates of $f(x, y)$ with respect to the basis $\mathbf{e}_{a,b}=\binom{x/2}{a}\binom{y/2}{b}$. Then, by Lemma III.2,

$$\sum_{(x,y)\in X}(-1)^y f(x,y)\equiv(-1)^{\alpha}\sum_{a=0}^{r-1}\left\{\binom{1/4}{a+1}-\binom{3/4}{a+1}\right\}(f_{a,r-1}-f_{r-1,a}).$$

However, with respect to the basis $\{\binom{x}{a}|0\leq a<r\}$ for polynomials in x, the coefficient of $\binom{x}{r-1}$ in x^a is

$$\equiv\begin{cases}0, & \text{for } 0\leq a<r-1,\\ -1, & \text{for } a\equiv 0(r-1) \text{ and } a\neq 0.\end{cases}$$

Thus

$$f_{a,r-1}-f_{r-1,a}\equiv\sum_{j=1}^{\infty}\left(\text{coeff. of }\binom{x}{a}y^{j(r-1)}\text{ in } f(2y,2x)-f(2x,2y)\right)$$

$$=-\text{coeff. of }\binom{x}{a}\text{ in } F(x).$$

However, for any polynomial g,

$$\sum_{a=0}^{\infty}\left\{\binom{3/4}{a+1}-\binom{1/4}{a+1}\right\}\left(\text{coeff. of }\binom{y}{a}\text{ in } g(y)\right)=\int_{B+1/4}^{B+3/4}g(y)dy=\frac{g(\bar{B})}{2},\qquad\text{(III.4)}$$

using (II.6), while the summand is $\equiv 0$ for $a \geq r$. The lemma now follows by combining these results. ■

In particular, if f has degree less than $r-1$ in x and y independently, then

$$\sum_{(x,y)\in X} (-1)^y f(x,y) \equiv 0.$$

IV. ASYMPTOTIC EXPANSIONS

In this section we use the results of Sec. III to derive an asymptotic expansion for $Z_r(M_{n,t})$. Throughout this section r will be an odd prime while n and t are integers with n odd and $t \neq -n$. Also, $q = \mathbf{e}^k$.

A. Proof of integrality

Theorem IV.1. *For any odd prime r, $\bar{Z}_r(M_{n,t})$ may be represented by a polynomial in q with integer coefficients.*

Proof: By (II.14), we know that $\bar{Z}_r(M_{n,t})$ can be written in the form

$$\bar{Z}_r(M_{n,t}) = (-1)^{t+\alpha} A^{2n} \frac{\Sigma_{(a,b)\in X}(-1)^b q^{G(a,b)+1/4(3a+b)}}{(q^{-\delta}-1)\Sigma_{a=0}^{r-1} q^{\delta a^2}}. \tag{IV.2}$$

Let R_r denote the quotient of the ring of polynomials in q with rational coefficients by the ideal generated by $(q^r-1)/(q-1)$ and let $f_r(q)$ denote the element given by the right-hand side of (IV.2). Next observe that in R_r, $(\Sigma_{a=0}^{r-1} q^{\delta a^2}) \cdot (\Sigma_{a=0}^{r-1} q^{-\delta a^2}) = r$ so that

$$f_r(q) = \frac{(-1)^{t+\alpha}}{r(q^{-\delta}-1)} \sum_{(a,b)\in X} \sum_{c=0}^{r-1} (-1)^b q^{G(a,b)+1/4(3a+b)+n/2-\delta c^2}. \tag{IV.3}$$

Since $q^r = 1$ in R_r, the sum on the right-hand side of this last expression may be considered as a polynomial in q of degree at most $r-1$ and such that the coefficient of q^p, for $0 \leq p < r$, is

$$\sum_{(a,b)\in X} (-1)^b \cdot \# \left\{ c \in \mathbf{Z}/r\mathbf{Z} \,\middle|\, G(a,b) + \frac{1}{4}(3a+b) + \frac{n}{2} - \delta c^2 \equiv p \right\}$$
$$\equiv \sum_{(a,b)\in X} (-1)^b \delta^{(r-1)/2} \left(G(a,b) + \frac{1}{4}(3a+b) + \frac{n}{2} - p \right)^{(r-1)/2}$$

since $\#\{c \in \mathbf{Z}/r\mathbf{Z} | c^2 \equiv x\} \equiv 1 + x^{(r-1)/2}$. Using Lemma III.3 it follows that these coefficients are all integers divisible by r. Also, the sum of all these coefficients is

$$\sum_{(a,b)\in X} (-1)^b \cdot r = 0.$$

Therefore, $(q^{-\delta}-1) f_r(q)$ is equal in R_r to a polynomial in q with integer coefficients of sum zero. Hence $f_r(q)$ is equal in R_r to a polynomial in q whose coefficients are rationals which have denominators not divisible by r. ■

The polynomial in q for $\bar{Z}_r(M_{n,t})$, whose existence is given by this theorem, when considered as a polynomial in $h = q-1$, has the coefficients of h^i uniquely determined, modulo r, for $0 \leq i \leq (r-3)/2$. Indeed, Theorem IV.1 is a special case of a theorem of Murakami;[20] however, it is instructive to use the techniques of Sec. III to verify its validity for our particular manifolds.

B. Computation of asymptotic expansion

Consider the summation in the numerator of the right-hand side of (IV.2). The coefficient of k^i in this sum is

$$\frac{1}{i!}\sum_{(x,y)\in X}(-1)^y\left(G(x,y)+\frac{1}{4}(3x+y)\right)^i, \tag{IV.4}$$

the summand of which is a sign, times a polynomial in x and y of degree $2i$ with coefficients in $\mathbf{Z}[1/2]$. Suppose that $i<r$. Then by Lemma III.3, the sum in (IV.4) is congruent modulo r to

$$\frac{(-1)^\alpha}{2^{i+1}\cdot i!}\{\text{coeff. of } y^{r-1} \text{ in } (2G(2\bar{B},2y)+3\bar{B}+y)^i-(2G(2\bar{B},2y)+\bar{B}+3y)^i\},$$

and in particular is $\equiv 0$ when $2i\leqslant r-1$. Put $i=j+(r+1)/2$ and observe that $i!\equiv((r+1)/2)!(3/2)\cdots(j+1/2)$. Hence the sum in the numerator of (IV.2) has the property that coefficients of k^i for $i\leqslant r-1$ are congruent to those in

$$\frac{(-1)^\alpha(k/2)^{(r+1)/2}}{2((r+1)/2)!}\sum_{j=0}^{(r-3)/2}\frac{(k/2)^j}{(3/2)\cdots(j+1/2)}(\text{coeff. of } y^{r-1}$$
$$\text{in } (2G(2\bar{B},2y)+3\bar{B}+y)^{j+(r+1)/2}-(2G(2\bar{B},2y)+\bar{B}+3y)^{j+(r+1)/2}). \tag{IV.5}$$

Lemma IV.6: Suppose that $g(y)=ay^2+by+c$ for some $a,b,c\in\mathbf{Z}[1/2]$ with $a\not\equiv 0$ and $0\leqslant j\leqslant(r-3)/2$. Then

$$\sum_{j=0}^{(r-3)/2}\frac{(k/2)^j}{(3/2)\cdots(j+1/2)}(\text{coeff. of } y^{r-1} \text{ in } g(y)^{j+(r+1)/2})$$
$$=O(k^{(r-1)/2})+\frac{a^{(r-1)/2}}{k}(q^{1/2(c-b^2/4a)}-1).$$

Proof: The coefficient of y^{r-1} in $g(y)^{j+(r+1)/2}$ is easily seen to be

$$\sum_{p=0}^{j+1}a^{(r-1)/2-p}b^{2p}c^{j+1-p}\binom{j+p+1}{2p}\binom{j+(r+1)/2}{j+p+1},$$

and so the left-hand side of the equality in the statement of the lemma is

$$a^{(r-1)/2}c\sum_{j=0}^{(r-3)/2}\frac{(ck/2)^j}{(3/2)\cdots(j+1/2)}\sum_{p=0}^{j+1}\left(\frac{b^2}{ac}\right)^p\binom{j+p+1}{2p}\binom{j+1/2}{j+p+1}.$$

However,

$$\sum_{j=0}^{\infty}\sum_{p=0}^{j+1}\frac{x^jy^p}{(3/2)\cdots(j+1/2)}\binom{j+p+1}{2p}\binom{j+1/2}{j+p+1}=\frac{1}{2x}(e^{x(1-y/4)}-1),$$

from which the result is derived by combining it with the last statement. ■

Considering both $2G(2x,2y)+3x+y$ and $2G(2x,2y)+x+3y$ as quadratic functions of y in the form of $g(y)$ in the above lemma, we find that both have $a=(t+n)/2$. Applying the above lemma to (IV.5) and combining it with (III.1), we deduce that the coefficients of k^i in an expansion of the right-hand side of the expression for $\bar{Z}_r(M_{n,t})$ in (IV.2), for $0\leq i\leq(r-3)/2$, are congruent modulo r to those in

$$(-1)^t A^{2n}|t+n|^{(r-1)/2}\,\frac{q^{\Delta_1(\bar{B})}-q^{\Delta_2(\bar{B})}}{2(1-q^{-\delta})},\tag{IV.7}$$

where $\Delta_1(x)$ and $\Delta_2(x)$ are the values of $1/2(c-b^2/4a)$ for the two quadratics. This holds for $t+n\not\equiv 0$; when $t+n\equiv 0$ the first $(r-1)/2$ coefficients of k^i are all divisible by r. Note that in general $|t+n|^{(r-1)/2}$ is congruent to the quadratic residue of $|t+n|$ modulo r. Indeed,

$$\begin{aligned}2(t+n)\Delta_1(x)&=4n(t-n)x^2+4tx-\tfrac{1}{4}n(t+n)-\tfrac{1}{2},\\ 2(t+n)\Delta_2(x)&=4n(t-n)x^2+4(t-2n)x-\tfrac{1}{4}n(t+n)-\tfrac{9}{2}.\end{aligned}\tag{IV.8}$$

The expression in (IV.7) should be interpreted purely formally as representing a power series in $h=q-1$ or in k. To find the coefficient of a certain power of h in the series, that coefficient is found in the power series for the exponentials in the numerator and the resulting polynomial in $\bar{B}$ is evaluated with $\bar{B}^n$ replaced by $\bar{B}_n$. We have now arrived at the following theorem.

Theorem IV.9: *When r an odd prime, the polynomial in* $h=q-1$ *representing* $\bar{Z}_r(M_{n,t})$ *given by Theorem IV.1 has the property that the coefficients of* h^i *in it, for* $0\leq i\leq(r-3)/2$, *are congruent modulo r to those in* $|t+n|^{(r-1)/2}Z_\infty(M_{n,t})$ *where*

$$Z_\infty(M_{n,t})=(-1)^t A^{2n}\,\frac{q^{\Delta_1(\bar{B})}-q^{\Delta_2(\bar{B})}}{2(1-q^{-\delta})}\tag{IV.10}$$

and $\delta=\mathrm{sgn}(t+n)$ *while* $\Delta_1(x)$ *and* $\Delta_2(x)$ *are given by (IV.8).*

Example IV.11: The coefficient of h^0 in $Z_\infty(M_{n,t})\in\mathbf{Q}[[h]]$ is

$$\frac{1}{2}(-1)^t\delta(\Delta_1(\bar{B})-\Delta_2(\bar{B}))=(-1)^t\delta\,\frac{2n\bar{B}+1}{t+n}=\frac{(-1)^t}{|t+n|},$$

since $\bar{B}_0=1$ and $\bar{B}_1=0$.

Example IV.12: The coefficient of h^1 in $Z_\infty(M_{n,t})$ is

$$(-1)^t\,\frac{\delta}{8}(\Delta_1(\bar{B})-\Delta_2(\bar{B}))(n-t+3\delta+2\Delta_1(\bar{B})+2\Delta_2(\bar{B}))$$

$$=\frac{(-1)^t\delta}{4|t+n|^2}(3n^2-5-(n+t)^2+3\delta(t+n)).$$

Example IV.13: The manifold $M_{n,t}$ is a **Z**-homology sphere for $|t+n|=1$. In this case, $t+n=\delta$ and so $Z_r(M_{n,t})=\bar{Z}_r(M_{n,t})$. The formal power series in h given by $Z_\infty(M_{n,t})$ now has the property that the first $(r-1)/2$ coefficients are congruent modulo r to those in a rational polynomial representation of $Z_r(M_{n,t})$ not involving denominators divisible by r. This property uniquely identifies $Z_\infty(M_{n,t})$ and it is therefore the rational formal power series whose existence was proved by Ohtsuki in Ref. 19. By Example IV.11, the coefficient of h^0 is just 1. The coefficient of h^1 given in Example IV.12 reduces to $\delta(3n^2-3)/4$, which for odd integers n is always divisible by 6. Indeed, it is 6 times the Casson invariant of $M_{n,t}$, as was shown for general **Z**-homology spheres in Ref. 18.

Example IV.14: The expression (IV.10) may be used to determine the asymptotic growth of the coefficients of powers of k in the formal power series for $Z_\infty(M_{n,t})$. Observe first that if $B_m(x)$ denotes the nth Bernoulli polynomial, then

$$(\bar{B}+\alpha)^m=\frac{2}{m+1}\left\{B_{m+1}\left(\frac{3}{4}+\alpha\right)-B_{m+1}\left(\frac{1}{4}+\alpha\right)\right\}\sim 8(2\pi)^{-m-1}m!\,\sin\left\{\frac{\pi}{2}(m+1)-2\pi\alpha\right\},$$

for large m. The second step uses $B_m(x)=-m!(2\pi i)^{-m}\Sigma_{s\neq 0}s^{-m}e^{2\pi isx}$ which is valid for $0\leqslant x<1$ so that this step is valid for $0<|\alpha|<1/4$. However, $B_m(x+1)=B_m(x)+mx^{m-1}$ and therefore whenever $4\alpha\notin\mathbf{Z}$, the above estimate on the growth of $\bar{B}_m$ remains valid. The coefficient of k^m in the expansion of $q^{a(\bar{B}+\alpha)^2}$ is therefore

$$\sim\frac{4}{\pi}\frac{(2m)!}{m!}\left(-\frac{a}{4\pi^2}\right)^m\cos 2\pi\alpha, \tag{IV.15}$$

for $\alpha\notin\frac{1}{4}+\frac{1}{2}\mathbf{Z}$, while the coefficient of k^m in the expansion of $q^{f(x)}$ where $f(x)=ax^2+bx+c$ grows as in (IV.15) with $\alpha=b/2a$. Since the leading terms in $\Delta_1(x)$ and $\Delta_2(x)$ are identical and nonzero for $n\neq t$, the coefficient of k^{m-1} in the expansion of $Z_\infty(M_{n,t})$ is therefore

$$\sim\frac{2(-1)^t}{\pi}\frac{(2m)!}{m!}\left(\frac{n(n-t)}{2\pi^2(n+t)}\right)^m\left(\cos\frac{\pi t}{n(t-n)}-\cos\frac{\pi(t-2n)}{n(t-n)}\right).$$

The ratio of the mth to the $(m-1)$-th coefficient in the expansion in powers of k (and hence also for the expansion in powers of h) therefore grows with m as

$$\sim\frac{2n(n-t)}{\pi^2(n+t)}m,\quad \text{for } n\neq t. \tag{IV.16}$$

When $n=t$, $\Delta_1(x)$ and $\Delta_2(x)$ are both linear functions and the formal power series defined by (IV.10) has a positive radius of convergence; see Sec. VI for a closed formula for the holomorphic function so defined.

Note that Theorem IV.9 constructively demonstrates the existence of Ohtsuki-type formal power series even for manifolds which are not integral homology spheres; see Ref. 22. However, in such cases care must be taken over the normalization of $Z_r(M_{n,t})$ which is employed. In the case of the manifolds $M_{n,t}$ discussed in this paper, we have seen that the appropriate normalization is

$$\frac{c_r(\delta)}{c_r(n+t)}Z_r(M_{n,t}),$$

where $c_r(m)=A^{-m}(1+A^{-mr^2})$, the remaining dependence on r being by a factor which is the quadratic residue of $|t+n|$.

C. Integrality for homology spheres

It follows immediately from (IV.10) that the coefficients in the expansion of $Z_\infty(M_{n,t})$ in powers of h are all rational. In this section the denominators in these coefficients will be discussed.

In order to do this, it will be convenient to express polynomials in a single variable, x say, in the form of linear combinations of binomial coefficients $\binom{x}{m}$, for $m\geqslant 0$. For any ring $R\subset\mathbf{Q}$, let

$$P(x,R)=\left\{\sum_{m=0}^{\infty}a_m\binom{x}{m}\,\middle|\,a_m\in R \text{ are almost all zero}\right\}.$$

Since $f(x+1)-f(x)=\Sigma_{a=0}^{\infty}c_{a+1}\binom{x}{a}$ when $f(x)=\Sigma_{a=0}^{\infty}c_a\binom{x}{a}$, the following lemma may be obtained by induction.

Lemma IV.17: A polynomial $f(x)$ lies in $P(x, R)$ if, and only if, $f(x)\in R$ for all $x\in\mathbf{Z}$.

Lemma IV.18: If $a>0$, u and $v \neq 0$ are integers, then $v^{2a}\binom{u/v}{a}\in\mathbf{Z}$.

Proof (I thank G. Freiman for providing this proof): Now $v^{2a}\binom{u/v}{a}=(v^a/a!)u(u-v)\cdots(u-(a-1)v)$. Suppose that p is a prime dividing both v and $a!$. The greatest power of p dividing $a!$ is p^t where

$$t=\left[\frac{a}{p}\right]+\left[\frac{a}{p^2}\right]+\cdots\leqslant\frac{a}{p}+\frac{a}{p^2}+\cdots=\frac{a}{p-1}\leqslant a.$$

Thus any prime dividing v will appear in $a!$ with a power which is at most that in which it appears in v^a.

Suppose now that p is a prime dividing $a!$ but not v. Then the number of elements of $\{u,u-v,\ldots,u-(a-1)v\}$ divisible by p^m will be either $[a/p^m]$ or $[a/p^m]+1$, for any $m\in\mathbf{N}$. The number of powers of p dividing $u(u-v)\cdots(u-(a-1)v)$ is therefore at least $[a/p]+[a/p^2]+\cdots$, which is the greatest power of p dividing $a!$. We have now shown that any prime dividing $a!$ will appear in $a!$ with a power which is at most that in which it appears in $v^a u(u-v)\cdots(u-(a-1)v)$. ■

Combining the last two lemmas, we deduce that whenever $v,a\in\mathbf{N}$ and $k,l\in v^{-1}\mathbf{Z}$, $v^{2a}\binom{kx+lx^2}{a}\in P(x,\mathbf{Z})$, while $\binom{1/4}{a}-\binom{3/4}{a}\in 2^{-4a}\mathbf{Z}$. Recalling (III.4), we deduce that

$$\binom{k\bar{B}+l\bar{B}^2}{a}\in 2^{-4a-3}v^{-2a}\mathbf{Z},$$

so that $q^{k\bar{B}+l\bar{B}^2}\in\mathbf{Z}[1/2,\ 1/v][[h]]$. The following theorem now follows immediately from (IV.10), since $8(t+n)\Delta_i(x)$ is a quadratic polynomial with integer coefficients for $i=1,2$.

Theorem IV.19: *The coefficients of powers of h in $Z_\infty(M_{n,t})$, when considered as a formal power series in h, lie in $\mathbf{Z}[1/2,\ 1/|t+n|]$.*

For $\mathbf{Z}$-homology spheres, Theorem IV.19 gives $Z_\infty\in\mathbf{Z}[1/2][[h]]$. However, in this case, it is believed that one can obtain the following stronger result.

Conjecture IV.20: When $M_{n,t}$ is a $\mathbf{Z}$-homology sphere, $Z_\infty(M_{n,t})$ is a formal power series in $h=q-1$ with integer coefficients.

Discussion: Suppose that $M_{n,t}$ is an integral homology sphere, so that $|t+n|=1$ while $\delta=n+t$. Then by (IV.10), it suffices to show that $\frac{1}{2}A^{2n}(q^{\Delta_1(\bar{B})}-q^{\Delta_2(\bar{B})})\in\mathbf{Z}[[h]]$. By (IV.8), $-\delta(\Delta_i(x\delta)+n/2)=f_i(x)$ depends only on $\bar{n}=n\delta$ while

$$f_1(x)=2\bar{n}(2\bar{n}-1)x^2+2(\bar{n}-1)x-\frac{3\bar{n}}{8}+\frac{1}{4},$$

$$f_2(x)=2\bar{n}(2\bar{n}-1)x^2+2(3\bar{n}-1)x-\frac{3\bar{n}}{8}+\frac{9}{4}.$$

Since in our symbolic notation $f(\bar{B})=f(-\bar{B})$, (IV.10) may be reduced to

$$Z_\infty(M_{n,t})=\frac{\bar{q}^{f_2(\bar{B})}-\bar{q}^{f_1(\bar{B})}}{2(\bar{q}-1)},$$

where $\bar{q}=q^{-\delta}$. The conjecture now reduces to the question, does $q^{f_2(\bar{B})}-q^{f_1(\bar{B})}$ lie in $2\mathbf{Z}[[h]]$ for all odd integers $\bar{n}$?

Example IV.21: In the course of the discussion of Conjecture IV.20 it was observed that for the subfamily $\{M_{n,t} \| n+t|=1\}$ consisting of **Z**-homology spheres, $Z_\infty(M_{n,t})$ depends only on $\bar{n}=n(n+t)$, as a function of $\bar{q}=q^{-(n+t)}$. As a power series in $\bar{h}=\bar{q}-1$ one can obtain

$$Z_\infty(M_{n,t})=\alpha_0+\alpha_1\bar{h}+\alpha_2\bar{h}^2+\cdots,$$

where $\alpha_0=1$ and α_m is a polynomial in $\bar{n}$ of degree $2m$. By geometric arguments it follows that $Z_\infty=1$ when $\bar{n}=\pm 1$ and so one may write $\alpha_m=(-1)^m(\bar{n}^2-1)\beta_m/8$, for $m\in\mathbf{N}$, where β_m is a polynomial in $\bar{n}$ of degree $2m-2$. Indeed, it may be computed that $\beta_1=6$, $\beta_2=(25\bar{n}^2-16\bar{n}+3)/4$, and $\beta_3=(427\bar{n}^4-528\bar{n}^3+230\bar{n}^2-48\bar{n}+15)/48$. It can now be seen explicitly that these first few coefficients are integral. As was mentioned already in Example IV.13, $\alpha_1/6$ is always an integer, namely, the Casson invariant of the 3-manifold. From the above explicit calculation it can be seen that α_2 is divisible by 3 and is an odd multiple of $\alpha_1/6$.

Example IV.22: For the Poincaré homology sphere $M_{-3,2}$ we have $\delta=-1$ so that $\bar{q}=q$ and the first 14 terms of the expansion are

$$\begin{aligned}Z_\infty(M_{-3,2})=&1-6h+45h^2-464h^3+6224h^4-102\,816h^5+2\,015\,237h^6-45\,679\,349h^7\\&+1\,175\,123\,730h^8-33\,819\,053\,477h^9+1\,076\,447\,743\,008h^{10}\\&-37\,544\,249\,290\,614h^{11}+1\,423\,851\,232\,935\,885h^{12}\\&-58\,335\,380\,481\,272\,491h^{13}+\cdots.\end{aligned}$$

Some other computations of Z_∞ for a number of **Z**-homology 3-spheres in our series are collected below:

$$\begin{aligned}Z_\infty(M_{3,-4})=&1-6h+69h^2-1064h^3+20\,770h^4-492\,052h^5+13\,724\,452h^6\\&-440\,706\,098h^7+\cdots,\end{aligned}$$

$$Z_\infty(M_{-5,4})=1-18h+411h^2-12\,900h^3+523\,445h^4-\cdots,$$

$$Z_\infty(M_{5,-6})=1-18h+531h^2-21\,180h^3+1\,074\,975h^4-\cdots,$$

$$Z_\infty(M_{-7,6})=1-36h+1674h^2-106\,884h^3+8\,799\,855h^4-\cdots,$$

$$Z_\infty(M_{7,-8})=1-36h+2010h^2-152\,244h^3+14\,703\,739h^4-\cdots.$$

It can be seen that the coefficients grow very rapidly with the complexity of the manifold. By (IV.16), the ratio between mth and $(m-1)$-th coefficients in the expansion of $Z_\infty(M_{n,t})$ in powers of $\bar{h}$ for $|n+t|=1$ is asymptotically $2\bar{n}(1-2\bar{n})m/\pi^2$.

V. RECONSTRUCTION OF WRT INVARIANTS

In this section we show how the values of $\bar{Z}_r(M_{n,t})$ for odd primes r which are not factors of $t+n$ may be reconstructed from the formal power series $Z_\infty(M_{n,t})$. Throughout, r is an odd prime and all congruences are modulo r unless explicitly stated to the contrary. Let $\mathbf{Z}_r$ denote the set of those rationals whose denominators are not divisible by r. Let $\mathbf{Z}_r[[h]]$ denote the ring of formal power series in $h=q-1$ with rational coefficients whose denominators are coprime to r. We also use the notation $\{c\}_r$ to denote the quadratic residue of c modulo r; that is,

$$\{c\}_r=\begin{cases}-1, & \text{if } c \text{ is not a square in } \mathbf{Z}/r\mathbf{Z},\\ 0, & \text{if } c\equiv 0,\\ 1, & \text{otherwise.}\end{cases}$$

Thus the number of solutions in $\mathbf{Z}/r\mathbf{Z}$ to $x^2=c$ is $1+\{c\}_r$.

Lemma V.1: For any $a,s\in\mathbf{Z}/r\mathbf{Z}$,

$$\sum_x \{s+ax^2\}_r=\begin{cases} r\{s\}_r, & \text{if } a\equiv 0,\\ -\{a\}_r, & \text{if } a\not\equiv 0 \text{ and } s\neq 0,\\ (r-1)\{a\}_r, & \text{if } a\not\equiv 0 \text{ and } s\equiv 0,\end{cases}$$

where the sum is over a complete set of residues modulo r.

Proof: For $a\equiv 0$ or $s\equiv 0$, the statement is immediate. Assuming $a,s\not\equiv 0$, let $\bar{a}$ be the inverse to a in $\mathbf{Z}/r\mathbf{Z}$. Then

$$\begin{aligned}\sum_x \{s+ax^2\}_r&=\{a\}_r\sum_x \{s\bar{a}+x^2\}_r\\ &=\{a\}_r(\#\{(x,\ y)\in\mathbf{Z}/r\mathbf{Z}\times\mathbf{Z}/r\mathbf{Z}|x^2+s\bar{a}=y^2\}-r)\\ &=\{a\}_r(\#\{(u,\ v)\in\mathbf{Z}/r\mathbf{Z}\times\mathbf{Z}/r\mathbf{Z}|uv=-s\bar{a}\}-r)=-\{a\}_r,\end{aligned}$$

where in the penultimate step the change of variables $u=x-y$, $v=x+y$ has been employed. ■

For $a\not\equiv 0$, the result of the sum in the statement of Lemma V.1 can be written as $(r\delta_{s\equiv 0}-1)\{a\}r$, using the Dirac delta function δ_T which is 1 if T is true and 0 otherwise.

Lemma V.2: Suppose that $f(y)$ is a $\mathbf{Z}_r$-valued polynomial function and that a and b are rationals whose denominators are not divisible by r and whose difference is divisible by r. Then,

$$\frac{b-a}{r}\,(q^{f(0)}+\cdots+q^{f(r-1)})-\int_{B+a}^{B+b} q^{f(y)}\,dy$$

is divisible by q^r-1 in $\mathbf{Z}_r[[h]]$.

Proof: Without loss of generality, we may assume that $a<b$. Suppose first that $a,b\in\mathbf{Z}$ with $\epsilon=(b-a)/r\in\mathbf{N}$. Then

$$\int_{B+a}^{B+b} q^{f(y)}\,dy=\sum_{m=a}^{a-1+\epsilon r} q^{f(m)}=\sum_{l=0}^{\epsilon-1}\sum_{k=0}^{r-1} q^{f(a+lr+k)}=\epsilon(q^{f(0)}+\cdots+q^{f(r-1)}),$$

using $q^r=1$ along with the fact that $x\equiv y$ implies $f(x)\equiv f(y)$. As formal power series in h, the above equalities hold modulo q^r-1. The result thus holds whenever a and b are integers.

Next observe that if $K(h)\in\mathbf{Z}_r[[h]]$, then the statement that $K(h)$ is divisible by q^r-1 in $\mathbf{Z}_r[[h]]$ can be written as a countable sequence of $\mathbf{Q}$-congruences (modulo r), each of which involves only a finite number of coefficients of powers of h in $K(h)$. Since the coefficient of h^i in the expression in the statement of the lemma is a polynomial in a and b (whose degree depends linearly on i) and the statement of the lemma holds for integer values of a and b, it therefore holds for all $a,b\in\mathbf{Z}_r$. ■

Lemma V.3: Suppose that $Q(x,\ y)$ is a quadratic polynomial in x and y whose homogeneous part of degree 2 is symmetric in x and y. Assume that the common coefficient of x^2 and y^2 in $Q(x,\ y)$ is $a\not\equiv 0$. Let $\Delta(x)$ and $\Delta'(x)$ denote the values in $\mathbf{Z}/r\mathbf{Z}$ of the discriminant $c-b^2/4a$ for the two quadratics $Q(2x,\ 2y)$ and $Q(2y,2x)$, considered as quadratics in y with coefficients dependent on x. Then

$$\frac{q^{\Delta(\bar{B})}-q^{\Delta'(\bar{B})}}{2}\{a\}_r+\frac{1}{r}\sum_{p=0}^{r-1} q^p\sum_{(x,y)\in X}(-1)^y\{p-Q(x,\ y)\}_r$$

is divisible by q^r-1 in $\mathbf{Z}_r[[h]]$.

Proof: By definition, X consists of integral points in $[-\alpha,\alpha]\times[-\alpha,\alpha]$ for which the parities of the two coordinates are contrary. Thus

$$\sum_{(x,y)\in X}(-1)^y\{p-Q(x,y)\}_r=\sum_{\substack{x\ \text{odd}\\ y\ \text{even}}}\{p-Q(x,y)\}_r-\{p-Q(y,x)\}_r$$

$$=\sum_{\substack{x\\ y\ \text{even}}}\{p-Q(x,y)\}_r-\{p-Q(y,x)\}_r,$$

where the two sums on the right-hand side are over domains which are subsets of the set of all integer points in $[-\alpha,\alpha]\times[-\alpha,\alpha]$; the last step uses the antisymmetry of the summand. By Lemma V.1, the last sum can be evaluated to give

$$\sum_{y\ \text{even}}(r\delta_{p-\Delta'(y/2)\equiv 0}-1)\{-a\}_r-(r\delta_{p-\Delta(y/2)\equiv 0}-1)\{-a\}_r,$$

and therefore,

$$\frac{1}{r}\sum_{p=0}^{r-1}q^p\sum_{(x,y)\in X}(-1)^y\{p-Q(x,y)\}_r=\{-a\}_r\sum_{y\ \text{even}}q^{\Delta'(y/2)}-q^{\Delta(y/2)}$$

$$=\{-a\}_r\sum_{y=-[\alpha/2]}^{[\alpha/2]}q^{\Delta'(y)}-q^{\Delta(y)}.$$

By (II.3), the sum on the right-hand side may be rewritten as

$$S=\int_{B-[\alpha/2]}^{B+1+[\alpha/2]}(q^{\Delta'(y)}-q^{\Delta(y)})dy.$$

Next observe that

$$-\left[\frac{\alpha}{2}\right]=\frac{1}{4}-\frac{r}{4}\ \text{or}\ \frac{3}{4}-\frac{r}{4},$$

$$1+\left[\frac{\alpha}{2}\right]=\frac{3}{4}+\frac{r}{4}\ \text{or}\ \frac{1}{4}+\frac{r}{4},$$

according as $r\equiv 1$ or $3(4)$. Combining this with the fact that $\{-1\}_r=(-1)^{(r-1)/2}$ we obtain

$$S=\{-1\}_r\int_{B+1/4+(-1)^{\alpha+1}r/4}^{B+3/4+(-1)^{\alpha}r/4}(q^{\Delta'(y)}-q^{\Delta(y)})dy$$

$$=\{-1\}_r\int_{B+1/4}^{B+3/4}(q^{\Delta'(y)}-q^{\Delta(y)})dy+\frac{1}{2}\sum_{m=0}^{r-1}(q^{\Delta'(m)}-q^{\Delta(m)}),$$

up to the addition of a multiple of q^r-1 in $\mathbf{Z}_r[[h]]$, using Lemma V.2. However, it follows from their definition in terms of $Q(x,y)$ that $\Delta(x)$ and $\Delta'(x)$ are quadratics in x which share the same coefficient of x^2 and the same discriminant. The case when the coefficients of x^2 in $\Delta(x)$ and

$\Delta'(x)$ vanish modulo r may be dealt with independently. Working modulo r in this case, the linear terms differ by at most a sign and when they vanish $\Delta(x)=\Delta'(x)$. In all cases we conclude that the sum on the right-hand side of the last equation vanishes; the result follows using (II.6). ■

Theorem V.4: *Assume that n is an odd integer, r an odd prime, and $t\in\mathbf{Z}$ is such that $t+n$ is not divisible by r. That is, we assume that $M_{n,t}$ is a $\mathbf{Z}/r\mathbf{Z}$-homology sphere. Then $\bar{Z}_r(M_{n,t})-\{|t+n|\}_r Z_\infty(M_{n,t})$ is divisible by $(q^r-1)/(q-1)$ in the ring of formal power series in $h=q-1$ with rational coefficients whose denominators are not divisible by r.*

Proof: By (IV.2) and (IV.10), it is required to show that

$$(-1)^t A^{2n}\{|t+n|\}_r \frac{q^{\Delta_1(\bar{B})}-q^{\Delta_2(\bar{B})}}{2(1-q^{-\delta})}-(-1)^{t+\alpha}A^{2n}\frac{\Sigma_{(a,b)\in X}(-1)^b q^{G(a,b)+1/4(3a+b)}}{(q^{-\delta}-1)\Sigma_{a=0}^{r-1}q^{\delta a^2}}$$

is a multiple of $(q^r-1)/(q-1)$ in $\mathbf{Z}_r[[h]]$. Rewriting the expressions, it suffices to show that

$$\frac{q^{\Delta_1(\bar{B})}-q^{\Delta_2(\bar{B})}}{2}\{t+n\}_r+\frac{1}{r}\sum_{p=0}^{r-1}q^p\sum_{(a,b)\in X}(-1)^b\left\{p-G(a,b)-\frac{1}{4}(3a+b)\right\}_r$$

is a multiple of q^r-1 in $\mathbf{Z}_r[[h]]$. We have here used the fact that the quadratic residue is multiplicative and $\{-\delta\}_r=(-\delta)^\alpha$ for $\alpha=(r-1)/2$. Recall also that by construction, $\Delta_1(x/2)$ and $\Delta_2(x/2)$ are the values of $c-b^2/4a$ obtained by viewing $G(x,y)+(3x+y)/4$ and $G(x,y)+(x+3y)/4$ as quadratics in y. The result now follows immediately from Lemma V.3 when it is noted that the coefficient of the square terms in either of the quadratics just mentioned is $(t+n)/16$.■

Note that Theorem IV.9 may be deduced from Theorem V.4. The result of this theorem may be expressed alternatively as follows. Consider the natural map

$$\theta_r:\mathbf{Z}\left[\frac{1}{2},\frac{1}{t+n}\right][[h]]\rightarrow\frac{\mathbf{Z}_r[[h]]}{((q^r-1)/(q-1))\mathbf{Z}_r[[h]]},$$

given by dividing out by the ideal generated by the relation imposed on q by requiring it to be a root of unity of order r. Then Theorem V.4 states that

$$\bar{Z}_r(M_{n,t})=\{|t+n|\}_r\theta_r(Z_\infty(M_{n,t})),$$

so that if $Z_\infty(M_{n,t})\in\mathbf{Z}[1/2,\ 1/(t+n)][[h]]$ is known, then $\bar{Z}_r(M_{n,t})$ may be obtained as a polynomial function of q for any odd prime r which is not a divisor of

$$|t+n|=|H_1(M_{n+t},\mathbf{Z})|.$$

VI. EXTENSION TO HOLOMORPHIC FUNCTIONS

In this section we will investigate the extent to which $Z_\infty(M_{n,t})$ which is defined in (IV.10) as a formal series can be viewed as an asymptotic expansion of a holomorphic function of q.

Write $q=\mathbf{e}^k$. As a warm-up, we first investigate $q^{f(\bar{B})}$ where $f(x)$ is a real linear function, say, $ax+b$ for $a,b\in\mathbf{R}$. Using (II.4),

$$q^{a\bar{B}+b}=q^b\sum_{n=0}^{\infty}\frac{(ak)^n}{n!}\bar{B}_n=\frac{2q^b}{q^{a/4}+q^{-a/4}}.$$

Thus, for linear functions, f, $q^{f(\bar{B})}$ can be extended from a formal power series to a (possibly multivalued) holomorphic function. More generally, for any function $f(x)$, one may use the integral presentation (II.7) to view $q^{f(\bar{B})}$ as representing a (multivalued) holomorphic function of q, or a single-valued holomorphic function $g(k)$ of $k=\ln q$, defined by

$$q^{f(\bar{B})}=4\int_{-\infty}^{\infty}\frac{e^{kf(iz)}\,dz}{e^{2\pi z}+e^{-2\pi z}}, \tag{VI.1}$$

on the domain of q for which this integral converges. When f is a quadratic function, say, $f(x)=ax^2+bx+c$ for some $a,b,c\in\mathbf{R}$, the integral converges for $a=0$ and for $\Re(ak)>0$, that is, for $|q|>1$ or $|q|<1$ according to whether a is positive or negative.

However, it is also possible to consider the sum

$$2\sum_{m\in\mathbf{Z}+1/2}e^{\pi i(m+1/2)+kf(m/2)}, \tag{VI.2}$$

for functions f for which this sum converges. Observe that the integrand in (VI.1) has poles at $(m/2+1/4)i$ for $m\in\mathbf{Z}$ and the residues there are $e^{kf(-m/2-1/4)}/4\pi i(-1)^m$ so that a naive application of Cauchy's residue theorem would indicate that (VI.1) and (VI.2) represent the same function. In fact, though, for quadratic f as above, the sum (VI.2) converges for $\Re(ak)<0$, that is, for $|q|<1$ or $|q|>1$ according to whether a is positive or negative, so that the regions of definition of (VI.1) and (VI.2) are disjoint, for $a\neq 0$. Indeed, when $a\neq 0$, both (VI.1) and (VI.2) define holomorphic functions of $\ln q$ for which $\ln q=0$ is contained in the common boundary of their domains of definition and their asymptotic expansions about this point are both precisely the formal power series $q^{f(\bar{B})}$ previously defined.

Combining this with (IV.10) we obtain the following theorem.

Theorem VI.3: *For any odd integer n and integer $t\neq -n$, the formal power series $Z_\infty(M_{n,t})$ in $h=q-1$ may be viewed as the asymptotic expansion of the following holomorphic function of* $\ln q$:

$$Z_\infty(M_{n,t})=\begin{cases}2\,\dfrac{(-1)^t q^{n/2}}{1-q^{-\delta}}\displaystyle\int_{-\infty}^{\infty}\frac{q^{\Delta_1(iz)}-q^{\Delta_2(iz)}}{e^{2\pi z}+e^{-2\pi z}}\,dz,\\[2ex] \dfrac{(-1)^t q^{n/2}}{1-q^{-\delta}}\displaystyle\sum_{m=-\infty}^{\infty}(-1)^m\left(q^{\Delta_1(m/2-1/4)}-q^{\Delta_2(m/2-1/4)}\right),\end{cases}$$

according to whether the integral or sum converges, where $\delta=\mathrm{sgn}(t+n)$ and $\Delta_1(x),\Delta_2(x)$ are given by (IV.8). The domain of definition of Z_∞ is $\ln q\in\mathbf{C}\setminus i\mathbf{R}$, that is, $0<|q|<\infty$ with $|q|\neq 1$. The integral converges when $1<|q|<\infty$ or $0<|q|<1$ and the sum converges when $0<|q|<1$ or $1<|q|<\infty$, according to whether $n(t-n)/(t+n)$ is positive or negative, respectively. When it is zero so that $n=t$, the following closed expression may be obtained:

$$Z_\infty(M_{n,t})=\frac{(-1)^n q^{3n/8}(q^{-1/8n}-q^{-9/8n})}{(1-q^{-\delta})(q^{1/4}+q^{-1/4})},$$

where $q\neq 1$ and in addition $q=0$ or $q=\infty$ is excluded from the domain according to whether $n>0$ or $n<0$.

VII. CONCLUSIONS

In this paper, a detailed analysis of the $\mathfrak{sl}_2$ Witten–Reshetikhin–Turaev invariants for a specific family of 3-manifolds has been carried out. Although it may appear at first sight that the methods employed depended heavily on the simple state-sum form for the invariants which only exists for these particular manifolds, the author believes that similar results hold more generally.

*Conjecture VII.1: For an appropriate normalization $\bar{Z}_r(M)$ of the $\mathfrak{sl}_2$-WRT invariant, there is an invariant, $Z_\infty(M)$, of **Q**-homology 3-spheres, M, which takes values in formal power series in $h=q-1$ with coefficients in **Z** $[1/2, |H_1(M, Z)|^{-1}]$, such that $\bar{Z}_r(M)-Z_\infty(M)$ is divisible by $(q^r-1)/(q-1)$ in $\mathbf{Z}_r[[h]]$, for all odd primes r. Furthermore, $Z_\infty(M)$ can be expressed as an asymptotic expansion around* $\ln q=0$ *of a holomorphic function of* $\ln q$ *with domain* $\mathbf{C}\backslash i\mathbf{R}$.

By Ref. 22, it is known that for **Q**-homology spheres, a formal power series $Z_\infty(M)$ in $h=q-1$ exists with rational coefficients and such that the coefficient of h^i in it is congruent to that in $\bar{Z}_r(M)$ for almost all primes r and $0\leq i\leq(r-3)/2$. By Ref. 20, the first two terms of an expansion of $Z_\infty(M)$ are known for **Q**-homology spheres. Conjecture 1 would imply that *all* the information on $\mathfrak{sl}_2$-WRT invariants at prime roots of unity is contained in a new invariant which is a holomorphic function. It should also be possible to reconstruct from this function the values of $Z_r(M)$ when r is composite, but this is likely to involve some deep theory. When $\mathfrak{sl}_2$ is replaced by another Lie algebra, a conjecture of the same form as that above should hold and it is expected that, much as for Vassiliev invariants, there will be universal invariants out of which those for particular Lie algebras may be constructed.

*Conjecture VII.2: When M is a **Z**-homology 3-sphere,* $Z_\infty(M)\in\mathbf{Z}[[h]]$.

The main obstacle that must be overcome before these conjectures can be verified is a better understanding of the quantum $6j$-symbols entering state-sum expressions for the WRT invariant. The coefficients of powers of h in $Z_\infty(M)$ should match those coming from a formal perturbation expansion of the Witten's functional integral arising from Chern–Simons theory; see Ref. 25. That is, they should be given by a combinatorial sum, involving the graph cohomology of M, of finite-dimensional integrals over the configuration space of points in M. In particular, such coefficients should be 3-manifold invariants of *finite type*, playing a role similar to that played by Vassiliev invariants in the theory of link invariants in S^3. I refer the reader to Ref. 6 for the close connection between perturbative Chern–Simons theory for knots in S^3 with Vassiliev invariants, to Ref. 26 for a self-contained account of the pretty algebraic structures related to Vassiliev invariants, and to Refs. 27 and 28 for definitions of finite type for 3-manifold invariants.

Since, for **Z**-homology spheres, the coefficients of powers of h in $Z_\infty(M)$ are conjectured to be integral, it is natural to expect them to be counting some sets, with appropriate signs, much as the first coefficient, the Casson invariant, counts representations of the fundamental group into SU(2). If this is indeed so, it is interesting to speculate on how the objects being counted depend on the manifold (perhaps only via the fundamental group and signature information), on the power of h, and on the Lie group involved (perhaps via affine Lie groups at a suitable level).

A better combinatorial understanding of the WRT invariants for manifolds is undoubtedly needed and long overdue. For example, unlike the related invariants of Turaev and Viro for which a description is given in Ref. 29 in terms of a triangulation of the manifold, this has not yet been carried out for the WRT invariants. The generalizations and connections outlined here will be the subject of future work.

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