On the self-adjointness of the Lorentz generator for $(:\varphi^4:)_{1+1}$

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An alternative proof to that provided by Jaffe and Cannon of the self-adjointness of the local Lorentz generator for the $(:\varphi^4:)_{1+1}$ quantum field theory is given. The proof avoids the use of second-order estimates and a singular perturbation theory.

In this brief note, we establish the self-adjointness of the local Lorentz generator for the two-dimensional $:\varphi^4:$ interaction by the method of Ref. 1. This result has been previously obtained by Cannon and Jaffe[2] using first- and second-order estimates, and a singular perturbation theory. Here we avoid the use of second-order estimate and the Glimm—Jaffee singular perturbation theory.[3] It is hoped that a new proof may lead to some new results and insights.

The $(:\varphi^4:)_{1+1}$ quantum field theory has been brought to a very satisfactory stage mainly by the work of Glimm and Jaffe.[4] On the Fock space, they constructed a densely defined bilinear form $\varphi(x,t)$, continuous in x and t, which gives rise to a unique self-adjoint operator

$$\varphi(f)=\int dxdt\,\varphi(x,t)f(x,t) \tag{1}$$

for a real function $f\in C_0^\infty(R^2)$. The C^*-algebra of local observables is defined as the norm closure

$$=(\bigcup_B \ (B))^- . \tag{2}$$

Here the union is taken over bounded regions B of space—time and (B) is the weakly closed (von Neumann) algebra generated by

$$\{\exp[i\varphi(f)]: f=\bar{f}\in C_0^\infty(R)\}. \tag{3}$$

The Poincaré group $P=\{a,\Lambda\}$ is the semidirect product of R^2 with R^1,

$$\{a,\Lambda\}\{a',\Lambda'\}=\{a,\Lambda a',\Lambda\Lambda'\}, \tag{4}$$

where $a\in R^2$ is a space—time translation, $a=(\alpha,\tau)$, and Λ is the one-parameter Lorentz rotation

$$\Lambda_\beta: (x,t)\rightarrow(x\cosh\beta+t\sinh\beta, x\sinh\beta+t\cosh\beta). \tag{5}$$

Poincaré covariance means that there exists a representation

$$\sigma_{\{a,\Lambda\}}(\ (B))=\ (\{a,\Lambda\}B) \tag{6}$$

for all bounded open sets B and all $\{a,\Lambda\}\in P$. The covariance of the local algebras ensures the covariance of the field operators, namely

$$\sigma_{\{a,\Lambda\}}(\varphi(f))=\varphi(f_{\{a,\Lambda\}}) \tag{7}$$

with

$$f_{\{a,\Lambda\}}(x,t)=f(\{a,\Lambda\}^{-2}(x,t)). \tag{8}$$

Space—time covariance was proven by Glimm and Jaffe.[5] The time translation is implemented locally by a unitary operator $U(t;B)$, i.e.,

$$\sigma_t(\ (B))=U(t;B)\ (B)U^{-1}_{(t;B)} \tag{9}$$

with

$$U(t;B)=\exp[itH(g)], \tag{10}$$

where $H(g)$ is the Hamiltonian with a space cutoff $g(x)\in C_0^\infty(R)$, $g(x)\equiv 1$ on a sufficiently large set depending on B. The space translation is implemented by $\exp(-ixP$, where P is the free field momentum operator.

The pure Lorentz transformation is locally implemented by a unitary operator $U(\Lambda_\beta;B)$, i.e.,

$$\sigma_{\Lambda_\beta}(\ (B))=U(\Lambda_\beta;B)\ (B)U^{-1}(\Lambda_\beta;B). \tag{11}$$

The formal infinitesimal generator of Lorentz transformations in a region B is

$$\begin{aligned}M(g)&=M_0+M_I(g)\\&=\int xH_0(x)dx+\int xH_I(x)g(x)dx,\end{aligned} \tag{12}$$

where the space cutoff function $g=1$ on a sufficiently large interval. Here, $H(x)=H_0(x)+H_I(x)$ is the energy density. Using space—time covariance, Cannon and Jaffe showed that it suffices to consider region B of space—time in the domain $x>0$. Also, it is technically convenient to use different spatial cutoffs in the free and the interaction part of M. Thus, for a region B in $x>0$, we take

$$M=M(g_0,g)=M_0+M_1, \tag{13a}$$

$$M_0=\alpha H_0, \tag{13b}$$

$$M_1=H_0(xg_1)+H_I(xg_2), \tag{13c}$$

where $\alpha>0$, $xg_i(x)$, $xg_2(x)\geqslant 0$, $g_0(x), g(x)\in C_0^\infty(R^+)$, and

$$\alpha+xg_1(x)=x=xg_2(x) \tag{14}$$

for x in a sufficiently large interval of the positive x axis. Here we have defined $g_0(x)=xg_1(x)$, and $g(x=xg_2(x)$. The first step toward proving that $M=M(g_0,g)$ is the infinitesimal generator for local Lorentz rotations, is to prove the self-adjointness of M.

We write

$$\begin{aligned}M=\alpha H_0+H_{0,\kappa}(g_0)+H_{I,\kappa}(g)+[H_0(g_0)-H_{0,\kappa}(g_0)]\\+[H_I(g)-H_{I,\kappa}(g)],\end{aligned} \tag{15}$$

where as usual κ is an upper momentum cutoff. We first estimate each term in (15). By undoing the Wick ordering we obtain

$$H_{0,\kappa}(g_0)\geqslant -c_1\kappa^2, \tag{16}$$

$$H_{I,\kappa}(g)\geqslant -c_2(\ln\kappa)^2, \tag{17}$$

where c_1, c_2 are positive constants independent of κ. By a standard N_τ estimate[6]

$$\begin{aligned}&\|(N+1)^{-1}(H_\perp(g)-H_{I,\kappa}(g))(N+I)^{-1}\|\leqslant c_3\kappa^{-1/2},\\&\times c_3>0.\end{aligned} \tag{18}$$

To estimate the difference $H_0(g_0)-H_{0,\kappa}(g_0)$, we write

$$H_0(g_0)=H_0^{(1)}(g_0)+H_0^{(2)}(g_0), \tag{19}$$

with

$$H_0^{(1)}(g_0)=\frac{1}{2(2\pi)}\int dk_1dk_2\,\hat{g}_0(k_1-k_2)\frac{\mu(k_1)\mu(k_2)+K_1k_2+\mu_0^2}{[\mu(k_1)\mu(k_2)]^{1/2}}\times a(k_1^*)a(k_2) \tag{20}$$

$$H_0^{(2)}(g_0)=\frac{2}{2(2\pi)}\frac{1}{2}\int dk_1dk_2\,\hat{g}_0(k_1-k_2)\frac{-\mu(k_1)\mu(k_2)+K_1k_2+\mu_0^2}{[\mu(k_1)\mu(k_2)]^{1/2}}\times(a^*(k_1)a^*(-ka)+a(-k_1)a(k_2)). \tag{21}$$

$H_0^{(1)}(g_0)$ is a sum of three terms having the form $A^*K(g_0)A$ in configuration space, where $K(g_0)$ is a multiplication operator with a nonnegative kernel. Therefore, $H_0^{(1)}(g_0)$, and, similarly, $H_0^{(1)}(g_0)=H_{0,\kappa}^{(1)}(g_0)$ are nonnegative operators. Jaffe and Cannon proved that $H_0^{(2)}(g_0)$ has an L_2 kernel and

$$\|(N+I)^{-1/2}(H_I^{(2)}(g_0)-H_{0,\kappa}^{(2)}(g_0))(N+I)^{-1/2}\|\leqslant c_4\kappa^{-1/4},\ c_4>0. \tag{22}$$

Finally, we estimate the free term αH_0 by

$$\alpha H_0\geqslant\alpha\mu_0N. \tag{23}$$

Let P_n be the projection onto states with numbers of particles lying in the range

$$n^\beta\leqslant N<(n+2)^\beta,\quad \beta\geqslant 4. \tag{24}$$

We note

$$\sum_{n=\text{even}}P_n=\sum_{n=\text{odd}}P_n=I. \tag{24'}$$

Picking $\kappa_n=\exp[(1/c_2)n^{\beta/2}]$, and using (16), (17), (18), (22) and (23), we quickly obtain

$$P_n\alpha H_0P_n\geqslant\alpha\mu_0P_nNP_n\geqslant\alpha\mu_0n^\beta, \tag{25}$$

$$P_nH_{0,\kappa_n}(g_0)P_n\geqslant -C,\quad \exp(2/c_2),\ n^{\beta/2}, \tag{26}$$

$$P_nH_{I,\kappa_n}(g)P_n\geqslant -n^\beta, \tag{27}$$

$$P_n(H_0^{(1)}(g_0)-H_{0,\kappa}^{(1)}(g_0))P_n\geqslant 0, \tag{28}$$

$$\|P_n(H_I(g)-H_{I,\kappa}(g))P_n\|\leqslant d_1\exp(-d_1'N^{\beta/2},\quad d_1,d_1'>0,$$

$$\|P_n(H_0^{(2)}(g_0)-H_{0,\kappa}^{(2)}(g_0))P_n\|\leqslant d_2\exp(-d_2'N^{\beta/2}), \tag{29}$$

$$d_2,d_2'>0. \tag{30}$$

Using (25) through (30) and choosing an appropriate α, we get

$$M_n=P_nMP_n\geqslant dn^\beta P_n, \tag{31}$$

where d is a positive constant. For d large enough we get

$$b+M_n\geqslant dn^\beta. \tag{32}$$

Let M' be obtained from M by replacing αH_0 by αN, a multiple of the particle number operator. Then

$$b+M_n\geqslant b+M_n'=b+P_n(\alpha N+H_0(g_0)+H_1(g))Pn. \tag{33}$$

By a standard N_τ estimate

$$\|b+M_n'\|\leqslant d'N^{2\beta} \tag{34}$$

for some constant d'.

Following Ref. 7, we define P_e and P_d as the projection operators onto states with number of particles in the ranges

$$\underset{n=\text{even}}{U}(n^\beta-4\leqslant N\leqslant n^\beta+4) \tag{35}$$

and

$$\underset{n=\text{odd}}{U}(n^\beta-4\leqslant N\leqslant n^\beta+4), \tag{36}$$

respectively. We define M_e and M_d by

$$M_e=\sum_{n=\text{even}}M_n=\alpha H_0+\sum_{n=\text{even}}P_nM_1P_n, \tag{37}$$

$$M_d=\sum_{n=\text{odd}}M_n=\alpha H_0+\sum_{n=\text{odd}}P_nM_1P_n. \tag{38}$$

We write M in two different forms

$$M=M_e+L_e=M_d+L_d, \tag{39a}$$

$$L_e=M_1-\sum_{n=\text{even}}P_nM_1P_n, \tag{39b}$$

$$L_d=M_1-\sum_{n=\text{odd}}P_nM_1P_n, \tag{39c}$$

where the ranges (35) and (36) have been chosen so that

$$P_eL_eP_e=L_eP_e=P_eL_e=L_e, \tag{40}$$

$$P_dL_dP_d=L_dP_d=P_dL_d=L_d. \tag{41}$$

Federbush's expansion of the resolvent is

$$\begin{aligned}R(-b;M)&=R(-b;M_e)\\&\quad-R(-b;M_d)L_eR(-b;M_e)\\&\quad+R(-b;M_e)L_dR(-b;M_d)L_eR(-b;M_e)\\&\quad-\cdots,\end{aligned} \tag{42a}$$

$$\begin{aligned}&=R(-b;M_e)\\&\quad-R(-b;M_d)P_eL_eP_eR(-b;M_e)\\&\quad+R(-b;M_e)P_dL_dP_dR(-b;M_d)P_eL_eP_e\\&\qquad\times R(-b;M_d)\\&\quad-\cdots.\end{aligned} \tag{42b}$$

Our main result is the following theorem.

Theorem 1: Let g_0,g satisfy (14), with $g_0,g\geqslant 0$, $g_0,g\in C_0^\infty(R^+)$. Then there is a finite constant δ such that (42) converges in the uniform operator topology for $b>\delta$. The limit $R(-b)$ is the resolvent of a self-adjoint operator M such that $M>\delta$.

The proof of this theorem follows from the following three lemmas:

Lemma 1: For n large enough, there exist positive constants c_1, c_2 independent of n such that

$$\|P_eP_nR(-b;M_n)P_d\|\leqslant c_1\exp(-c_2n^{\beta/2}). \tag{43}$$

Proof: Since $\alpha H_0\geqslant\alpha\mu_0N$, it is enough to prove (43) with M_n' replacing M_n. Estimates (32) and (34) permit us to apply Theorem 2.4 of Ref. 1, with $|a_n\rangle=P_nP_d|a\rangle$, $|\beta_n\rangle$ $P_nP_e|b\rangle$, $\mu_n=dn^\beta$, $A=b+M_n'-dn^\beta D_n=d'n^2\beta$ (this corresponds to M in the notation of Ref. 1), and $N<\{[(n+1)^\beta-4]-(n^\beta+4)\}/4$. Thus we obtain (43).

Lemma 2: Let $\epsilon>0$ be smal enough. Then there exist constant $c(\epsilon)$ such that

$$\|R(-b;M_e)\|,\ \|R(-b;M_d)\|\leq 1/b, \tag{44}$$

$$\|L_eP_dR(-b;M_d)P_e\|,\ \|L_eP_eR(-b;M_e)P_d\|<\tfrac{1}{2}, \tag{45}$$

$$\|R(-b;M_d)L_eR(-b;M_e)\|\leq c(\epsilon)b^{-1-\epsilon}, \tag{46}$$

$$\|L_dR(-b;M_d)LR(-b;M_e)\|\leq c(\epsilon)^2b^{-1-\epsilon}. \tag{47}$$

Proof: Inequality (44) is an easy consequence of estimate (31). Let $|a\rangle$ and $|b\rangle$ be two normalized states in the Fock space. To prove (45), we consider

$$\begin{aligned}\langle a|\,L_dP_dR(-b;M_d)P_e|\,b\rangle &= \sum_{n=\text{odd}}\langle a|\,L_dP_dR(-b;M_c)P_eP_n|\,b\rangle\\ &= \sum_{n=\text{odd}}\langle a|\,L_dP_dP_nR(-b;M_n)P_e|\,b\rangle\\ &= \sum_{n=\text{odd}}\langle a|\,(1-P_n)M_1P_nP_dP_n\\ &\qquad\times R(-b;M_n)P_e|\,b\rangle\\ &\leq \sum_{n=\text{odd}}c_1(n+2)^{2\beta}\exp(-c_2n^{\beta/2})<\tfrac{1}{2}.\end{aligned}$$

In the last step above, we have used estimate (43) and a standard N_τ estimate. Similar arguments establish estimated (46) and (47).

Lemma 3: For b large enough, the series (42) converges uniformily to an operator $R(-b)$ which is a pseudoresolvent and satisfies

$$\lim_{b\to+\infty}(-b)R(-b)=I \tag{48}$$

in the norm operator topology.

Proof: Estimates (44) through (47) imply that the nth term in (42) is bounded by $c(\epsilon)^nb^{-1-n\epsilon}$. Therefore, the series converges for $b>c(\epsilon)^{1/\epsilon}$ to an operator $R(-b)$. Clearly,

$$\lim_{b\to+\infty}b\,[\|R(-b;M_d)L_eR(-b;M_e)\|+\cdots]=0. \tag{49}$$

This implies (48). It is not hard to prove that $R(-b)$ is a pseudoresolvent.

Proof of Theorem 1: We follow the proof of Theorem 2.2 in Ref. 1 Equality (48) implies that $R(-b)$ is an inversible operator. Then $-b-(R(-b)^{-1}$ defines M whose domain is independent of b because of the pseudoresolvent property of $R(-b)$. The self-adjointness follows from the next lemma.[8]

Lemma 4: If $T:\mathcal{H}\to\mathcal{H}$ is an operator with dense domain on the Hilbert space $\mathcal{H}$, and if T^{-1} exists and has a dense domain, then $(T^*)^{-1}=(T^{-1})^*$.

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