Modulational instability in nonlinear periodic structures: Implications for "gap solitons"

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A nonlinear periodic structure can convert an input continuous wave beam into a train of pulses. This modulational instability implies that the nonlinear spatial resonance ("gap solitons") of distributed feedback structures are generally unstable. Stability is assured only for low coupling strengths or large detunings from the Bragg condition.

Wave propagation in periodic structures is characterized by the presence of stop bands and pass bands.[1] In the linear theory, a wave whose frequency lies within a stop band is strongly reflected. Its amplitude decays exponentially with propagation distance into the medium and hence it is known as a localized state. On the other hand a wave whose frequency lies outside the stop band can pass through the structure unimpeded. In 1979, Winful *et al.* first studied the effect of a nonlinear dielectric constant on the transmission properties of a periodic structure.[2,3] They found that an intense wave could alter the refractive index of the structure enough to tune itself out of the stop band. For certain intensities total transmission occurs. At these intensities a spatial resonance is excited within the distributed feedback resonator,[3] much the same way as nonlinear resonances are excited in a Fabry–Perot resonator.[4] These spatial resonances have been termed "gap solitons" because they have the characteristic shape of sech^2 solitons and because they "reside" within the band gap.[5] We prefer to simply call these structures nonlinear resonances.

The nonlinear reflection of distributed feedback structures has been studied in cholesteric liquid crystals[6] (where the periodicity arises from the natural helical ordering) and in the four-wave mixing process[7] (where the periodic structure is created by the interacting waves themselves). Among the phenomena that can occur in nonlinear periodic structures are optical bistability,[2] pulse compression,[8] soliton propagation,[9–11] self-pulsations, and chaos.[12]

In this letter we show that the nonlinear resonances of periodic structures are generally unstable above a threshold intensity. The instability is such that a continuous-wave input beam is converted into a train of pulses. The pulsation frequency increases linearly with intensity. The self-pulsation can be understood in terms of four-wave mixing gain with feedback supplied by the periodicity.[13] The structure, in effect, acts a four-photon parametric oscillator with distributed feedback. This instability requires a certain threshold intensity.

We consider a medium whose refractive index varies periodically as

$$n = n_0 + n_1 \cos(2\beta_0 z), \qquad (1)$$

where $n_1 \ll n_0$, $\beta_0 = 2\pi n_0/\lambda_0$, and λ_0 is the free-space wavelength that satisfies the Bragg condition associated with periodicity. The periodic modulation introduces coupling between forward and backward propagating waves and the strength of this coupling is measured by the parameter $\kappa = \pi n_1/\lambda_0$. If the refractive index also depends on the local intensity, then the slowly varying amplitudes of the forward and backward waves satisfy the evolution equations:[8]

$$\frac{\partial E_F}{\partial z} + \frac{n_0}{c}\frac{\partial E_F}{\partial t} = i\kappa E_B e^{-i2\Delta\beta z} + i\gamma(|E_F|^2 + 2|E_B|^2)E_F, \qquad (2a)$$

$$\frac{\partial E_B}{\partial z} - \frac{n_0}{c}\frac{\partial E_B}{\partial t} = -i\kappa E_F e^{i2\Delta\beta z} - i\gamma(|E_B|^2 + 2|E_B|^2)E_B. \qquad (2b)$$

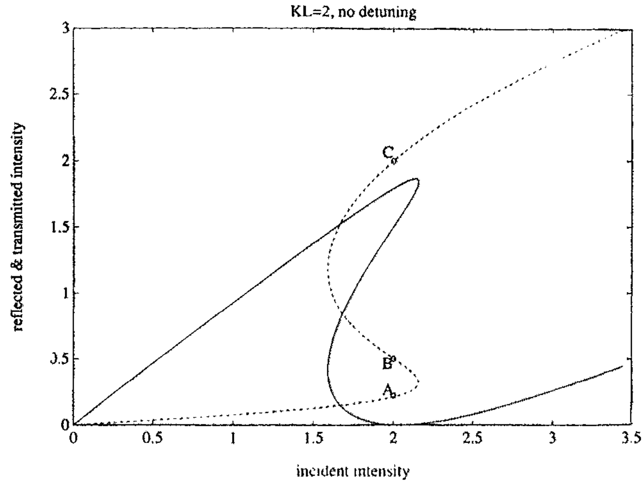
Here $\Delta\beta = \beta - \beta_0$, β is the propagation constant of the incident wave, $\gamma = \pi n_2/\lambda$, and n_2 is the nonlinear index coefficient.

At steady state and in the absence of nonlinearity, Eqs. (2) can be readily solved for the linear transmission properties of a distributed feedback structure.[14] For wavelengths such that $|\Delta\beta| < \kappa$, the structure is highly reflective and the field decays exponentially with propagating distance into the medium. Outside this stop band, there are particular values of $\Delta\beta$ for which the structure is totally transmissive. Considerable enhancement of the cavity field occurs at these transmission resonances as a result of constructive interference. The total intensity $|E_F|^2 + |E_B|^2$ within the structure (normalized by the input intensity) is given by

$$I = \frac{(\Delta\beta L)^2 - (\kappa L)^2 \cos[2\delta L(\zeta - 1)]}{(\Delta\beta L)^2 - (\kappa L)^2 \cos^2(\delta L)}, \qquad (3)$$

where $\delta L = [(\Delta\beta L)^2 - (\kappa L)^2]^{1/2}$, $\zeta = z/L$ and L is the length of the structure. Clearly, the transmission resonances occur at $\delta L = m\pi$, which corresponds to $\Delta\beta L = [(\kappa L)^2 + (m\pi)^2]^{1/2}$, with $m = 1,2,3,\ldots$. At the resonances, the intensity distribution has a number of peaks equal to m and an enhancement factor of $1 + 2(\kappa L/m\pi)^2$ over the input intensity.

Inclusion of nonlinearity makes it possible for an intense beam to tune itself into or out of these resonances. The nonlinear equations can still be solved for the steady-state intensity distribution within the structure. For the case where the light wave at low intensity satisfies the Bragg condition $\Delta\beta = 0$, the intensity distribution of the forward wave in the nonlinear periodic structure is given by [2,3]



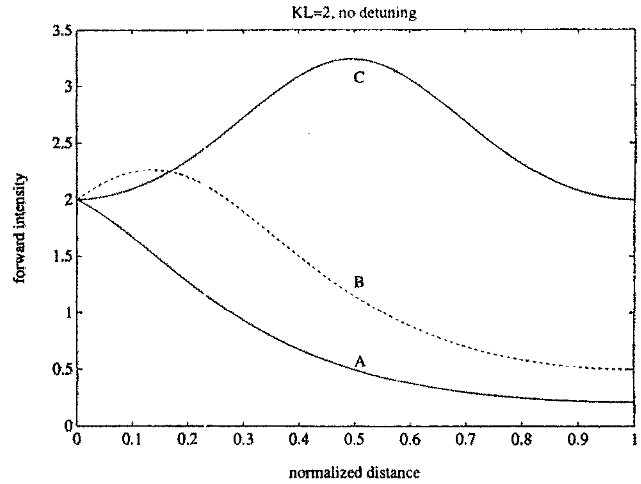


FIG. 1. Reflected (solid curve) and transmitted (dotted curve) intensities vs incident intensity for a nonlinear periodic structure with $\kappa L = 2$ and $\Delta\beta = 0$. The points labeled *A*, *B*, and *C* are the three transmitted intensities corresponding to an input intensity $I = 2$.

FIG. 2. Three spatial distributions of cavity intensity corresponding to an input of $I = 2$. Curve *C* is the nonlinear resonance or gap soliton.

$$y(\zeta) = \frac{|E_F(\zeta)|^2}{|E_C|^2} = \frac{J}{1 + [y_-/y_+]\,\mathrm{sn}^2[(\zeta - 1)y_+ \,|\, (y_-/y_+)^2]}, \quad (4)$$

where

$$y_\pm = J/2 \pm [(J/2)^2 + (\kappa L)^2]^{1/2},$$

sn is an elliptic function and J is the normalized transmitted intensity. The intensities are normalized by $|E_C|^2 = 2/3\gamma L$.

The relation between transmitted intensity and incident intensity is shown in Fig. 1 for a nonlinear distributed feedback (DFB) structure with $\kappa L = 2$ and $\Delta\beta L = 0$. For this value of coupling, it is seen that a transmission resonance occurs at $I = 2$ (where the reflected intensity is zero). In fact for an input intensity of $I = 2$, there are three possible transmitted intensities labeled *A*, *B*, and *C* in Fig. 1. Corresponding to each of these output values is a distinct spatial distribution of the light field. These distributions are shown in Fig. 2. The distribution marked *A* is the expected near-exponential decay of a light wave whose frequency satisfies the Bragg condition. Distribution *C* is the spatial resonance corresponding to that same input intensity. In order to reach that resonance, the input intensity must be increased beyond the knee of the S-shaped curve and then decreased to a value $I = 2$. Distribution *B* is unstable under steady-state condition.

We now investigate the stability of the spatial resonance by solving the time-dependent coupled mode equations numerically. Figure 3 shows the result of such a calculation. The input intensity starts at a value of $I = 2.5$ and is reduced slowly to $I = 2$ and then kept steady at that value. It is seen that the output exhibits sustained pulsations. The nonlinear spatial resonance is thus unstable. If, on the other hand, the input intensity is increased slowly from $I = 0$ to $I = 2$, it is found that the lower branch of the hysteresis loop is stable.

For sufficiently high input intensities, higher order resonances can be excited. These possess more than one hump. Figure 4 shows the higher order "gap solitons" that are excited at input intensities of $I = 5.6$ and $I = 9.1$. We have found that these higher order resonances are also unstable. These nonlinear resonances may be compared with the linear mode distributions of DFB structures shown in Ref. 14.

The origin of the instability is four-wave mixing which provides gain to two sidebands symmetrically disposed about the pump wavelength. These sidebands enjoy feedback if they are located at the resonances of the periodic structure. Because of the tuning action of the strong pump beam, it is possible for all three waves to be simultaneously resonant.[15] This situation is highly unstable and results in self-pulsations. At even higher input intensities the pulsations are chaotic.

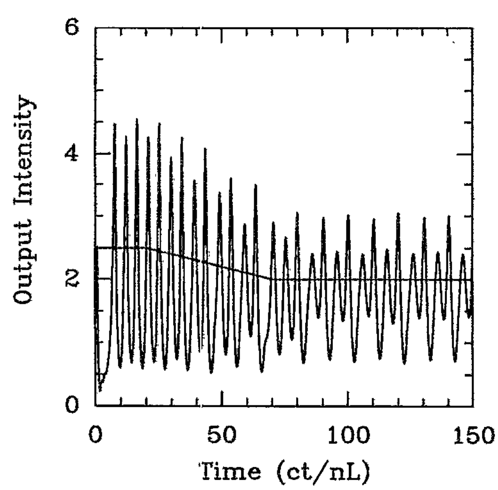


FIG. 3. Self-pulsing solution of the coupled-mode equations. The input intensity (straight line segments) starts at $I = 2.5$ and is reduced slowly to $I = 2$. Here $\kappa L = 2$, $\Delta\beta = 0$.

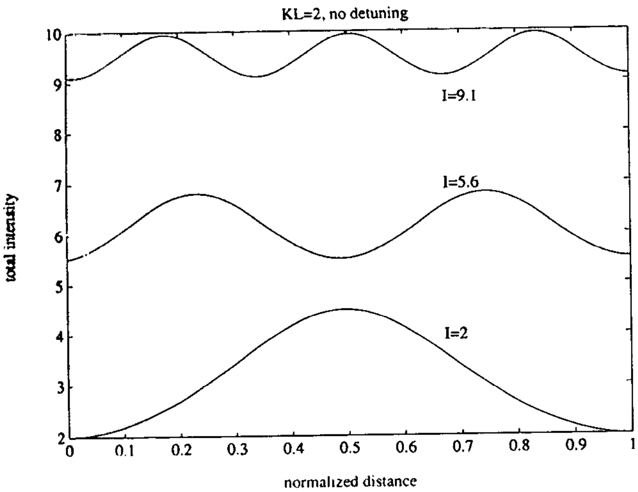


FIG. 4. Higher order nonlinear resonances of a distributed feedback structure.

Practical applications may be envisioned for such a pulse generator. In that case the instability is a desirable feature. We seek conditions under which highly nonlinear behavior such as hysteresis and self-pulsations occur. From the steady-state solution (4) we find that in the absence of detuning, i.e., $\Delta\beta = 0$, the minimum coupling strength required for hysteresis is $\kappa L \cong \pi/2$. Further, one can show that the normalized critical intensity required for switching to the upper branch is $I_s \sim \kappa L$. In the presence of detuning, this switching intensity is reduced to $I_s = \kappa L - \Delta\beta L$. The four-wave mising instability occurs when $\gamma |E|^2 L = \pi/2$. If the switching intensity is less than this value, gap solitons may be observed stably. However such stable structures require that κL be small ($\kappa L < 1$). In that case the nonlinear resonance is extremely weak. The "gap soliton" in that case is essentially flat with a very low contrast ratio between its peak and its wings. Recall that the enhancement factor for the DFB resonances is $1 + 2(\kappa L/n\pi)^2$. For $\kappa L < 1$, the enhancement is less than 20%.

The problem of nonlinear propagation in periodic structures also shows up in stimulated Brillouin scattering which can be interpreted in terms of moving acoustic gratings. This model predicts self-pulsations and chaos in that interaction as well.[16]

In conclusion, we have shown that gap solitons are unstable in regimes of practical interest. However the instability may form the basis of a distributed feedback pulse generator.

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