

Thermal stability of radiating fluids: Asymmetric slot problem

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The buoyancy driven thermal stability of a radiating nongray gas between two infinitely long vertical plates is studied analytically. The Squire theorem is shown to hold, and two-dimensional disturbances are more dangerous than the corresponding three-dimensional ones. A combination of the Galerkin and Chandrasekhar methods are used to solve the eigenvalue problem associated with two-dimensional disturbances. In contrast to the Benard problem, the effect of radiation on the onset of stationary cells diminishes as the optical thickness of the gas increases indefinitely.

I. INTRODUCTION

The instability of natural convection in a slot involving two infinitely long vertical plates at different isothermal temperatures appears in two regimes (the so-called conduction and convection regimes) which are distinguished by the temperature of the initial state; the initial temperature of the conduction regime is independent of and that of the convection regime depends linearly on the vertical direction. In each regime the instability sets in the form of stationary cells or that of traveling waves, depending on the physical parameters associated with the problem. In the past two decades, several papers dealing with this problem have been published. For a complete list of references, we may refer the reader to the recent article by Hart.¹

Although the extension of the problem including the effect of rotation or that of non-Newtonian behavior have recently been considered,^{2,3} the effect of radiation has apparently been left untreated and is the concern of this study. For the sake of simplicity, we restrict ourselves to the conduction regime alone. So far, investigations on the convection regime have all been based on the temperature profiles determined experimentally for the initial steady problem. Therefore, further analytical studies are needed on the initial state before the addition of radiative effects to the instability of the convection regime. Otherwise, the algebraic involvement with the solution of disturbance equations for both regimes are known to be of the same order.

II. FORMULATION

The formulation of the problem, except its modification by radiative terms, is well known. Before giving this formulation, let us briefly comment on the radiative effects. Clearly, the exact (integral) formulation of radiative transfer becomes rather inconvenient to use when it is considered in the formulation of a problem involving the combined effects of radiation and diffusion. Consequently, approximate theories have been developed in which the integral transfer equation is replaced by an approximate differential equation. Such a theory, the so-called Eddington approximation, is based on the first-order expansion of the transfer

equation into spherical harmonics. The first two moments of the transfer equation coupled with the assumption of local isotropy give the same result which is

$$\partial_i \partial_j q_j^R - 3\alpha_P \alpha_R q_i^R = 4\alpha_P \partial_i E_b, \quad (1)$$

where q_i^R denotes the radiant flux, α_P and α_R are the Planck and Rosseland means of absorption coefficients, respectively, $E_b = \sigma T^4$ is the blackbody radiation, σ is the Stefan-Boltzmann constant, and T is the absolute temperature. The foregoing equation may be rearranged in terms of the first moment of intensity as

$$\partial_i \partial_j j - 3\alpha_P \alpha_R j = -12\alpha_P \alpha_R E_b, \quad (2)$$

where

$$j = \int_{4\pi} \int_0^\infty I_\nu d\nu d\Omega = c_0 u^R,$$

u^R denoting the radiative energy density and c_0 the velocity of light. Equation (2) proves convenient in the formulation of the disturbance equations. Following Traugott,⁴ we included, in terms of α_P and α_R , the weighted effect of nongrayness into these equations (see Refs. 5-8 for details on the three-dimensional transfer equation, the weighted nongrayness, and the elaboration on this nongrayness).

Neglecting the contribution of the radiative energy density to the energy equation and that of radiative stress to the momentum equation, assuming a constant property fluid with the exception of the density variation in the buoyancy term, and linearizing the radiation term in view of the interest in high temperature levels but not the large temperature differences, the non-dimensionalized Boussinesq formulation may be written as

$$\begin{aligned} \partial_i u_i + u_k \partial_k u_i &= -\partial_i p + \theta \delta_i / G + \partial_k \partial_k u_i / G, \\ \partial_i \theta + u_k \partial_k \theta &= \partial_k \partial_k \theta / R + (\chi/R) \partial_k \partial_k j, \\ \partial_i u_i &= 0, \\ \partial_k \partial_k j - 3\tau^2 j &= -\tau^2 [\theta_m + 4(\theta - \theta_m)], \end{aligned} \quad (3)$$

where $\delta_i = 1$ for $i = 1$ and $\delta_i = 0$ for $i \neq 1$, $i = 1, 2, 3$, u_i is the i th component of velocity. Next, we apply the foregoing formulation to a viscous fluid confined between

two infinitely long vertical plates at different isothermal temperatures. Here, velocities have the scale $\bar{U}_m = g\alpha_0 d^2 \Delta T / \nu$, lengths d , pressure $\rho \bar{U}_m^2$, temperature ΔT , and the first moment of intensity $12\sigma T_m^3 \Delta T$, g denotes the gravitational acceleration, α_0 denotes the coefficient of thermal expansion, ΔT denotes the magnitude of the difference between plate temperatures, d is the distance between the plates, ν is the kinematic viscosity, ρ is the density, T_m is the mean temperature, $g\alpha_0 d^2 \Delta T / \kappa \nu$ is the Rayleigh number, $G = R/P$ is the Grasshof number, $P = \nu/\kappa$ is the Prandtl number, $\Phi_0 = \alpha_m k / 4\sigma T_m^3$ is the Planck number, κ is the thermal diffusivity, k is the thermal conductivity, and $\alpha_m = (\alpha_P \alpha_R)^{1/2}$ is the mean absorption. Furthermore, the optical thickness and the degree of nongrayness of the fluid are, respectively, defined by $\tau = \alpha_m d$, $\eta = (\alpha_P / \alpha_R)^{1/2}$, and $\chi = \eta / \Phi_0$.

Although fluid dynamical boundary conditions remain the same, the usual thermal boundary conditions require additional radiative boundary conditions. Following Goody⁹ (see also Arpaci and Gözümlü¹⁰), these conditions may be stated, in terms of q_w^R , as

$$q_w^R = (\lambda / \eta \tau) \partial q_w^R / \partial z,$$

or, repeating the same steps in terms of j_w , as

$$j_w = 4E_{bw} + (\eta / 3\lambda \tau) \partial j_w / \partial z,$$

where $z = x_3/d$, x_3 being the coordinate normal to the boundary, $1/\lambda = 4(1/\epsilon - \frac{1}{2})$, ϵ being the hemispherical (diffuse) emissivity of the wall, and subscript w denoting the wall values. For mirror boundaries $q_w^R = 0$ which implies

$$\partial j_w / \partial z = 0.$$

III. THE INITIAL PROBLEM

The formulation of the initial problem may readily be obtained from the one-dimensional form of the general formulation. The result, in terms of the radiative heat flux nondimensionalized by $k\Delta T/d$, is

$$D^2 \bar{u}_1 = \bar{\theta} - \bar{\theta}_m, \quad D^2 \bar{\theta} = D\bar{q}^R, \\ D^2 \bar{q}^R - 3\tau^2 \bar{q}^R = 4\chi \tau^2 D\bar{\theta}, \quad (4)$$

$$\bar{u}_1(-\frac{1}{2}) = \bar{u}_1(\frac{1}{2}) = 0, \quad \bar{\theta}(-\frac{1}{2}) = -\frac{1}{2} + \bar{\theta}_m, \quad \bar{\theta}(\frac{1}{2}) = \frac{1}{2} + \bar{\theta}_m,$$

$$\bar{q}^R(-\frac{1}{2}) - (\lambda / \eta \tau) D\bar{q}^R(-\frac{1}{2}) = 0, \quad (5)$$

$$\bar{q}^R(\frac{1}{2}) + (\lambda / \eta \tau) D\bar{q}^R(\frac{1}{2}) = 0,$$

where $D \equiv d/dz$. The solution of Eq. (4), subject to Eq. (5) is somewhat lengthy, but trivial. The dimensionless result is

$$\bar{u}_1 = [M/24 + (2K/\varphi^3 \tau^3) \sinh(\varphi \tau/2)]z \\ - (M/6)z^3 - (K/\varphi^3 \tau^3) \sinh(\varphi \tau z), \\ \bar{\theta} - \bar{\theta}_m = Mz + (K/\varphi \tau) \sinh(\varphi \tau z), \quad (6)$$

where

$$M = H/[H + (8\chi/3\varphi \tau) \sinh(\varphi \tau/2)], \\ K = (4\chi/3)/[H + (8\chi/3\varphi \tau) \sinh(\varphi \tau/2)], \\ H = \frac{[1 + (\lambda \varphi / \eta)^2] \sinh(\varphi \tau) + 2(\lambda \varphi / \eta) \cosh(\varphi \tau)}{2 \sinh(\varphi \tau/2) + 2(\lambda \varphi / \eta) \cosh(\varphi \tau/2)}, \\ \varphi = (3 + 4\chi)^{1/2}.$$

Figures 1 and 2 show the general suppressing effect of radiation depending on Φ_0 , τ , ϵ and η . A similar problem corresponding to plate temperatures linearly increasing upward has recently been solved by Greif *et al.*¹¹ However, the work is restricted to the initial steady solution for thin gas, and ignores the stability aspects of the problem.

IV. THE STABILITY PROBLEM

Having determined the initial state, we now proceed to the stability of these profiles which is the main concern of our study. Following the usual steps of the infinitesimal stability theory, we readily obtain the disturbance equations as follows:

$$[(D^2 - \alpha^2 - \beta^2)/G - (c + i\alpha \bar{u}_1)]U - (D\bar{u}_1)W \\ = i\alpha P' - \Theta/G, \quad (7)$$

$$[(D^2 - \alpha^2 - \beta^2)/G - (c + i\alpha \bar{u}_1)]V = i\beta P', \quad (8)$$

$$[(D^2 - \alpha^2 - \beta^2)/G - (c + i\alpha \bar{u}_1)]W = DP', \quad (9)$$

$$[(D^2 - \alpha^2 - \beta^2)/G - (c + i\alpha \bar{u}_1)]\Theta - (D\bar{\theta})W \\ = -(\chi/R)(D^2 - \alpha^2 - \beta^2)J, \quad (10)$$

$$i\alpha U + i\beta V + DW = 0, \quad (11)$$

$$(D^2 - \alpha^2 - \beta^2 - 3\tau^2)J = -4\tau^2 \Theta, \quad (12)$$

where U, V, W, P', Θ and J are the amplitude functions of u_1, u_2, u_3, p, θ and j , and α, β the wavenumber of disturbances in the x_1 and x_2 directions, c is the wave speed.

Next, applying the Squire theorem, we show that the two-dimensional disturbances are more dangerous than the corresponding three-dimensional ones. Thus, multiplying Eq. (7) by α and Eq. (8) by β , summing the result, and introducing the transformations

$$\alpha^2 + \beta^2 = \tilde{\alpha}^2, \quad \alpha G = \tilde{\alpha} \tilde{G}, \\ P' = \tilde{P}, \quad \alpha J = \tilde{\alpha} \tilde{J}, \quad P'G = \tilde{P} \tilde{G}, \\ \alpha U + \beta V = \tilde{\alpha} \tilde{U}, \quad \alpha \Theta = \tilde{\alpha} \tilde{\Theta}, \\ W = \tilde{W}, \quad \chi = \tilde{\chi}, \quad \tau = \tilde{\tau},$$

we may reduce the system (7)–(12) to that of an equivalent two-dimensional problem for which $\tilde{\alpha} > \alpha$ and $\tilde{G} < G$ (superscript \sim prescribing the two-dimensional problem). Hence, it is sufficient to consider only two-dimensional disturbances in determining the critical state of neutral stability. The knowledge accumulated

so far on the classical slot problem¹ indicates the onset of stationary instability for $P \leq 1$. Since the majority of fluid flow problems with pronounced effects of radiation are associated with gaseous media, and since always $P < 1$ for these media, the instability of these problems occurs in the form of stationary cells for which $c = 0$.

Introducing the stream function,

$$u_1 = \partial\psi/\partial x_2, \quad u_2 = -\partial\psi/\partial x_1,$$

denoting the amplitude function of ψ by Φ , the stability problem may be reduced, for $c = 0$, to

$$(D^2 - \alpha^2)^2 \Phi / G + i\alpha[(D^2 \bar{u}_1)\Phi - \bar{u}_1 D^2 \Phi + \alpha^2 \bar{u}_1 \Phi] = -D\Theta / G, \quad (13)$$

$$(D^2 - \alpha^2)\Theta / R - i\alpha[\bar{u}_1 \Theta - (D\bar{\theta})\Phi] = -(\chi / R)(D^2 - \alpha^2)J, \quad (14)$$

$$(D^2 - \alpha^2 - 3\tau^2)J = -4\tau^2\Theta, \quad (15)$$

subject to

$$\Phi(-\frac{1}{2}) = \Phi(\frac{1}{2}) = D\Phi(-\frac{1}{2}) = D\Phi(\frac{1}{2}) = 0, \quad (16)$$

$$\Theta(-\frac{1}{2}) = \Theta(\frac{1}{2}) = 0, \quad (16)$$

$$J(-\frac{1}{2}) - (\eta/3\lambda\tau)DJ(-\frac{1}{2}) = 0, \quad (17)$$

$$J(\frac{1}{2}) + (\eta/3\lambda\tau)DJ(\frac{1}{2}) = 0. \quad (17)$$

In the foregoing formulation the superscript \sim is dropped for convenience. Such a system, excluding the

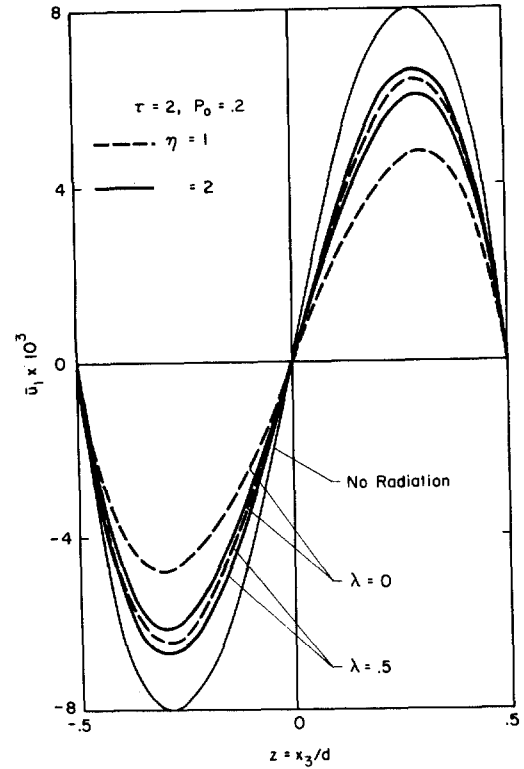


FIG. 2. The effect of weighted nongrayness on the base flow between mirror or black surfaces.

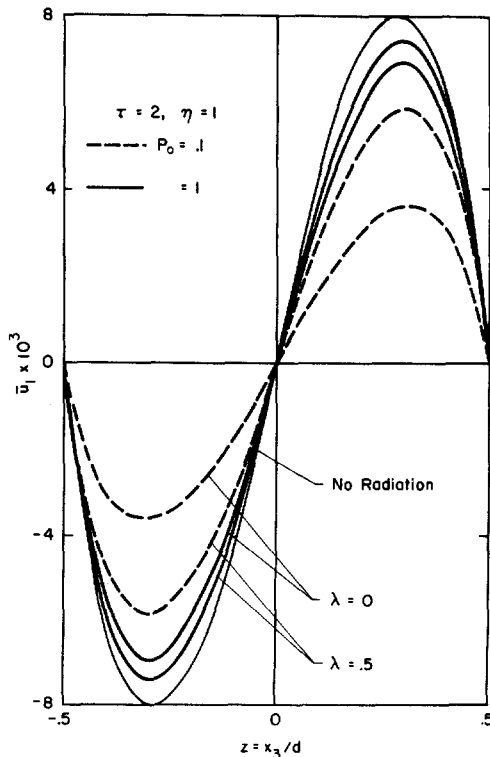


FIG. 1. The effect of the ratio of conduction to radiation on the base flow between mirror or black surfaces.

radiation effects, was first given by Gershuni.¹² Note that, before we set $c = 0$, Eq. (13) was the usual Orr-Sommerfeld equation modified by an additional buoyancy term. Equation (13) is coupled with the thermal disturbance equation (14) which in turn is coupled with the radiative disturbance equation (15).

The eigenvalues of the system (13)–(17), which specify the states of neutral stability, form a relationship, $f(G, \alpha, P, \Phi_0, \eta, \tau, \epsilon) = 0$. This relationship must next be determined. The foregoing system is not self-adjoint because of the variable coefficients resulting from the initial state, but a variational formulation based on the concept of adjoint systems can always be constructed following Roberts.¹³ However, because of these coefficients, a numerical solution based on the concept of adjoint systems requires the integration of differential equations with variable coefficients. For the present problem this integration appears to be somewhat involved. Consequently, we adopt a convenient but simple (but maybe less accurate) method of solution by combining the Galerkin technique, used extensively for the slot problem, with the method successfully employed by Chandrasekhar¹⁴ for the Bénard and Couette problems. First, following Vest and Arpaci¹⁵ we assume the sets

$$\Theta = \sum_{n=1}^N (a_n \sin \Pi_n z + i b_n \cos \Pi_n z), \quad (18)$$

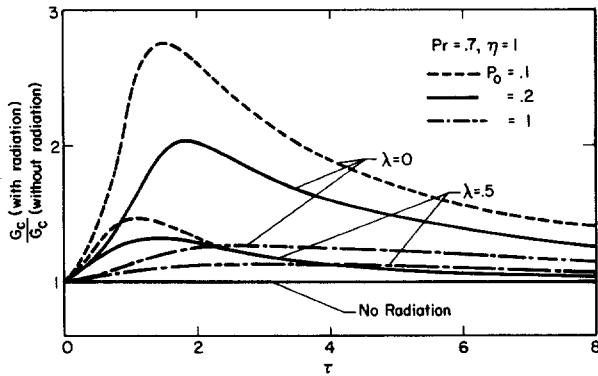


FIG. 3. The effect of the ratio of conduction to radiation on the normalized critical Grashof number versus optical thickness for mirror or black surfaces.

where $\Pi_n = 2n\pi$, $\Omega_n = (2n-1)\pi$, $n = 1, 2, \dots$, and

$$\Phi = \sum_{n=1}^N (c_n C_n + i d_n S_n), \quad (19)$$

where C_n and S_n denote the functions already discussed and tabulated by Harris and Reid¹⁶ and Reid and Harris.¹⁷ Next, inserting Eq. (18) into Eq. (15) and integrating the result we obtain J subject to Eq. (17) as

$$J = \sum_{n=1}^N a_n (f_n \sin \Pi_n z - T_n \sinh \gamma z) + i b_n (t_n \cos \Omega_n z - F_n \cosh \gamma z), \quad (20)$$

where

$$\gamma^2 = \alpha^2 + 3\tau^2, \quad f_n = 4\tau^2 / (\Pi_n^2 + \gamma^2), \quad t_n = 4\tau^2 / (\Omega_n^2 + \gamma^2),$$

$$F_n = \frac{(-1)^n (4\tau/3) \Omega_n \eta}{(\Omega_n^2 + \gamma^2) [\lambda \cosh \gamma/2 + (\eta\gamma/3\tau) \sinh \gamma/2]},$$

$$T_n = \frac{(-1)^n (4\tau/3) \Pi_n \eta}{(\Pi_n^2 + \gamma^2) [\lambda \sinh \gamma/2 + (\eta\gamma/3\tau) \cosh \gamma/2]}.$$

Then, introducing Eq. (20) into Eq. (14), we reduce the problem to Eq. (13) and

$$(D^2 - \alpha^2) \Theta / R + i \alpha [(D \bar{\Theta}) \Phi - \bar{u}_1 \Theta] = (\chi \tau^2 / R) [4 \Theta - 3 J(\Theta)] \quad (21)$$

which are only in terms of Θ and Φ . Now, Eqs. (13) and (21) are orthogonalized with respect to Φ_n and Θ_n ,

$$\int_{-1/2}^{1/2} \Phi_m L_1(\Phi_n, \Theta_n) dz = 0,$$

$$\int_{-1/2}^{1/2} \Theta_m L_2(\Phi_n, \Theta_n) dz = 0,$$

where L_1 and L_2 represent the linear operators of Eqs. (13) and (21). Inserting Eqs. (18) and (19) into Eqs. (22) and (23) yields the usual secular determinant of infinite order. The inner products of this determinant,

additional to those already given by Harris and Reid¹⁶ and Reid and Harris¹⁷ and Vest,¹⁸ are evaluated by exact integrations. The eigenvalues are determined by searching the $(G-\alpha)$ plane for zeros of the determinant with fixed values of the other variables. The results are given in Figs. 3 and 4 for the critical value of G (relative to that of G in the absence of radiation) versus τ while keeping other variables as parameters.

V. DISCUSSION

An important outcome of the present study is the behavioral difference between the neutral curves of the Bénard and the slot problems corresponding to large optical thicknesses. It is well known from the literature on the Bénard problem^{9,19,20} that, as the optical thickness increases, the onset of stationary instability is monotonically delayed. For the slot problem, however, as we see from Figs. (3) and (4), the onset of stationary instability approaches that of the classical problem with no radiation. This result may be interpreted as follows: Since both problems for a thick gas may be assumed to be those for a nonradiating gas with a Prandtl number reduced by $1+4\chi/3$, and since the neutral curve for the stationary instability of the slot problem is known to be independent of the Prandtl number (see, for example, Vest and Arpaci,¹⁴ and the references cited therein), the thick gas radiation, reducing only the Prandtl number, does not affect the onset of instability of the slot problem.

Another difference between the foregoing problems appears to be that, as the optical thickness increases, the onset of instability is monotonically increased for the Bénard problem, while it assumes a maximum for the slot problem. However, the extremum appearing in the slot problem is appreciable only for cases corresponding to nongray gas and nonblack boundaries whose effects on the Bénard problem so far have not been studied. The latest study by Arpaci and Gözümlü¹⁰ on the Bénard problem also leads to curves with a maximum when these effects are included.

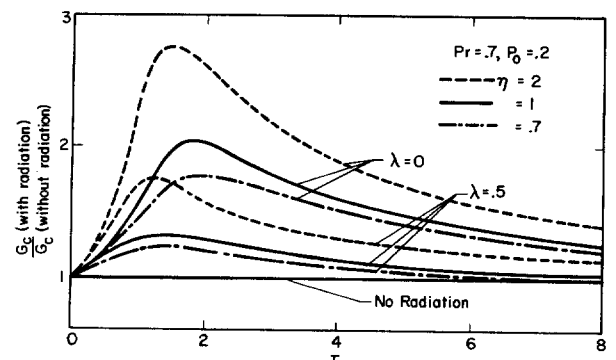


FIG. 4. The effect of weighted nongrayness on the normalized critical Grashof number versus optical thickness for mirror or black surfaces.

The decreasing ϕ_0 which implies an increased effect of radiation or decreasing λ which corresponds to the change of surface color from black to mirror, flattens the base temperature which, in turn, delays the onset of instability (Figs. 3 and 4). Clearly, the parameter $\chi = \eta/\phi_0$ of the thermal disturbance equation and the radiative boundary conditions, $J \pm (\eta/3\lambda\tau)DJ = 0$, imply the effect of η as being opposite to ϕ_0 or to λ .

We did not evaluate specific values of η for different gases. It is well known that the Planck mean depends only on the temperature, while the Rosseland mean depends both on the temperature and the pressure. Since the stability problems associated with natural convection inherently require small temperature differences and small pressure changes, the assumption of a constant η appears to be reasonable. According to Sampson,²¹ a conservative and approximate range for η appears to be between 0.7 and 2 corresponding, respectively, to $\alpha_P/\alpha_R \approx \frac{1}{2}$ and 4. It should be stressed, however, that the nongrayness characterized by η may only show first-order and weighted effects. Without altering the fluid mechanical flavor of our study, a more realistic model such as the one suggested by Liu and Clarke,²² or even more realistic models along the same lines, may always be employed for a better representation of nongrayness. These models, although increasing the algebraic involvement of the problem, create no major obstacle for its solution. On the other hand, an astrophysical investigation that includes the spectral effects of nongrayness is also possible after the excellent work of Gille and Goody²³ on the Bénard problem.

We did not make a thorough search for the possibility of the occurrence of overstability. Clearly, for a thin gas the problem approaches the nonradiating problem, and for a thick gas it effectively approaches the nonradiating problem with a Prandtl number reduced by $1+4\chi/3$. Since the onset of instability for $P \leq 1$ is

known to be in the form of stationary cells, no overstability is expected for the thin gas and thick gas limits. Following the method recently elaborated by Finlayson,²⁴ an attempt to find overstability in the most likely neighborhood, $1 \leq \tau \leq 5$, revealed no sign of this kind of instability.

The convergence of the method used was checked by performing computations with secular determinants of order 4, 8, and 12. The order of the determinant was increased in steps of 4 so that equal numbers of approximating functions for the real and imaginary parts of both the stream function and temperature were considered. This avoided artificial weighting of the momentum or energy equation. The change in the critical Grashof number as the order of determinant is increased from 8 to 12 is approximately 3%.

We based our formulation on the Eddington approximation which is known to give exact results for $\tau \rightarrow 0$ and $\tau \rightarrow \infty$. As already indicated by Unno and Spiegel,⁷ and later by Arpaci and Gözüüm,¹⁰ the same approximation appears to give adequate results for intermediate optical thicknesses.

For the stream function we selected the simplest possible trial functions proposed by Harris and Reid¹⁶ and Reid and Harris.¹⁷ More complex functions have already been used by Rudakov,²⁵ and even more complex ones by Dolph and Lewis²⁶ with no apparent advantage. However, a recent article by Orszag²⁷ on the Orr-Sommerfeld equation clearly shows the advantage of using Chebyshev polynomials for considerably improved convergence.

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¹J. E. Hart, *J. Fluid Mech.* **47**, 547 (1971).

²S. A. Korpela, Ph.D. dissertation, University of Michigan, 1972.

³D. Gözüüm, Ph. D. dissertation, University of Michigan, 1972.

⁴S. C. Traugott, in *Proceedings of the 1963 Heat Transfer and Fluid Mechanics Institute*, edited by A. Roshko, B. Sturtevant, and D. R. Bartz (Stanford University Press, Stanford, California, 1963), p. 1.

⁵S. C. Traugott, *AIAA J. (Am. Inst. Aeronaut. Astronaut.)* **4**, 541 (1966).

⁶W. Unno and E. A. Spiegel, *Publ. Astron. Soc. Jap.* **18**, 85 (1966).

⁷A. C. Cogley, W. G. Vincenti, and S. E. Gilles, *AIAA J. (Am. Inst. Aeronaut. Astronaut.)* **6**, 551 (1968).

⁸S. E. Gilles, A. C. Cogley, and W. G. Vincenti, *Int. J. Heat Mass Transfer* **12**, 445 (1969).

⁹R. M. Goody, *J. Fluid Mech.* **1**, 424 (1956).

¹⁰V. S. Arpaci and D. Gözüüm, *Phys. Fluids*, (to be published).

¹¹R. Greif, I. S. Habib, and J. C. Lin, *J. Fluid Mech.* **46**, 513 (1971).

¹²G. Z. Gershuni, *Zh. Tekh. Fiz.* **23**, 1838 (1953).

¹³P. H. Roberts, *J. Math. Anal. Appl.* **1**, 195 (1960).

¹⁴S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, Oxford, 1961).

¹⁵C. M. Vest and V. S. Arpaci, *J. Fluid Mech.* **36**, 1 (1969).

¹⁶D. L. Harris and W. H. Reid, *Astrophys. J. Suppl. Ser.* **3**, 429 (1958).

¹⁷W. H. Reid and D. L. Harris, *Astrophys. J. Suppl. Ser.* **3**, 448 (1958).

¹⁸C. M. Vest, Ph.D. dissertation, University of Michigan, 1967.

¹⁹E. A. Spiegel, *Astrophys. J.* **132**, 716 (1960).

²⁰C. Christophorides and S. H. Davis, *Phys. Fluids* **13**, 222 (1970).

²¹D. H. Sampson, *Radiative Contributions to Energy and Momentum in a Gas* (Wiley, New York, 1965), p. 101.

²²J. T. C. Liu and J. H. Clarke, *Phys. Fluids* **10**, 2088 (1967).

²³J. Gille and R. M. Goody, *J. Fluid Mech.* **20**, 47 (1964).

²⁴B. A. Finlayson, *J. Fluid Mech.* **33**, 201 (1968).

²⁵R. N. Rudakov, *Prikl. Mat. Mekh.* **30**, 362 (1966) [*Sov. J. Appl. Math. Mech.* **30**, 439 (1967)].

²⁶C. L. Dolph and D. C. Lewis, *Q. Appl. Math.* **16**, 97 (1958).

²⁷S. Orszag, *J. Fluid Mech.* **50**, 689 (1971).