Invariant Imbedding and Case Eigenfunctions*

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(Received 20 June 1968)

A new approach to the solution of transport problems, based on the ideas introduced into transport theory by Ambarzumian, Chandrasekhar, and Case, is discussed. To simplify the discussion, the restriction to plane geometry and one-speed isotropic scattering is made. However, the method can be applied in any geometry with any scattering model, so long as a complete set of infinite-medium eigenfunctions is known. First, the solution for the surface distributions is sought. (In a number of applications this is all that is required.) By using the infinite-medium eigenfunctions, a system of singular integral equations together with the uniqueness conditions is derived for the surface distributions in a simple and straightforward way. This system is the basis of the theory. It can be reduced to a system of Fredholm integral equations or to a system of nonlinear integral equations, suitable for numerical computations. Once the surface distributions are known, the complete solution can be found by quadrature by using the full-range completeness and orthogonality properties of the infinite-medium eigenfunctions. The method is compared with the standard methods of invariant imbedding, singular eigenfunctions, and a new procedure recently developed by Case.

I. INTRODUCTION

In the past 50 years or so, a number of methods have been devised for solving the neutron (or radiation) transport equations. Excluding strictly approximation procedures such as spherical-harmonics expansions, discrete-ordinate methods, etc.,[1–3] the most important schemes are the Wiener–Hopf method, which is described in detail in Ref. 2, the invariant-imbedding technique, first introduced to transport theory by Ambarzumian[4] and developed extensively by Chandrasekhar[1] and others,[5] and the Case eigenfunction-expansion method.[3,6]

Historically, the first exact method was the Wiener–Hopf method. Because it was basically simpler, the invariant-imbedding method became more popular after its introduction. Eventually, the eigenfunction-expansion approach became more widely used than either of those methods for a number of reasons which are discussed below. (The Wiener–Hopf method is in fact identical with Case's method in the sense that any problem which can be solved by one method can be solved also by the other. Because Case's method is simpler and more familiar, we will not discuss the Wiener–Hopf method further.)

We first note that the traditional derivations of the equations of invariant imbedding are based upon intuitive physical arguments which, by virtue of the known existence of unique solutions of the transport equations,[7] are, in fact, spurious. However, this approach has some real advantages for numerical computation. On the other hand, it does not give complete knowledge of the neutron distribution in a given medium, but only the reflected and transmitted intensities. (Admittedly, in a number of applications these are all that are required.)

A really more serious disadvantage of the invariant-imbedding equations is that they are, in general, not uniquely soluble. To guarantee a unique solution, additional conditions must be imposed.[1] These conditions cannot be obtained from the original invariant-imbedding arguments, and so must be introduced in a somewhat arbitrary manner.

The Case method,[6] on the other hand, has the virtue of simplicity and familiarity, since it is based on an eigenfunction-expansion technique which is already well known to physicists from applications in "classical" boundary-value problems. Furthermore, no intuitive arguments and no extraneous conditions are necessary in order to derive the equations and to guarantee unique solutions. However, by straightforward application of this method, more information is frequently obtained than is really required (as, for example, the neutron distribution everywhere rather than at a surface) and reducing the results to numerics is highly nontrivial.[8]

The major purpose of the present paper is to rederive the nonlinear integral equations of invariant

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[1] S. Chandrasekhar, *Radiative Transfer* (Oxford University Press, London, 1950).
[2] B. Davison, *Neutron Transport Theory* (Oxford University Press, London, 1957).
[3] K. M. Case and P. F. Zweifel, *Linear Transport Theory* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1967).
[4] V. A. Ambarzumian, *Theoretical Astrophysics* (Pergamon Press, Inc., New York, 1958).
[5] R. E. Bellman, H. H. Kagiwada, R. E. Kalaba, and M. C. Prestrud, "Invariant Imbedding and Time-Dependent Transport Processes," The Rand Corporation, R-423-ARPA, 1964.
[6] K. M. Case, Ann. Phys. (N.Y.) 9, 1 (1960).
[7] K. M. Case and P. F. Zweifel, J. Math. Phys. 4, 1367 (1963).
[8] M. R. Mendelson, thesis, The University of Michigan, 1964.

imbedding and the uniqueness conditions in a fashion which does not suffer from the deficiencies noted above. This is accomplished by using the Case infinite-medium eigenfunctions. The nonlinear integral equations follow from a system of singular integral equations, which are themselves derived in a simple and straightforward way from Case's eigenfunctions. It is interesting to compare our derivation with those of Sobolev,[9] Busbridge,[10] and Mullikin.[11]

We deal primarily with slab problems—in the limit, of course, half-space results are obtained. The familiar restriction to plane geometry and one-speed isotropic scattering is made. However, the method can be applied in any geometry with any scattering model (e.g., multivelocity anisotropic scattering) so long as a complete set of infinite-medium eigenfunctions is known.

The results we obtain are not new. However, we do feel that our approach yields a coherent, mathematically satisfying, and simple derivation of singular integral equations and equivalent invariant-imbedding nonlinear integral equations, together with the conditions which guarantee unique solution.

In Sec. II, we give a brief review of Case's eigenfunctions and their properties. Then, in Sec. III, the system of singular integral equations and the nonlinear integral equations—together with the conditions guaranteeing uniqueness—are derived for the slab. In Sec. IV, some remarks are made for the half-space problems.

II. THE CASE EIGENFUNCTIONS

We begin with the Case eigenfunctions of the one-speed one-dimensional transport equation with isotropic scattering

$$\left(\mu \frac{\partial}{\partial x} + 1\right)\psi(x, \mu) = \frac{c}{2}\int_{-1}^{1}\psi(x, \mu')\, d\mu'. \tag{1}$$

These eigenfunctions may be written in the form[3,6]

$$\psi_\nu(x, \mu) = \phi(\nu, \mu)e^{-x/\nu}, \tag{2a}$$

with

$$\phi(\nu, \mu) = \frac{c\nu}{2} P\frac{1}{\nu - \mu} + \lambda(\nu)\delta(\nu - \mu), \quad \nu \in (-1, 1), \tag{2b}$$

$$\phi(\pm\nu_0, \mu) = \frac{c\nu_0}{2}\frac{1}{\nu_0 \mp \mu}, \tag{2c}$$

$$\int_{-1}^{1}\phi(\nu, \mu)\, d\mu = 1, \quad \nu \in (-1, 1), \quad \nu = \pm\nu_0. \tag{2d}$$

[9] V. V. Sobolev, *A Treatise on Radiative Transfer* (D. Van Nostrand Inc., Princeton, N.J., 1963).
[10] I. W. Busbridge, *The Mathematics of Radiative Transfer* (Cambridge University Press, London, 1960).
[11] T. W. Mullikin, Astrophys. J. **136**, 627 (1962); **139**, 379, 1267 (1964).

Here the discrete eigenvalue ν_0 is a root of the dispersion function

$$\Lambda(z) = 1 - \frac{cz}{2}\int_{-1}^{1}\frac{d\mu}{z - \mu}. \tag{3}$$

The quantity $\lambda(\nu)$ which appears in Eq. (2b) is related to the boundary values of the dispersion function $\Lambda(z)$ on the branch cut $(-1, 1)$. In fact,

$$\lambda(\nu) = \tfrac{1}{2}[\Lambda^+(\nu) + \Lambda^-(\nu)], \tag{4}$$

where

$$\Lambda^\pm(\nu) = \lim_{0<\epsilon\to 0}\Lambda(\nu \pm i\epsilon), \quad \nu \in (-1, 1). \tag{5}$$

We note that c, the mean number of neutrons emitted per collision, will always be assumed to be such that the slab is "subcritical." For $c < 1$, this is certainly true for all slab thickness.

The eigenfunctions are orthogonal in the sense that

$$\int_{-1}^{1}\mu\phi(\nu, \mu)\phi(\nu', \mu)\, d\mu = 0, \quad \nu \neq \nu'. \tag{6}$$

In fact, the normalization integrals are also known:

$$\int_{-1}^{1}\mu\phi^2(\pm\nu_0, \mu)\, d\mu = \pm N(\nu_0), \tag{7a}$$

$$\int_{-1}^{1}\mu\phi(\nu, \mu)\phi(\nu', \mu)\, d\mu = N(\nu)\delta(\nu - \nu'), \quad \nu \in (-1, 1), \tag{7b}$$

where

$$N(\nu_0) = \frac{c}{2}\nu_0^2\left.\frac{\partial\Lambda(z)}{\partial z}\right|_{z=\nu_0}, \tag{8a}$$

$$N(\nu) = \nu\Lambda^+(\nu)\Lambda^-(\nu). \tag{8b}$$

(All of the above results, which are well known, are restated merely for convenience.)

We now consider the solution of the so-called albedo problem for a slab. This is the problem of determining the distribution of neutrons everywhere in a source-free slab due to an incident beam. We seek the solution, denoted as $\psi(0, \mu_0 \to x, \mu; \tau)$, to the homogeneous transport equation (1) subject to the boundary conditions

$$\begin{aligned} \psi(0, \mu_0 \to 0, \mu; \tau) &= \delta(\mu_0 - \mu), \quad \mu_0 > 0, \quad \mu > 0, \\ \psi(0, \mu_0 \to \tau, -\mu; \tau) &= 0, \quad \mu > 0, \end{aligned} \tag{9}$$

where τ is the thickness of a slab.

We expand the solution $\psi(0, \mu_0 \to x, \mu; \tau)$ in terms of the eigenfunctions. That is,

$$\begin{aligned}\psi(0, \mu_0 \to x, \mu; \tau) &= A(\nu_0)\phi(\nu_0, \mu)e^{-x/\nu_0} + A(-\nu_0)\phi(-\nu_0, \mu)e^{x/\nu_0} \\ &\quad + \int_{-1}^{1}A(\nu)\phi(\nu, \mu)e^{-x/\nu}\, d\nu. \end{aligned} \tag{10}$$

The Case procedure is to determine the expansion coefficient as discussed earlier.

However, by using the set of eigenfunctions in another way, we are led at once to a system of singular integral equations for the reflected and transmitted intensities.

III. DERIVATION OF INVARIANT-IMBEDDING EQUATIONS

A. Albedo Problem for a Slab

Let us first consider the solution of the albedo problem defined in the previous section, for it will be shown that the solution of any problem can be expressed in terms of the albedo solution (Sec. III.B). We consider a slab whose left-hand surface is at $x = 0$, and whose right-hand surface is at $x = \tau$.

We are primarily interested in the reflected and the transmitted distributions $\psi(0, \mu_0 \to 0, -\mu; \tau)$ and $\psi(0, \mu_0 \to \tau, \mu; \tau)$, $\mu > 0$. From the reciprocity theorem for one-speed theory[3,12] it follows that these distributions satisfy the relations

$$\mu\psi(0, \mu_0 \to 0, -\mu; \tau) = \mu_0\psi(0, \mu \to 0, -\mu_0; \tau), \quad (11)$$

$$\mu\psi(0, \mu_0 \to \tau, \mu; \tau) = \mu_0\psi(0, \mu \to \tau, \mu_0; \tau), \quad \mu_0 > 0, \quad \mu > 0. \quad (12)$$

In view of these relations, it is convenient to introduce so-called Ambarzumian–Chandrasekhar's S and T functions, defined as[1]

$$(1/2\mu)S(\tau; \mu_0, \mu) = \psi(0, \mu_0 \to 0, -\mu; \tau) \quad (13)$$

and

$$(1/2\mu)T(\tau; \mu_0, \mu) + \delta(\mu_0 - \mu)e^{-\tau/\mu_0} = \psi(0, \mu_0 \to \tau, \mu; \tau). \quad (14)$$

Both functions are symmetric:

$$S(\tau; \mu_0, \mu) = S(\tau; \mu, \mu_0), \quad (15)$$

$$T(\tau; \mu_0, \mu) = T(\tau; \mu, \mu_0). \quad (16)$$

The reflected and transmitted distributions of an albedo problem $\psi(0, -\mu; \tau)$ and $\psi(\tau, \mu; \tau)$, $\mu > 0$, for a given incident distribution $\psi(0, \mu; \tau)$, $\mu > 0$, can be then expressed as

$$\psi(0, -\mu; \tau) = \frac{1}{2\mu}\int_0^1 S(\tau; \mu', \mu)\psi(0, \mu'; \tau)\, d\mu', \quad (17)$$

$$\psi(\tau, \mu; \tau) = \psi(0, \mu; \tau)e^{-\tau/\mu} + \frac{1}{2\mu}\int_0^1 T(\tau; \mu', \mu)\psi(0, \mu'; \tau)\, d\mu', \quad \mu > 0. \quad (18)$$

We now derive a system of singular integral equations for S and T by using the intuitive invariant-imbedding arguments.[13,14]

Let us take any exponentially decreasing infinite-medium eigenfunction

$$\phi(\nu, \mu)e^{-x/\nu}, \quad \nu \in (0, 1), \quad \nu = \nu_0. \quad (19)$$

The function $\phi(\nu, \mu)e^{-x/\nu}$ describes a distribution of neutrons for the infinite medium. At $x = 0$, the angular density $\phi(\nu, -\mu)$, $\mu > 0$, can be thought of as resulting from the reflection of the "incident" distribution $\phi(\nu, \mu)$, $\mu > 0$, on the slab of thickness τ, and from the transmission of the "incident" distribution $\phi(\nu, -\mu)e^{-\tau/\nu}$, $\mu > 0$, at $x = \tau$, through the same slab. Therefore, in view of Eqs. (17) and (18), we have

$$[1 - e^{-\tau(1/\nu + 1/\mu)}]\phi(\nu, -\mu) = \frac{1}{2\mu}\int_0^1 S(\tau; \mu', \mu)\phi(\nu, \mu')\, d\mu' + \frac{e^{-\tau/\nu}}{2\mu}\int_0^1 T(\tau; \mu', \mu)\phi(\nu, -\mu')\, d\mu',$$
$$\mu > 0, \quad \nu \in (0, 1), \quad \nu = \nu_0. \quad (20)$$

Similarly, by taking any exponentially increasing eigenfunction

$$\phi(-\nu, \mu)e^{x/\nu}, \quad \nu \in (0, 1), \quad \nu = \nu_0, \quad (21)$$

and reasoning as before, we get

$$(e^{-\tau/\nu} - e^{-\tau/\mu})\phi(\nu, \mu) = \frac{e^{-\tau/\nu}}{2\mu}\int_0^1 S(\tau; \mu', \mu)\phi(-\nu, \mu')\, d\mu' + \frac{1}{2\mu}\int_0^1 T(\tau; \mu', \mu)\phi(\nu, \mu')\, d\mu',$$
$$\mu > 0, \quad \nu \in (0, 1), \quad \nu = \nu_0. \quad (22)$$

For $\nu \in (0, 1)$, the above equations constitute a system of singular integral equations for S and T, while for $\nu = \nu_0$ we obtain two conditions which must be satisfied by S and T.

Because Eqs. (20) and (22) are the basis for our further discussion, we now rederive them rigorously, without appealing to the above intuitive invariant-imbedding arguments. Actually, the rigorous derivation is even simpler than the intuitive one given above.

To see this, let us define an albedo problem by the following boundary conditions:

$$\psi(0, \mu; \tau) = \phi(\nu, \mu),$$
$$\psi(\tau, -\mu; \tau) = \phi(\nu, -\mu)e^{-\tau/\nu},$$
$$\mu > 0, \quad \nu \in (0, 1), \quad \nu = \nu_0, \quad 0 \le x \le \tau. \quad (23)$$

[12] K. M. Case, Rev. Mod. Phys. **29**, 651 (1957).

[13] S. Pahor and I. Kuščer, Astrophys. J. **143**, 888 (1966).

[14] S. Pahor, Nucl. Sci. Eng. **29**, 248 (1967).

It can be easily verified by inspection that the *unique* solution of this particular problem is simply

$$\psi(x, \mu; \tau) = \phi(\nu, \mu)e^{-x/\nu} \tag{24}$$

(because it is a solution of the transport equation and obeys the boundary conditions).

By applying Eqs. (17) and (18) to this solution, we get Eqs. (20) and (21). This represents a rigorous derivation of Eqs. (20) and (21). The same system of singular integral equations, including anisotropic scattering, was already derived by Sobolev[9] and Mullikin.[11] However, our derivation of these equations is much simpler than that of Sobolev and Mullikin; furthermore, it is evident how the described technique could be applied to any geometry and scattering model, once the complete set of infinite medium eigenfunctions is known. (Even if the set is not complete, we obtain in this way some information on the surface distribution. However, the resulting equations are not uniquely soluble.)

It is interesting to compare the present approach with the approach recently developed by Case,[15] where the infinite-medium Green's function is used as a starting point. In both cases, first the integral equations for the surface distributions are derived. However, the corresponding equations are different, though equivalent, and the kernels of Eqs. (20) and (21), yielded by the present method, are somewhat simpler.

The functions $S(\tau; \mu_0, \mu)$ and $T(\tau; \mu_0, \mu)$ can be expressed in terms of Ambarzumian–Chandrasekhar's $X(\mu)$ and $Y(\mu)$ functions of a single variable, with τ as a parameter, which are more suitable for numerical computations than $S(\tau; \mu_0, \mu)$ and $T(\tau; \mu_0, \mu)$.

Let us integrate Eqs. (20) and (22) over μ from 0 to 1. Defining new $X(\mu)$ and $Y(\mu)$ functions as

$$X(\mu) = 1 + \frac{1}{2}\int_0^1 S(\tau; \mu', \mu)\frac{d\mu'}{\mu'}, \tag{25}$$

$$Y(\mu) = e^{-\tau/\mu} + \frac{1}{2}\int_0^1 T(\tau; \mu', \mu)\frac{d\mu'}{\mu'}, \tag{26}$$

and using the normalization condition (2d), we get a system of equations for $X(\mu)$ and $Y(\mu)$:

$$1 = \int_0^1 X(\mu')\phi(\nu, \mu')\, d\mu' + e^{-\tau/\nu}\int_0^1 Y(\mu')\phi(\nu, -\mu')\, d\mu', \tag{27}$$

$$1 = \int_0^1 X(\mu')\phi(\nu, -\mu')\, d\mu' + e^{\tau/\nu}\int_0^1 Y(\mu')\phi(\nu, \mu')\, d\mu', \quad \nu \in (0, 1), \quad \nu = \nu_0. \tag{28}$$

By introducing new functions $Z(\mu)$ and $W(\mu)$ as

$$Z(\mu) = X(\mu) + Y(\mu), \tag{29}$$

$$W(\mu) = X(\mu) - Y(\mu), \tag{30}$$

we obtain for these functions two uncoupled equations

$$1 + e^{-\tau/\nu} = \int_0^1 Z(\mu')\phi(\nu, \mu')\, d\mu' + e^{-\tau/\nu}\int_0^1 Z(\mu')\phi(\nu, -\mu')\, d\mu', \tag{31}$$

$$1 - e^{-\tau/\nu} = \int_0^1 W(\mu')\phi(\nu, \mu')\, d\mu' - e^{-\tau/\nu}\int_0^1 W(\mu')\phi(\nu, -\mu')\, d\mu', \quad \nu \in (0, 1), \quad \nu = \nu_0. \tag{32}$$

Singular integral equations, such as Eq. (31) or Eq. (32), with the condition for $\nu = \nu_0$ included, are equivalent to certain Fredholm integral equations.[16] These Fredholm integral equations were studied in detail by Leonard and Mullikin[17] and they derived conditions which guarantee the existence and uniqueness of the solution. In our case of isotropic scattering, these conditions are satisfied for all subcritical c and certainly for $c < 1$. Therefore, also Eqs. (31) and (32), Eqs. (27) and (28), and Eqs. (20) and (22) are uniquely soluble.

What remains to be done is to express $S(\tau; \mu_0, \mu)$ and $T(\tau; \mu_0, \mu)$ in terms of $X(\mu)$ and $Y(\mu)$. In deriving these relations, we obtain for $X(\mu)$ and $Y(\mu)$ a system of nonlinear integral equations which are convenient for numerical computations.

We introduce two new functions $R(\tau; \mu_0, \mu)$ and $U(\tau; \mu_0, \mu)$ as

$$\left(\frac{1}{\mu_0} + \frac{1}{\mu}\right)S(\tau; \mu_0, \mu) = cR(\tau; \mu_0, \mu), \tag{33}$$

$$\left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)T(\tau; \mu_0, \mu) = cU(\tau; \mu_0, \mu), \tag{34}$$

and we substitute them for $S(\tau; \mu_0, \mu)$ and $T(\tau; \mu_0, \mu)$ in Eqs. (20) and (22). By using the explicit form of eigenfunctions (2b) and (2c) for $\nu \in (0, 1)$ and $\nu = \nu_0$, we get, after a partial-fraction analysis and taking into

[15] K. M. Case, *Proceedings of the Symposium on Transport Theory, April, 1967* (American Mathematical Society, Providence, R.I.) (to be published).

[16] N. I. Muskelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

[17] A. Leonard and T. W. Mullikin, J. Math. Phys. 5, 399 (1964).

account Eqs. (25) and (26), the following four equations:

$$X(\mu) - e^{-\tau/\nu}Y(\mu) = \lambda(\nu)R(\tau; \nu, \mu) + \frac{c\nu}{2} P\int_0^1 \frac{R(\tau; \mu', \mu)}{\nu - \mu'} d\mu' - \frac{c\nu}{2}\int_0^1 \frac{U(\tau; \mu', \mu)}{\nu + \mu'} d\mu' e^{-\tau/\nu}, \quad (35)$$

$$e^{-\tau/\nu}X(\mu) - Y(\mu) = -\lambda(\nu)U(\tau; \nu, \mu) - \frac{c\nu}{2} P\int_0^1 \frac{U(\tau; \mu', \mu)}{\nu - \mu'} d\mu' - \frac{c\nu}{2} e^{-\tau/\nu}\int_0^1 \frac{R(\tau; \mu', \mu)}{\nu + \mu'} d\mu', \quad (36)$$

$$X(\mu) - e^{-\tau/\nu_0}Y(\mu) = \frac{c\nu_0}{2}\int_0^1 \frac{R(\tau; \mu', \mu)}{\nu_0 - \mu'} d\mu' - \frac{c\nu_0}{2}\int_0^1 \frac{U(\tau; \mu', \mu)}{\nu_0 + \mu'} d\mu' e^{-\tau/\nu_0}, \quad (37)$$

$$e^{-\tau/\nu_0}X(\mu) - Y(\mu) = -\frac{c\nu_0}{2}\int_0^1 \frac{U(\tau; \mu', \mu)}{\nu - \mu'} d\mu' - \frac{c\nu_0}{2} e^{-\tau/\nu_0}\int_0^1 \frac{R(\tau; \mu', \mu)}{\nu + \mu'} d\mu'. \quad (38)$$

Now, if we first multiply Eq. (27) by $X(\mu)$ and subtract Eq. (28) multiplied by $Y(\mu)$, then multiply Eq. (27) by $Y(\mu)$ and subtract Eq. (28) multiplied by $X(\mu)$, we get equations identical to Eqs. (35) to (38), except that

$$R(\tau; \mu_0, \mu) \to X(\mu)X(\mu_0) - Y(\mu)Y(\mu_0), \quad (39)$$

$$U(\tau; \mu_0, \mu) \to Y(\mu)X(\mu_0) - Y(\mu_0)X(\mu). \quad (40)$$

Thus, the above bilinear expressions are solutions of Eqs. (35)–(38). These solutions are also unique, because Eqs. (35)–(38) uniquely determine $R(\tau; \mu_0, \mu)$ and $U(\tau; \mu_0, \mu)$.

By expressing $S(\tau; \mu_0, \mu)$ and $T(\tau; \mu_0, \mu)$ in Eqs. (25) and (26) in terms of $X(\mu)$ and $Y(\mu)$ [via Eqs. (39), (40), (33), and (34)], we get a system of nonlinear integral equations for $X(\mu)$ and $Y(\mu)$,

$$X(\mu) = 1 + \frac{c\mu}{2}\int_0^1 \frac{X(\mu)X(\mu') - Y(\mu)Y(\mu')}{\mu + \mu'} d\mu', \quad (41)$$

$$Y(\mu) = e^{-\tau/\mu} + \frac{c\mu}{2}\int_0^1 \frac{Y(\mu)X(\mu') - X(\mu)Y(\mu')}{\mu - \mu'} d\mu', \quad (42)$$

with the conditions, which must be satisfied by $X(\mu)$ and $Y(\mu)$,

$$1 = \frac{c\nu_0}{2}\int_0^1 \frac{X(\mu')}{\nu_0 - \mu'} d\mu' + \frac{c\nu_0}{2} e^{-\tau/\nu_0}\int_0^1 \frac{Y(\mu')}{\nu_0 + \mu'} d\mu', \quad (43)$$

$$1 = \frac{c\nu_0}{2}\int_0^1 \frac{X(\mu')}{\nu_0 + \mu'} d\mu' + \frac{c\nu_0}{2} e^{\tau/\nu_0}\int_0^1 \frac{Y(\mu')}{\nu_0 - \mu'} d\mu', \quad (44)$$

following from Eqs. (27) and (28) for $\nu = \nu_0$.

Let us now also show that the system of nonlinear integral equations (41) and (42), together with the conditions (43) and (44), uniquely determine $X(\mu)$ and $Y(\mu)$.

First, we note that $X(\mu)$ and $Y(\mu)$ can be analytically continued outside the interval (0, 1), by using Eqs. (41) and (42). It can be easily verified[10] that if $X(\mu)$ and $Y(\mu)$ satisfy Eqs. (41) and (42), but not necessarily (43) and (44), they also satisfy the integral equations

$$\Lambda(z)X(z) = 1 - \frac{cz}{2}\int_0^1 \frac{X(\mu)}{z - \mu} d\mu - \frac{zc}{2} e^{-\tau/z}\int_0^1 \frac{Y(\mu)}{z + \mu} d\mu, \quad (45)$$

$$\Lambda(z)Y(z) = e^{-\tau/z}\left[1 - \frac{cz}{2}\int_0^1 \frac{X(\mu)}{z + \mu} d\mu\right] - \frac{zc}{2}\int_0^1 \frac{Y(\mu)}{z - \mu} d\mu, \quad z \notin (-1, 1). \quad (46)$$

By applying the Plemelj formula[16] to the above equations for $z \in (0, 1)$, we get the singular integral equations (27) and (28). Since these singular integral equations, together with the conditions (43) and (44), uniquely determine $X(\mu)$ and $Y(\mu)$, the same is true for the nonlinear integral equations (41) and (42) combined with the conditions (43) and (44).

We can now easily prove that $X(z)$ and $Y(z)$ are analytic functions in the whole complex plane, except at $z = 0$, where they have an essential singularity.

Since $\Lambda(z) = \Lambda(-z)$, we see at once from Eqs. (45) and (46) that $X(z)$ and $Y(z)$ satisfy the relation[10]

$$Y(z) = e^{-\tau/z}X(-z), \quad (47)$$

which is valid in the whole complex plane. In view of Eqs. (41), (42), (45), (46), and the conditions (43) and (44), the $X(z)$ and $Y(z)$ could be singular only for $z = -\nu_0$ and $z \in (-1, 0)$. However, since $X(z)$ and $Y(z)$ are analytic for $z = \nu_0$ and $z \in (0, 1)$, $z = 0$ excluded, the same is true also for $z = -\nu_0$ and $z \in (-1, 0)$, because of Eq. (47), while it follows from Eqs. (41) and (42) that, for $z = 0$, the functions $X(z)$ and $Y(z)$ have an essential singularity.

B. Green's Function for a Slab

We now show how other slab problems can be solved with the help of the solution for the albedo problem. Evidently, what we need is the solution of the Green's function problem, defined by the non-homogeneous transport equation

$$\left(\mu \frac{\partial}{\partial x} + 1\right) G(x_0, \mu_0 \to x, \mu; \tau) = \frac{c}{2} \int_{-1}^{1} G(x_0, \mu_0 \to x, \mu'; \tau)\, d\mu' + \delta(\mu_0 - \mu)\delta(x_0 - x), \quad 0 < x_0 < \tau, \tag{48}$$

with boundary conditions

$$G(x_0, \mu_0 \to 0, \mu; \tau) = 0, \quad \mu > 0,$$
$$G(x_0, \mu_0 \to \tau, -\mu; \tau) = 0, \quad \mu > 0. \tag{49}$$

In order to determine the emergent distributions $G(x_0, \mu_0 \to 0, -\mu; \tau)$ and $G(x_0, \mu_0 \to \tau, \mu; \tau)$, $\mu > 0$, we need the infinite-medium Green's function $G(x_0, \mu_0 \to x, \mu; \infty)$ which satisfies Eq. (48). This function can be solved in terms of Case's eigenfunctions[3,6] and is, therefore, considered as known.

Let us seek the solution of our problem in the form

$$G(x_0, \mu_0 \to x, \mu; \tau) = G(x_0, \mu_0 \to x, \mu; \infty) + \psi(x, \mu; \tau). \tag{50}$$

By substituting Eq. (50) into Eqs. (48) and (49), it follows that the unknown function $\psi(x, \mu; \tau)$ must satisfy the homogeneous transport equation (1) and the boundary conditions

$$\psi(0, \mu; \tau) = -G(x_0, \mu_0 \to 0, \mu; \infty), \quad \mu > 0,$$
$$\psi(\tau, -\mu, \tau) = -G(x_0, \mu_0 \to \tau, -\mu; \infty), \quad \mu > 0. \tag{51}$$

In this way, the Green's function problem for a slab is reduced to determining the infinite-medium Green's function and to solving two albedo problems discussed in the previous section.

C. Distribution Inside a Slab

Once the surface distributions for a slab problem are known, the inside distribution can be determined by using the full-range completeness and orthogonality relations of Case's eigenfunctions. In view of the results of the previous section, it is sufficient to show how the albedo problem, defined by Eqs. (1) and (9), can be solved completely in terms of the function $S(\tau; \mu_0, \mu)$ or $T(\tau; \mu_0, \mu)$ and Case's eigenfunctions.

We start with the eigenfunction expansion given by Eq. (10). By using the full-range orthogonality relations of Case's eigenfunctions (7a), (7b), and (6), we can determine the expansion coefficients with the help of the function $S(\tau; \mu_0, \mu)$, for instance, by setting $x = 0$ in Eq. (10). In this way we get

$$N(\nu)A(\nu) = \mu_0 \phi(\nu, \mu_0) - \frac{1}{2} \int_0^1 S(\tau; \mu_0, \mu)\phi(\nu, -\mu)\, d\mu, \quad \nu \in (-1, 1), \quad \nu = \pm\nu_0. \tag{52}$$

On the other hand, by using the function $T(\tau; \mu_0, \mu)$ and setting $x = \tau$ in Eq. (10), we obtain

$$N(\nu)A(\nu) = \mu_0 \phi(\nu, \mu_0) e^{\tau(1/\nu - 1/\mu_0)} + \tfrac{1}{2} e^{\tau/\nu} \int_0^1 T(\tau; \mu_0, \mu)\phi(\nu, \mu)\, d\mu, \quad \nu \in (-1, 1), \quad \nu = \pm\nu_0. \tag{53}$$

By using Eqs. (20) and (22), satisfied by $S(\tau; \mu_0, \mu)$ and $T(\tau; \mu_0, \mu)$, it can be easily verified that the rhs of Eqs. (52) and (53) are indeed identical.

IV. HALF-SPACE PROBLEMS

We now briefly discuss half-space problems and show how they can be solved exactly in closed form.

The equations for the half-space problems can be formally obtained from the slab equations of the previous section by limiting τ to infinity and writing

$$\lim S(\tau; \mu_0, \mu) = S(\mu_0, \mu), \tag{54}$$

$$\lim X(\mu) = H(\mu), \tag{55}$$

$$\lim T(\tau; \mu_0, \mu) = 0, \tag{56}$$

$$\lim Y(\mu) = 0. \tag{57}$$

(We assume, of course, that $c < 1$.) The resulting half-space equations are much simpler than the equations of the previous section. In fact, it will be shown that a closed-form solution for $H(\mu)$ can be obtained. Once $H(\mu)$ is known, all other half-space problems can be solved exactly in terms of $H(\mu)$ and Case's eigenfunctions.

To show that, let us consider the explicit form of the singular integral equation for the function $H(\mu)$:

$$\lambda(\nu)H(\nu) = 1 - \frac{c\nu}{2} P \int_0^1 \frac{H(\mu)}{\nu - \mu}\, d\mu, \tag{58}$$

together with the condition

$$0 = 1 - \frac{c\nu_0}{2} \int_0^1 \frac{H(\mu)}{\nu_0 - \mu}\, d\mu, \tag{59}$$

resulting from Eq. (27) of the previous section by letting τ approach infinity.

We assume that a solution of Eq. (58) exists and that it satisfies a Hölder condition[16] for $\mu \in (0, 1)$ and the condition (59). Guided by the form of the singular integral equation (58), we define an analytic function $F(z)$ in the complex plane cut from -1 to 1 as

$$\Lambda(z)F(z) = 1 + \frac{cz}{2}\int_0^1 \frac{H(\mu)}{\mu - z}\, d\mu. \qquad (60)$$

Since $\Lambda(z)$ is analytic in the whole cut plane, with $\Lambda(z) \neq 0$, except for $z = \pm\nu_0$, the same is true also for $F(z)$, in view of our assumption on $H(\mu)$ for $\mu \in (0, 1)$.[16] For $z = \pm\nu_0$, $\Lambda(z)$ has simple zeros, so $F(z)$ may have simple poles there. However, it follows from Eq. (59) that $F(z)$ is analytic also for $z = \nu_0$.

By applying the Plemelj formula[16] to Eq. (60), and taking into account Eqs. (58) and (4), it follows that

$$F^+(x) = F^-(x) = H(x), \quad x \in (0, 1), \qquad (61)$$

and

$$F^+(x)\Lambda^+(x) = F^-(x)\Lambda^-(x), \quad x \in (-1, 0). \quad (62)$$

We see from Eq. (61) that $F(z)$ is the analytic continuation of $H(\mu)$, $\mu \in (0, 1)$. Therefore,

$$H(z) = F(z) \qquad (63)$$

and $H(z)$ is analytic in whole complex plane, cut from -1 to 0, except for $z = -\nu_0$, where it may have a simple pole.

Let us now consider the product $H(z)H(-z)\Lambda(z)$. This is an even function of z, analytic in the whole complex plane cut from -1 to 1, since $H(z)$ has at most a simple pole for $z = -\nu_0$. Moreover, this product is, in view of Eq. (60), also continuous across the cut $(-1, 1)$, with $H^2(0)\Lambda(0) = 1$, as follows from Eqs. (58) and (3). Hence $H(z)H(-z)\Lambda(z)$ is analytic in the whole complex plane and we have

$$H(z)H(-z)\Lambda(z) = 1. \qquad (64)$$

Two important results follow at once from the above relation. First, we see that $H(z)$ has indeed a simple pole for $z = -\nu_0$. Second, by combining Eqs. (60) and (64), we get the nonlinear integral equation for the function $H(z)$:

$$H(z)\left[1 + \frac{cz}{2}\int_0^1 \frac{H(\mu)}{z + \mu}\, d\mu\right] = 1, \quad z \notin (-1, 0). \quad (65)$$

Now, we turn our attention to Eq. (62). We see that $H(z)$ is also the solution of the homogeneous Hilbert problem.[16] By requiring that the solution is analytic in the whole complex plane, cut from -1 to 0, with a simple pole at $z = -\nu_0$, we can solve this problem uniquely in a closed form. We obtain[18]

$$H(z) = \frac{1 + z}{1 + z/\nu_0} \exp\left[\frac{z}{2\pi i}\int_0^1 \ln \frac{\Lambda^+(x)}{\Lambda^-(x)} \frac{dx}{(z + x)x}\right]. \qquad (66)$$

[18] S. Pahor, Nucl. Sci. Eng. 26, 192 (1966).

In deriving the above solution we have also justified the assumptions, made in the beginning of this discussion, that a solution of Eq. (58) exists and satisfies a Hölder condition for $\mu \in (0, 1)$.

It is obvious now, from the results of the previous section, how to express the emergent distribution for the albedo problem in terms of the function $H(\mu)$ and how to determine the emergent distribution for the half-space Green's function problem.

However, there is the so-called Milne problem, characteristic for the half-space, which should be mentioned. It turns out that for the half-space the homogeneous transport equation (1) has solutions even for a zero incident distribution, if we drop the condition that solutions are bounded at infinity. We may say that in this case we have sources at infinity.

The Milne problems [whose solution is defined as $\psi(x, \mu; \nu)$] are conveniently defined by the homogeneous transport equation (1) and the boundary conditions

$$\psi(0, \mu; \nu) = 0, \quad \mu > 0,$$
$$\psi(x, \mu; \nu) \to \phi(-\nu, \mu)e^{x/\nu},$$
$$x \to \infty, \quad \nu \in (0, 1), \quad \nu = \nu_0. \quad (67)$$

We want to determine the emergent distribution $\psi(0, -\mu; \nu)$, $\mu > 0$. To do that, let us define the following "albedo" problem:

$$\psi(0, \mu) = \phi(-\nu, \mu), \quad \mu > 0,$$
$$\psi(x, \mu) \to \phi(-\nu, \mu)e^{x/\nu},$$
$$x \to \infty, \quad \nu \in (0, 1), \quad \nu = \nu_0. \quad (68)$$

Obviously, the solution of this problem is

$$\psi(x, \mu) = \phi(-\nu, \mu)e^{x/\nu}, \quad \mu \in (-1, 1). \qquad (69)$$

Let us decompose the solution $\psi(x, \mu)$ into two parts:

$$\psi(x, \mu) = \psi_1(x, \mu) + \psi_2(x, \mu), \qquad (70)$$

where

$$\psi_1(0, \mu) = 0, \quad \mu > 0,$$
$$\psi_1(x, \mu) \to \phi(-\nu, \mu)e^{x/\nu}, \quad x \to \infty, \qquad (71)$$

and

$$\psi_2(0, \mu) = \phi(-\nu, \mu), \quad \mu > 0,$$
$$\psi_2(x, \mu) \to 0, \quad x \to \infty, \qquad (72)$$

with $\psi_1(x, \mu)$ and $\psi_2(x, \mu)$ satisfying the transport equation (1). Evidently, $\psi_1(x, \mu)$ is just the solution of our Milne problem, while $\psi_2(x, \mu)$ is the solution of a "proper" albedo problem with $\psi_2(\infty, \mu) = 0$.

Therefore, we may apply Eq. (17), with $\tau = \infty$, to $\psi_2(0, \mu)$. In this way we get

$$\psi_2(0, -\mu) = \phi(\nu, \mu) - \psi(0, -\mu; \nu) = \frac{1}{2\mu}\int_0^1 S(\mu', \mu)\psi_2(0, \mu')\, d\mu', \quad \mu > 0, \quad (73)$$

or, in view of Eq. (72),

$$\psi(0, -\mu; \nu) = \phi(\nu, \mu) - \frac{1}{2\mu}\int_0^1 S(\mu', \mu)\phi(-\nu, \mu')\, d\mu', \quad \mu > 0, \quad \nu \in (0, 1), \quad \nu = \nu_0. \quad (74)$$

Now, we express the function $S(\mu', \mu)$ in Eq. (74) in terms in the function $H(\mu)$ by using Eqs. (33) and (39). By taking into account Eqs. (2c), (2d), and (65), it follows that $\psi(0, -\mu; \nu)$ can be expressed in terms of the function $H(z)$ as

$$\psi(0, -\mu; \nu) = \frac{c}{2} P \frac{\nu}{\nu - \mu} \frac{H(\mu)}{H(\nu)} + \lambda(\nu)\delta(\nu - \mu), \quad \mu > 0, \quad \nu \in (0, 1), \quad \nu = \nu_0. \quad (75)$$

Of course, the only physically meaningful solution is that for $\nu = \nu_0$. However, the other solutions are useful for constructing the half-space solutions inside the medium.

Once the surface distribution for any particular half-space problem is known, the complete solution can be obtained by using the full-range orthogonality relations (6), (7a), and (7b).

For instance, let us construct the complete solution of the albedo problem. This solution, denoted as $\psi(0, \mu_0 \to x, \mu)$, satisfies the transport equation (1) and the boundary conditions

$$\psi(0, \mu_0 \to 0, \mu) = \delta(\mu_0 - \mu), \quad \mu > 0,$$
$$\psi(0, \mu_0 \to x, \mu) \to 0, \quad x \to \infty. \quad (76)$$

The emergent distribution $\psi(0, \mu_0 \to 0, -\mu)$, $\mu > 0$, can be expressed in terms of the function $S(\mu_0, \mu)$, in view of Eq. (13), as

$$\psi(0, \mu_0 \to 0, -\mu) = (1/2\mu)S(\mu_0, \mu). \quad (77)$$

Because of the condition (76) at infinity, we expand $\psi(0, \mu_0 \to x, \mu)$ only in terms of the exponentially decreasing eigenfunctions

$$\psi(0, \mu_0 \to x, \mu) = A(\nu_0)\phi(\nu_0, \mu)e^{-x/\nu_0} + \int_0^1 A(\nu)\phi(\nu, \mu)e^{-x/\nu}\, d\nu. \quad (78)$$

By setting $x = 0$ and expressing $\psi(0, \mu_0 \to 0, -\mu)$ using Eq. (77), we determine the expansion coefficients as has been explained. Taking into account Eq. (74), we finally get

$$\frac{1}{\mu_0}\psi(0, \mu_0 \to x, \mu) = \frac{\psi(0, -\mu_0; \nu_0)}{N(\nu_0)}\phi(\nu_0, \mu)e^{-x/\nu_0} + \int_0^1 \frac{\psi(0, -\mu_0; \nu)}{N(\nu)}\phi(\nu, \mu)e^{-x/\nu}\, d\nu, \quad (79)$$

and this represents the complete solution of the half-space albedo problem.

V. CONCLUSION

The method presented in this paper is based on the ideas introduced into transport theory by Ambarzumian, Chandrasekhar, and Case. First, the solution for the surface distributions is sought. (In a number of applications this is all that is required.) By using the infinite-medium eigenfunctions, a system of singular integral equations together with the uniqueness conditions is derived for the surface distributions in a simple and straightforward way. This system is the basis of the whole theory.

One could stop there and determine the surface distributions by solving numerically the system of singular integral equations combined with the uniqueness conditions. Or, this system can be reduced to certain uncoupled Fredholm integral equations which can be then used for numerical computations. Finally, the surface distributions can be also computed by using the nonlinear integral equations. It is evident that the question of how to compute the surface distributions is the most important one, since once these are known, the complete solution can be found by quadrature.

As far as we know, the system of singular integral equations (27) and (28) ($\nu = \nu_0$ included) has not been used to compute $X(\mu)$ and $Y(\mu)$. For numerical computations, this system can be rearranged so that the principle-value integrals disappear. Then it could be solved approximately, for instance, by reducing it to a system of linear algebraic equations.

The other possibility, to solve numerically the above mentioned Fredholm integral equations, was considered by Leonard and Mullikin.[17] They showed that these Fredholm integral equations converge rapidly under iteration for all c and τ. Unfortunately, the kernels of these equations are not simple functions and to compute them requires quite a lot of work.

So, it seems that the simplest way to obtain numerical values for the surface distributions is the straightforward iteration of the nonlinear integral equations

(41) and (42). This was done successfully by Chandrasekhar and others.[1,19] Since the system (41) and (42) is not uniquely soluble, the conditions (43) and (44) should be used as a check. At the same time, this would give an estimate of the accuracy of the iterations.

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ACKNOWLEDGMENTS

We wish to thank Professor G. C. Summerfield for stimulating discussions. One of us (S. P.) wishes to express his gratitude to the members of the Department of Nuclear Engineering, The University of Michigan, for hospitality shown him and to the Institute of Science and Technology for financial support.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 10, NUMBER 4 APRIL 1969

Realization of Chiral Symmetry in a Curved Isospin Space

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(Received 14 September 1968)

The nonlinear realizations of the chiral group $SU(2) \otimes SU(2)$ are studied from a geometric point of view. The three-dimensional nonlinear realization, associated with the pion field, is considered as a group of coordinate transformations in a three-dimensional isospin space of constant curvature, leaving invariant the line element. Spinor realizations in general coordinates are constructed by combined coordinate–spin-space transformations in analogy to Pauli's method for spinors in general relativity. The description of vector mesons and possible chiral-invariant Lagrangians, which yield the various nonlinear models in specific frames of general coordinates, are discussed.

1. INTRODUCTION

Chiral-invariant Lagrangians are currently used as a practical tool to study the implications of current algebra.[1,2] The Lagrangians are to be constructed as functionals of fields, which have definite transformation properties under the chiral group $SU(2) \otimes SU(2)$. Because there does not exist a three-dimensional linear representation of the group, it has been suggested[3,4] that the pion field transforms according to the three-dimensional nonlinear realization. This implies that chiral symmetry is a pure interaction symmetry not shared by the asymptotic fields.

A systematic development of the nonlinear realizations can depart from different points of view. While the transformation laws are nonlinear, the transformations are still implemented by unitary operators in quantum theory. Weinberg has studied the most general form for the commutators of generators and fields.[5] On the other hand, for a better understanding of the mathematical nature of nonlinear realizations, it seems worthwhile to keep the analogy to linear representations as close as possible.

A nonlinear realization is a representation of the group in a curved instead of Euclidean space. We show in Sec. 2 that the chiral group is the invariance group of the metric in a three-dimensional space with constant curvature $K = 1/f^2$. This "fundamental" nonlinear realization is associated with the pion field. While the field components are the coordinates in the curved space, space–time derivatives are tangents and transform as contravariant vectors under coordinate transformations. The Riemannian geometry of the curved space replaces the Euclidean geometry of linear-representation spaces. Following Pauli's treatment of spinors in general relativity,[6] we study in Sec. 3 spinor realizations of the chiral group in general coordinates by combined coordinate–spin space transformations. The realizations associated with vector mesons are discussed in Sec. 4.

The various nonlinear models treated in the literature[7–12] result from a specific choice of general pion coordinates. This is in complete agreement with Weinberg,[5] but we think that our more geometric

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