General Coupling Coefficients for the Group $SU(3)$*

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(Received 21 February 1966)

A Hilbert space method, previously applied to the group $SU(2)$, is employed to examine the representations $\mathfrak{D}^{\lambda\mu}$ and the reduction of the direct-product representation of the group $SU(3)$. The base vectors $|\lambda\mu;\alpha>$, an orthogonal Hilbert space of homogeneous polynomials, are transformed to the base vectors $|\lambda\mu;\alpha>_c$, and are associated with the complex conjugate representation by an explicit R- conjugation operation. For the general direct-product representation $\mathfrak{D}^{\lambda_1\mu_1} \otimes \mathfrak{D}^{\lambda_2\mu_2}$, explicit expressions are derived for the vector coupling coefficients and the number of times the irreducible representation $\mathfrak{D}^{\lambda_3\mu_3}$ is contained in the direct product. Two methods of labeling the degenerate states are given, the reduction of the direct product is shown to be complete, and the symmetry relations of the $3(\lambda\mu)$ coefficients are discussed.

INTRODUCTION

THE purpose of this paper is to examine the representations $\mathfrak{D}^{\lambda\mu}$ and the reduction of the direct-product representation $\mathfrak{D}^{\lambda_1\mu_1} \otimes \mathfrak{D}^{\lambda_2\mu_2}$ in a concise, transparent manner. The particular approach is a Hilbert space method devised by Bargmann[1] to study the representations of the rotation group.

The essential ideas of the Bargmann method, the use of homogeneous polynomials of complex variables as the base vectors associated with irreducible representations, and the construction of an invariant that yields the coefficients which reduce the direct product, were employed by van der Waerden[2] in 1932, and known to Weyl[3] (1925). Bargmann's essential contribution was to combine these ideas, with his function space $\mathfrak{F}_n$,[4] in a clear and simple treatment of the many (seemingly diverse) properties of $SU(2)$. Moreover, the essential features of the method are in a form which may be generalized to $SU(3)$. The Bargmann method may also be generalized to $SU(n)$, and this problem is to be discussed in a subsequent paper.

The essential properties of the function space $\mathfrak{F}_n$ necessary to read the article have been included in Sec. 1. For a comprehensive treatment, with proofs, consult Bargmann's two papers.[1,4] The particular subspace $\mathfrak{Q}_{\lambda\mu}$ of $\mathfrak{F}_6$, the space of base vectors $|\lambda\mu;\ \alpha\rangle$, is defined in Sec. 2. The invariant Hilbert space $\mathfrak{Q}_{\lambda\mu}$ is now associated with a 2-rowed Young tableau, necessitating an antisymmetry operation with respect to the columns of the tableau. The row labels $\alpha \equiv (y,\ t,\ t_0)$ of the representations $\mathfrak{D}^{\lambda\mu}$ are defined in the standard manner using the two linear commuting operators Y, T_0, of the rank-two group, and the Casimir operator $\mathbf{T}^2$ of the $SU(2)$ subgroup. The base vectors $|\lambda\mu;\ \alpha\rangle_c$ associated with the complex conjugate representation are obtained from $|\lambda\mu;\ \alpha\rangle$ by an explicit change of variables. Section 3 is devoted to the reduction of the direct-product representation and the $3(\lambda\mu)$ symbols. Unlike the $SU(2)$ case, the condition that the invariant $h_k(k_i)$ lies in the triple-product space does not uniquely determine the parameters k_i. This is the degeneracy (or multiplicity) problem; there exists g direct-product vectors $|\lambda_3\mu_3;\ \alpha_3\rangle_k$, $k = 0, 1, \cdots, g-1$, associated with the irreducible representation $\mathfrak{D}^{\lambda_3\mu_3}$ contained in the representation $\mathfrak{D}^{\lambda_1\mu_1} \otimes \mathfrak{D}^{\lambda_2\mu_2}$, and Sec. 3C discusses two methods of handling this problem. The recoupling or $6(\lambda\mu)$ coefficients are discussed in an accompanying article.[5]

The notation follows, as closely as possible, that of Bargmann's article. In particular, the Hermitian adjoint of an operator, or matrix, B, is indicated by B^*, the transpose of a matrix B by ${}^tB$, and the complex conjugate of α by $\bar{\alpha}$.

1. THE HILBERT SPACE $\mathfrak{F}_n$

A. Definition of $\mathfrak{F}_n$

The elements of $\mathfrak{F}_n$ are entire analytic functions of $f(z)$, where $z = (z_1, z_2, \cdots, z_n)$ is a point of the

* This work was supported in part by the National Science Foundation. (Grant No. GP-1536). This paper is based on a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Michigan, Ann Arbor, Michigan.
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[1] V. Bargmann, Rev. Mod. Phys. **34**, 829 (1962).
[2] B. L. van der Waerden, *Die gruppentheoretische Methode in der Quantenmechanik* (Julius Springer-Verlag, Berlin, 1932).
[3] H. Weyl, Math. Z. **23**, 271 (1925); **24**, 377 (1926); and **24** 789 (1926). The author thanks the referee for bringing this reference to his attention.
[4] For the initial development of the function space, see V. Bargmann, Commun. Pure Appl. Math. **14**, 187 (1961).
[5] M. Resnikoff, following paper, J. Math. Phys. **8**, 79 (1967).

n-dimensional complex Euclidean space C_n. For two elements f, f' of $\mathfrak{F}_n$, the inner product (f, f') is defined

$$(f, f') = \int \overline{f(z)} f'(z)\, d\mu_n(z), \qquad (1.1a)$$

where $\overline{f(z)}$ is the complex conjugate of $f(z)$ and the measure $d\mu_n(z)$ is

$$d\mu_n(z) = \pi^{-n} \exp(-z\cdot\bar{z})\, d^n z, \qquad (1.1b)$$

$z\cdot\bar{z} = z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$, and

$$d^n z = \prod_{k=1}^{n} dx_k\, dy_k, \qquad z_k = x_k + iy_k.$$

The integral (1.1a) is taken over the whole space C_n. It may be shown that[1]

$$((z_i)^{h_i}, (z_j)^{h_j}) = \delta_{ij}\, \delta_{h_i,h_j}\, (h_i)!. \qquad (1.2)$$

The operators on $\mathfrak{F}_n$ may be constructed as functions of z_n and the differential operator, $d_k \equiv \partial/\partial z_k$. The commutation relations

$$[z_k, z_m] = [d_k, d_m] = 0, \qquad [d_k, z_m] = \delta_{km} \qquad (1.3)$$

are obvious. For any elements f, g of $\mathfrak{F}_n$, z_k and d_k are adjoint with respect to the inner product

$$(z_k f, g) = (f, d_k g), \qquad (1.4)$$

as may be shown by expanding $f(z)$ in a power series (see Bargmann[1]).

B. Bargmann Operators M_{ij}

On the Hilbert space $\mathfrak{F}_3$, define the differential operators M_{ij}

$$M_{ij} f(\zeta) = \tfrac{1}{2} \sum_{\alpha,\beta=1}^{3} \zeta_\alpha (m_{ij})_{\alpha\beta} (\partial/\partial\zeta_\beta) f(\zeta), \qquad (1.5)$$

where $\zeta = (\xi, \eta, \sigma)$ replaces $z = (z_1, z_2, z_3)$ as a point in the space C_3, and the matrices m_{ij} are linear combinations of the infinitesimal matrices b_j,

$$ib_j = (\partial/\partial\theta_j) T_u f \,|_{\theta_k=0} \text{ all } k, \quad j = 1, \cdots, 8, \qquad (1.6)$$

θ_k, $k = 1, \cdots, 8$, representing the eight parameters of the group.[6] The matrices m_{ij} differ by factors from the infinitesimal matrices of Behrends *et al.*[7]:

$$\tfrac{1}{2}(m_{12}) = 6^{\frac{1}{2}} E_1, \qquad \tfrac{1}{2}(m_{21}) = 6^{\frac{1}{2}} E_{-1},$$
$$\tfrac{1}{2}(m_{13}) = 6^{\frac{1}{2}} E_2, \qquad \tfrac{1}{2}(m_{31}) = 6^{\frac{1}{2}} E_{-2},$$
$$\tfrac{1}{2}(m_{23}) = 6^{\frac{1}{2}} E_3, \qquad \tfrac{1}{2}(m_{32}) = 6^{\frac{1}{2}} E_{-3},$$
$$\tfrac{1}{2}(t_0) = \sqrt{3} H_1, \qquad \tfrac{1}{2}(y) = 6 H_2,$$

y, t_0 being the two linear commuting matrices of the rank-two group.

The Bargmann differential operators, Eq. (1.5), are

$$M_{12} = \xi \frac{\partial}{\partial\eta}, \quad M_{21} = \eta \frac{\partial}{\partial\xi}, \quad M_{13} = \xi \frac{\partial}{\partial\sigma},$$
$$M_{31} = \sigma \frac{\partial}{\partial\xi}, \quad M_{23} = \eta \frac{\partial}{\partial\sigma}, \quad M_{32} = \sigma \frac{\partial}{\partial\eta},$$
$$T_0 = \frac{1}{2}\left(\xi \frac{\partial}{\partial\xi} - \eta \frac{\partial}{\partial\eta}\right), \quad Y = \xi \frac{\partial}{\partial\xi} + \eta \frac{\partial}{\partial\eta} - 2\sigma \frac{\partial}{\partial\sigma}. \qquad (1.7)$$

If $f(\zeta)$ is a homogeneous polynomial of degree m in ζ, then so is $M_{ij} f(\zeta)$, according to the definition Eq. (1.5). Since the invariant Hilbert space of base vectors is given by the degree in the variables ζ, and since the function $M_{ij} f$ has the same degree m, the operators M_{ij} are said to operate *within* the Hilbert space, as raising and lowering operators.

Note that, using Eq. (1.4),

$$(M_{ij} f, g) = (f, M_{ji} g). \qquad (1.8)$$

In particular, Y, T_0 are Hermitian with respect to the inner product.

2. THE REPRESENTATIONS $\mathfrak{D}^{\lambda\mu}$

A. Hilbert Space $\mathfrak{Q}_{\lambda\mu}$

Let an element f of the function space $\mathfrak{F}_6$ be written $f(\zeta_1, \zeta_2)$, where ζ_1, ζ_2 are points in a three-dimensional complex Euclidean space C_3. $\mathfrak{Q}_{\lambda\mu}$, the subspace of $\mathfrak{F}_6$, is the space of homogeneous polynomials $f(\zeta_1, \zeta_2)$ of degree $\lambda + \mu$ in ζ_1, and μ in ζ_2.

To put this in operator form, define the operator[8] T_{ij}

$$T_{ij} = \xi_i \frac{\partial}{\partial\xi_j} + \eta_i \frac{\partial}{\partial\eta_j} + \sigma_i \frac{\partial}{\partial\sigma_j}. \qquad (2.1)$$

An element $f(\zeta_1, \zeta_2)$ belongs to $\mathfrak{Q}_{\lambda\mu}$ if and only if the Euler equations

$$T_{11} f = (\lambda + \mu) f, \qquad T_{22} f = \mu f \qquad (2.2)$$

are satisfied.[9]

The spaces $\mathfrak{Q}_{\lambda\mu}$ and $\mathfrak{Q}_{\lambda'\mu'}$ are obviously orthogonal for $\lambda \neq \lambda'$, or $\mu \neq \mu'$, by Eq. (1.2). The function space $\mathfrak{F}_6$ may then be decomposed into the sum of mutually orthogonal subspaces

$$\mathfrak{F}_6 = \sum_{\lambda,\mu} \mathfrak{Q}_{\lambda\mu}. \qquad (2.3)$$

[6] The matrices b_j and m_{ij} may, of course, be obtained directly from the $SU(3)$ matrix as parameterized by F. D. Murnaghan, *The Unitary and Rotation Groups* (Spartan Books, Washington, D. C., 1962), in analogy to the $SU(2)$ case. See M. Resnikoff, University of Michigan preprint (1965).

[7] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. **34**, 1 (1962).

[8] The operator T_{ij} was first considered by V. Bargmann and M. Moshinsky, Nucl. Phys. **18**, 697 (1960); **23**, 177 (1961).

[9] T_{11} and T_{22} are analogous to Bargmann's operator N, $N\cdot v^j_m = j\cdot v^j_m$ (see Ref. 1).

The Euler equations (2.2) require that the homogeneous polynomials $f(\zeta_1, \zeta_2)$ be of degree $\lambda + \mu$ in ζ_1, and degree μ in ζ_2, the number of boxes in the first and second rows, respectively, of the Young tableau for $SU(3)$. The additional condition from the Young tableau is that $f(\zeta_1, \zeta_2)$ be antisymmetric in the μ columns. Since ζ_2 is of degree μ, ζ_2 may occur only in the antisymmetric forms

$$\begin{aligned}\delta_{12} &= (\delta_{12}^{(1)}, \delta_{12}^{(2)}, \delta_{12}^{(3)}) \\ &= (\eta_1\sigma_2 - \eta_2\sigma_1, \sigma_1\xi_2 - \sigma_2\xi_1, \xi_1\eta_2 - \xi_2\eta_1).\end{aligned} \tag{2.4}$$

That is, the homogeneous polynomials f must have the functional form $f(\zeta_1, \delta_{12})$.[10] The differential form of this antisymmetry requirement is that

$$T_{12}f(\zeta_1, \zeta_2) = 0, \tag{2.5}$$

where T_{ij} is given by Eq. (2.1). T_{12} serves as the Weyl alternation operator,[11] $\Sigma\delta_q\cdot\mathbf{g}$, the operator which antisymmetrizes an unsymmetrized tensor with respect to the columns of a Young tableau.

The unitary transformations T_U on $\mathfrak{F}_6$ may be defined

$$T_Uf(\zeta_1, \zeta_2) = f({}^tU\zeta_1, {}^tU\zeta_2), \tag{2.6}$$

where ${}^tU$ is the transpose of U, an element of $SU(3)$. When the variables ζ_1, ζ_2 in C_3 undergo a unitary transformation U, T_U defines a transformation of the elements $f(\zeta_1, \zeta_2)$ in the Hermitian space $\mathfrak{F}_6$. It may be shown that the transformations T_U form a unitary representation.[1] The spaces $\mathfrak{Q}_{\lambda\mu}$ are obviously invariant under a unitary transformation T_U, since the right side of Eq. (2.6) may again be expressed as a linear combination of polynomials $f(\zeta_1, \zeta_2)$ of the same degree in ζ_1, ζ_2.

For $\mathfrak{F}_6$, the Bargmann differential operators [Eqs. (1.5) and (1.7)] become

$$M_{ij}(\zeta_1, \zeta_2) = M_{ij}(\zeta_1) + M_{ij}(\zeta_2) \tag{2.7}$$

from Eq. (2.6).

B. Row Labels α

The row labels α of the representations $\mathfrak{D}^{\lambda\mu}$ are specified in the usual manner by the two linear commuting operators Y, T_0 of the rank-two group, and

$$\mathbf{T}^2 = T_0^2 + T_0 + M_{21}M_{12}, \tag{2.8}$$

the Casimir operator of the subgroup $SU(2)$. The base vector is uniquely specified by the conditions (2.2), (2.5), and

$$\begin{aligned}Y\,|\lambda\mu;\alpha\rangle &= y\,|\lambda\mu;\alpha\rangle, \\ T_0\,|\lambda\mu;\alpha\rangle &= t_0\,|\lambda\mu;\alpha\rangle, \\ \mathbf{T}^2\,|\lambda\mu;\alpha\rangle &= t(t+1)\,|\lambda\mu;\alpha\rangle,\end{aligned}$$

where $\alpha \equiv (y, t, t_0)$. The numbers t and t_0 are the isospin and the z component of the isospin, whereas y is 3 times the hypercharge quantum number. Since the operators (2.2), and $\mathbf{T}^2$, T_0, Y are Hermitian with respect to the inner product, the base vectors are orthogonal,

$$(|\lambda\mu;\alpha\rangle, |\lambda'\mu';\alpha'\rangle) = \delta_{\lambda\lambda'}\,\delta_{\mu\mu'}\,\delta_{\alpha\alpha'} \tag{2.9}$$

using Eq. (1.2).

The base vector $|\lambda\mu;\alpha\rangle$, as an explicit function of ζ_1, ζ_2, may be constructed with the appropriate raising and lowering operators of $SU(3)$ using the operators M_{ij} and appropriate products. The method has been used by Elliott,[12] Elliott and Harvey,[13] Hecht,[14] and Gel'fand and Zeitlin,[15] and the base vectors appear in the literature (see Bargmann and Moshinsky,[8] Moshinsky,[16] Baird and Biedenharn,[17] and Mukunda and Pandit[18]). Only the result is quoted here.

$$\begin{aligned}|\lambda\mu;\alpha\rangle &= N(\lambda\mu;\alpha)(-1)^q \\ &\times \sum_k \binom{r}{k} \frac{(\mu-q)!\,p!}{(\mu-q-k)!\,[p-(r-k)]!} \\ &\times \xi_1^{p-(r-k)}\eta_1^{r-k}\sigma_1^{\lambda-p}(\delta_{12}^{(1)})^k(-\delta_{12}^{(2)})^{\mu-q-k}(\delta_{12}^{(3)})^q,\end{aligned} \tag{2.10a}$$

where $N(\lambda\mu;\alpha)$ normalizes $|\lambda\mu;\alpha\rangle$ to unity (derived in Appendix A),

$$N(\lambda\mu;\alpha) = \left\{\frac{(\lambda+1)!\,(\mu+p-q+1)!}{p!\,q!\,(\mu-q)!\,(\lambda-p)!\,(\mu+p+1)!\,(\lambda+\mu-q+1)!} \times \frac{(2t-r)!}{(2t)!\,r!}\right\}^{\frac{1}{2}} \tag{2.10b}$$

and

$$\begin{aligned}y &= -(2\lambda+\mu) + 3(p+q), && 0 \le p \le \lambda, \\ t &= \tfrac{1}{2}\mu + \tfrac{1}{2}(p-q), && 0 \le q \le \mu, \\ t_0 &= t - r, && r = 0, 1, \cdots, 2t.\end{aligned} \tag{2.10c}$$

[10] $f(\zeta_1,\zeta_2)$ and $f(\zeta_1,\delta_{12})$ are used interchangeably in the article.

[11] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, Inc., New York, 1931), p. 359.

[12] J. P. Elliott, Proc. Roy. Soc. (London) **A245**, 128, 562 (1958).

[13] J. P. Elliott and M. Harvey, Proc. Roy. Soc. (London) **A272**, 557 (1963).

[14] K. T. Hecht, Nucl. Phys. **62**, 1 (1965).

[15] I. M. Gel'fand and M. L. Zeitlin, Doklady Akad. Nauk SSSR **71**, 825 (1950).

[16] M. Moshinsky, Nucl. Phys. **31**, 384 (1962).

[17] G. Baird and L. Biedenharn, J. Math. Phys. **4**, 1449 (1963).

[18] N. Mukunda and L. K. Pandit, J. Math Phys. **6**, 746 (1965).

The second factor in Eq. (2.10b) may be recognized as the Condon and Shortley[19] normalization for the lowering operator $T_- = M_{21}$. A specific phase convention has been assumed,

$$T_- |\lambda\mu; ytt_0\rangle = C_1 |\lambda\mu; ytt_0 - 1\rangle, \qquad (2.11a)$$

where C_1 is a positive constant. In addition, the requirements

$$(|\lambda\mu; y+3, (t+\tfrac{1}{2}), (t_0+\tfrac{1}{2})\rangle, M_{13} |\lambda\mu; \alpha\rangle) > 0, \qquad (2.11b)$$

$$(|\lambda\mu; y+3, (t-\tfrac{1}{2}), (t_0+\tfrac{1}{2})\rangle, M_{13} |\lambda\mu; \alpha\rangle) > 0 \qquad (2.11c)$$

specify the phase of $|\lambda\mu; \alpha\rangle$ with respect to p and q. This phase convention agrees with Elliott and Harvey,[13] and Hecht[14] (though their hypercharge is the negative of the above), but De Swart,[20] Biedenharn,[21] Kuriyan, Lurie, and Macfarlane,[22] and Mukunda and Pandit,[18] assume the matrix element [Eq. (2.11c)] to be negative, since the $SU(2)$ factor of Eq. (2.12c) is negative-definite.[23]

C. The Representations $\mathfrak{D}^{\lambda\mu}$

1. Irreducibility

T_U defines a transformation of the elements $f(z)$ in the Hermitian space $\mathfrak{F}_6$ [see Eq. (2.6)]. The unitary representations $\mathfrak{D}^{\lambda\mu}(U)$ are defined by restricting T_U to act in the subspace $\mathfrak{Q}_{\lambda\mu}$:

$$T_U |\lambda\mu; \alpha\rangle = \sum_{\alpha'} \mathfrak{D}^{\lambda\mu}_{\alpha'\alpha}(U) |\lambda\mu; \alpha'\rangle, \qquad (2.12)$$

$$\mathfrak{D}^{\lambda\mu}_{\alpha'\alpha}(U) = (|\lambda\mu; \alpha'\rangle, T_U |\lambda\mu; \alpha\rangle) \qquad (2.13)$$

using Eq. (2.9). The representations $\mathfrak{D}^{\lambda\mu}$ are irreducible.[24] By Schur's lemma, it is sufficient to prove that every linear operator A defined on $\mathfrak{Q}_{\lambda\mu}$ (which commutes with all T_U) is necessarily of the form $A = \alpha\cdot 1$. If A commutes with all T_U, then it must also commute with all the generators M_{ij}, by Eq. (1.6). The operators T_{11}, T_{22}, which define the invariant spaces $\mathfrak{Q}_{\lambda\mu}$, and the antisymmetry operator T_{12} (or T_{21}), commute with the generators of the group M_{ij} and are of the form $A = \alpha\cdot 1$. There are no other linear commuting operators.

2. Inequivalence

The representations $\mathfrak{D}^{\lambda\mu}(U)$ and $\mathfrak{D}^{\lambda'\mu'}(U)$ are inequivalent for $\lambda \neq \lambda'$ or $\mu \neq \mu'$. The proof follows from Schur's lemma.[25] If $e_1, \cdots, e_m$ and $f_1, \cdots, f_n$ are two sets of vectors in the spaces $\mathfrak{Q}$, $\mathfrak{Q}'$, respectively, and if V_α is a set of unitary operators defined on $\mathfrak{Q}$, $\mathfrak{Q}'$, then

$$V_\alpha e_i = \sum_{j=1}^{m} e_j \rho_{ji}(\alpha), \qquad V_\alpha f_r = \sum_{s=1}^{n} f_s \sigma_{sr}(\alpha),$$

where the matrices $\rho_{ji}(\alpha)$, $\sigma_{sr}(\alpha)$ are unitary and irreducible. Let $\beta_{ir} = (e_i, f_r)$ be the inner product matrix. Then it may be shown[1], employing matrix notation, that $\rho(\alpha)\beta = \beta\sigma(\alpha)$. That is, β is a mapping of the space $\mathfrak{D}$ onto $\mathfrak{D}'$. Schur's lemma implies either

(1) $\beta = 0$, i.e., $(e_i, f_r) = 0$, for all i, r, or

(2) the representations are equivalent and β is a multiple of the unit matrix

$$(e_i, f_r) = \beta_{ir} = \epsilon\, \delta_{ir}. \qquad (2.14)$$

Then, the dimensions of the representations are equal, and for $e_i = f_i$,

$$(e_i, e_j) = \epsilon\, \delta_{ij}. \qquad (2.15)$$

The representations are certainly inequivalent if the dimensions

$$N = \tfrac{1}{2}(\mu+1)(\lambda+1)(\lambda+\mu+2)$$

of $\mathfrak{Q}_{\lambda\mu}$, $\mathfrak{Q}_{\lambda'\mu'}$ are not the same. In the cases where the dimension N is the same for different spaces[26] $\mathfrak{Q}_{\lambda\mu}$, $\mathfrak{Q}_{\lambda'\mu'}$, the inner product is zero [see Eq. (2.9)] and an equivalence transformation β cannot be found.

D. Complex Conjugate Representation $\overline{\mathfrak{D}^{\lambda\mu}(u)}$

Since the $SU(3)$ transformation matrix is unimodular, the 3×3 determinant

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{vmatrix} = \zeta_3\cdot(\zeta_1 \times \zeta_2) = \zeta_3\cdot\delta_{12} \qquad (2.16)$$

is invariant under a unitary transformation. This implies

[19] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, England, 1935).

[20] J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963).

[21] L. C. Biedenharn, Phys. Letters **3**, 69 (1962).

[22] J. G. Kuriyan, D. Lurie, and A. J. Macfarlane, J. Math. Phys. **6**, 722 (1965).

[23] For the Biedenharn phase convention, the base vector Eq. (2.10) must be multiplied by the factor $(-1)^q$.

[24] A different proof of the irreducibility of the representations may be found in M. Moshinsky, J. Math. Phys. **4**, 1128 (1963).

[25] See the statement of Schur's lemma given by Bargmann in Ref. 1.

[26] The dimension is the same for $\mathfrak{Q}_{\lambda\mu}$ and $\mathfrak{Q}_{\mu\lambda}$, but there are other possibilities, e.g., for $N = 15$, the following partitions $[\lambda,\mu]$ exist: [2,1], [1,2], [4,0], [0,4].

$$\sum_{\alpha} {}^tU_{i\alpha}\, \mathfrak{D}'^{(0,1)}_{\alpha k}(U) = \delta_{ik}, \qquad (2.17)$$

since $\mathfrak{D}^{(1,0)}(U) = U$. $\mathfrak{D}'^{(0,1)}$ is the irreducible unitary representation associated with δ_{12}. From unitarity,

$$\mathfrak{D}'^{(0,1)}(U) = \bar{U}. \qquad (2.18)$$

When the variables ζ_1, ζ_2 undergo unitary transformations, the 2×2 antisymmetric forms δ_{12} transform with respect to the complex conjugate representation. Given a base vector of the functional form $f(\zeta_1, \delta_{12})$, the base vector $|\lambda\mu;\alpha\rangle_c$ associated with the complex conjugate representation

$$T_U\, |\lambda\mu;\alpha\rangle_c = \sum_{\alpha'} \overline{\mathfrak{D}^{\lambda\mu}_{\alpha',\alpha}(U)}\, |\lambda\mu;\alpha'\rangle_c \qquad (2.19)$$

may be obtained by exchanging $\zeta_1 \leftrightarrow \delta_{12}$, and this is the explicit R-conjugation transformation.[27,28] Define the transformation R as

$$Rf(\zeta_1, \delta_{12}) = f(\delta_{12}, \zeta_1). \qquad (2.20)$$

Then,

$$R\, |\lambda\mu;\alpha\rangle = C\, |\lambda\mu;\alpha\rangle_c, \qquad (2.21a)$$

where

$$C = \left\{\frac{(\lambda+1)!}{(\mu+1)!}\right\}^{\frac{1}{2}}. \qquad (2.21b)$$

The operation R, Eq. (2.19), is not a unitary transformation; the base vector $|\lambda\mu;\alpha\rangle_c$ must be normalized to unity, Eq. (2.21).

If the following changes are made in the base vector $|\lambda\mu;\alpha\rangle$: (a) $\lambda \leftrightarrow \mu$, $p \to \mu - q$, $q \to \lambda - p$, $r \to 2t - r$ and (b) $k \to k - p + r$, then, upon comparison with $|\lambda\mu;\alpha\rangle_c$, one obtains

$$|\lambda\mu;\alpha\rangle_c = (-1)^{y/2-t_0}\, |\mu\lambda; -\alpha\rangle, \qquad (2.22a)$$

where

$$-\alpha \equiv (-y, t, -t_0).^{29} \qquad (2.22b)$$

The operation R is thus a one-to-one mapping of $\mathfrak{Q}_{\lambda\mu}$ onto $\mathfrak{Q}_{\mu\lambda}$.

Relation (2.22) is analogous to the $SU(2)$ result[1]

$$w^j_m = (-1)^{j+m} v^j_{-m}. \qquad (2.23)$$

The important distinction between Eqs. (2.21) and (2.22) is that w^j_m was a member of the same Hilbert space as v^j_m, but $|\lambda\mu;\alpha\rangle_c$ is a member of $\mathfrak{Q}_{\mu\lambda}$ (not $\mathfrak{Q}_{\lambda\mu}$).

3. REDUCTION OF THE DIRECT-PRODUCT REPRESENTATION[30]

A. Reduction of $\mathfrak{D}^{\lambda_1\mu_1} \otimes \mathfrak{D}^{\lambda_2\mu_2}$

Let $\mathfrak{F}_6^{(1)}$, $\mathfrak{F}_6^{(2)}$ be the Hilbert spaces of analytic functions $f(\zeta_1, \zeta_2)$, $f(\zeta_3, \zeta_4)$, respectively, where $\zeta_i = (\xi_i, \eta_i, \sigma_i)$ is a member of C_3. $\mathfrak{F}_{12} = \mathfrak{F}_6^{(1)} \otimes \mathfrak{F}_6^{(2)}$ is a Hilbert space of analytic functions $f(\zeta_1, \zeta_2; \zeta_3, \zeta_4)$. The subspace

$$\mathfrak{Q}_{\lambda_1\mu_1\lambda_2\mu_2} = \mathfrak{Q}_{\lambda_1\mu_1} \otimes \mathfrak{Q}_{\lambda_2\mu_2}$$

of $\mathfrak{F}_{12}$ is spanned by the N_1N_2 direct-product vectors $|\lambda_1\mu_1;\alpha_1\rangle \mid \lambda_2\mu_2;\alpha_2\rangle$, where $N_i = \frac{1}{2}(\lambda_i + 1)(\mu_i + 1)(\lambda_i + \mu_i + 2)$, $i = 1, 2, 3$, the dimension of the ith space. For any $SU(3)$ transformation U, the operators $T_U^{(1)}$ and $T_U^{(2)}$ are defined on $\mathfrak{F}_6^{(1)}$ and $\mathfrak{F}_6^{(2)}$ respectively, by Eq. (2.6). For a function $f(\zeta_1, \zeta_2; \zeta_3, \zeta_4)$, a member of $\mathfrak{F}_{12}$, $T_U^{(1,2)}$ forms a unitary representation

$$(T_U^{(1,2)}f)(\zeta_1, \zeta_2; \zeta_3, \zeta_4) = f({}^tU\zeta_1, {}^tU\zeta_2; {}^tU\zeta_3, {}^tU\zeta_4).$$

Further, since for $\mathfrak{Q}_{\lambda_1\mu_1\lambda_2\mu_2}$,

$$f(\zeta_1, \zeta_2; \zeta_3, \zeta_4) = f(\zeta_1, \zeta_2) f(\zeta_3, \zeta_4),$$

$$(T_U^{(1,2)}f)(\zeta_1, \zeta_2; \zeta_3, \zeta_4) = f({}^tU\zeta_1, {}^tU\zeta_2) f({}^tU\zeta_3, {}^tU\zeta_4)$$
$$= [(T_U^{(1)}f)(\zeta_1, \zeta_2)][(T_U^{(2)}f)(\zeta_3, \zeta_4)],$$

the result follows that

$$T_U^{(1,2)} = T_U^{(1)} \otimes T_U^{(2)}. \qquad (3.1)$$

Thus, $T_U^{(1,2)}$, restricted to the space $\mathfrak{Q}_{\lambda_1\mu_1\lambda_2\mu_2}$, provides the direct-product representation

$$\mathfrak{D}^{\lambda_1\mu_1}(U) \otimes \mathfrak{D}^{\lambda_2\mu_2}(U).$$

The infinitesimal transformations on $\mathfrak{F}_{12}$ are

$$M_{ij}(\zeta_1, \zeta_2; \zeta_3, \zeta_4) = M_{ij}(\zeta_1, \zeta_2) + M_{ij}(\zeta_3, \zeta_4). \qquad (3.2)$$

The extension to $\mathfrak{F}_{18}$ is obvious. Define the transformation

$$(T_U^{(1,2,3)}f)(\zeta_1, \zeta_2; \zeta_3, \zeta_4; \zeta_5, \zeta_6)$$
$$= f({}^tU\zeta_1, {}^tU\zeta_2; {}^tU\zeta_3, {}^tU\zeta_4; {}^tU\zeta_5, {}^tU\zeta_6).$$

Then,

$$T_U^{(1,2,3)} = T_U^{(1,2)} \otimes T_U^{(3)} = T_U^{(1)} \otimes T_U^{(2)} \otimes T_U^{(3)}. \qquad (3.3)$$

[27] D. Lurie and A. J. Macfarlane, J. Math. Phys. 5, 565 (1964) give an implicit derivation of R-conjugation. The term "R-conjugation" is due to M. Gell-Mann, California Institute of Technology Report CTSL-20 (1961).

[28] See also G. E. Baird and L. C. Biedenharn, J. Math. Phys. 5, 1723 (1964) for a discussion of the conjugation operation for $SU(n)$ in terms of operator mappings.

[29] Since $y = 3 \times$ hypercharge (Y),

$$(-1)^{y/2+t_0} = (-1)^{y/2-t_0+2/3(\lambda+2\mu)}$$

which agrees with de Swart (Ref. 20).

[30] The discussion of Sec. 3A follows from that given by Bargmann (Ref. 1) for the group $SU(2)$. The functional space is now $\mathfrak{F}_6$ and the proof of the theorem on the reduction of the direct product must be altered to account for the degeneracy in direct product states.

The direct product representation may be reduced according to the formula

$$[\lambda_1\mu_1] \otimes [\lambda_2\mu_2] = \sum g(\lambda_i\mu_i)[\lambda_3\mu_3], \qquad (3.4)$$

where $g(\lambda_i\mu_i)$ is the degeneracy, the number of times the irreducible representation $\mathfrak{D}^{\lambda_3\mu_3}$ is contained in $\mathfrak{D}^{\lambda_1\mu_1} \otimes \mathfrak{D}^{\lambda_2\mu_2}$. For each partition $[\lambda_3\mu_3]$ there exists gN_3 independent products $|\lambda_1\mu_1;\ \alpha_1\rangle$ $|\lambda_2\mu_2;\ \alpha_2\rangle$. Associate the index k with the space of N_3 orthonormal base vectors[31]

$$\mathfrak{Q}^{(k)}_{\lambda_1\mu_1\lambda_2\mu_2}, \qquad k = 0, 1, \cdots, g-1.$$

If the irreducible representation $\mathfrak{D}^{\lambda_3\mu_3}$ is contained in the product representation $\mathfrak{D}^{\lambda_1\mu_1} \otimes \mathfrak{D}^{\lambda_2\mu_2}$, there exists N_3 orthonormalized product base vectors

$$|\lambda_3\mu_3; \alpha_3\rangle_k \quad \text{in} \quad \mathfrak{Q}^{(k)}_{\lambda_1\mu_1\lambda_2\mu_2}$$

such that

$$T_U^{(1,2)}\ |\lambda_3\mu_3;\alpha_3\rangle_k = \sum_{\alpha_3'} \mathfrak{D}^{\lambda_3\mu_3}_{\alpha_3',\alpha_3}(U)\ |\lambda_3\mu_3;\alpha_3'\rangle_k. \qquad (3.5)$$

(Note that the irreducible representation $\mathfrak{D}^{\lambda_3\mu_3}$ has no subindex k because the equivalence transformation $\beta_{kk'}$ [see Eq. (2.14)] equals $\delta_{kk'}$, since the base vectors $|\lambda_3\mu_3;\ \alpha_3\rangle_k$ are also assumed orthogonal with respect to k.) Consider the expression

$$a_k = \textstyle\sum_{\alpha_3} |\lambda_3\mu_3;\alpha_3\rangle_k\ |\lambda_3\mu_3;\alpha_3\rangle_c,$$
$$k = 0, 1, \cdots, g-1 \qquad (3.6)$$

as a member of the space

$$\mathfrak{Q}^{(k)}_{\lambda_1\mu_1\lambda_2\mu_2\mu_3\lambda_3}.$$

a_k, being the sum of orthonormal functions, is not equal to zero. If $\mathfrak{D}^{\lambda_3\mu_3}(U)$ is contained in the product representation $\mathfrak{D}^{\lambda_1\mu_1} \otimes \mathfrak{D}^{\lambda_2\mu_2}$, then a_k is invariant in the triple-product space

$$\mathfrak{Q}^{(k)}_{\lambda_1\mu_1\lambda_2\mu_2\mu_3\lambda_3}:$$

$$T_U^{(1,2,3)}a_k = \sum_{\alpha_3} (T_U^{(1,2)}\ |\lambda_3\mu_3;\alpha_3\rangle_k)(T_U^{(3)}\ |\lambda_3\mu_3;\alpha_3\rangle_c),$$

from Eq. (3.3)

$$= \sum_{\alpha_3} \{\sum_{\alpha_3'} |\lambda_3\mu_3;\alpha_3'\rangle_k\ \mathfrak{D}^{\lambda_3\mu_3}_{\alpha_3',\alpha_3}(U)\}$$
$$\times \{\sum_{\alpha_3''} |\lambda_3\mu_3;\alpha_3''\rangle_c\ \overline{\mathfrak{D}^{\lambda_3\mu_3}_{\alpha_3'',\alpha_3}(U)}\},$$

using Eqs. (3.5) and (2.19)

$$= \sum_{\alpha_3',\alpha_3''} |\lambda_3\mu_3;\alpha_3'\rangle_k\ |\lambda_3\mu_3;\alpha_3''\rangle_c$$
$$\times \sum_{\alpha_3} \mathfrak{D}^{\lambda_3\mu_3}_{\alpha_3',\alpha_3}(U)\ \overline{\mathfrak{D}^{\lambda_3\mu_3}_{\alpha_3'',\alpha_3}(U)}$$
$$= \sum_{\alpha_3',\alpha_3''} |\lambda_3\mu_3;\alpha_3'\rangle_k\ |\lambda_3\mu_3;\alpha_3''\rangle_c\ \delta_{\alpha_3',\alpha_3''},$$

since the representations are unitary. Hence,

$$T_U^{(1,2,3)}a_k = a_k. \qquad (3.7)$$

Conversely, let h_k be an orthonormal function in $\mathfrak{Q}^{(k)}_{\lambda_1\mu_1\lambda_2\mu_2\mu_3\lambda_3}$ such that

$$T_U^{(1,2,3)}h_k = h_k. \qquad (3.8)$$

Since the functions $|\lambda_3\mu_3;\ \alpha_3\rangle_c$ span the space $\mathfrak{Q}_{\mu_3\lambda_3}$, h_k has an expansion

$$h_k = \sum_{\alpha_3} \chi^{(k)}_{\alpha_3}\ |\lambda_3\mu_3;\alpha_3\rangle_c \qquad (3.9)$$

with $\chi^{(k)}_{\alpha_3}$ uniquely determined in

$$\mathfrak{Q}^{(k)}_{\lambda_1\mu_1\lambda_2\mu_2}:$$

$$T_U^{(1,2,3)}h_k = \textstyle\sum_{\alpha_3} (T_U^{(1,2)}\chi^{(k)}_{\alpha_3})(T_U^{(3)}\ |\lambda_3\mu_3;\alpha_3\rangle_c)$$

using Eq. (3.3).

$$= \sum_{\alpha_3'} \{\sum_{\alpha_3} (T_U^{(1,2)}\chi^{(k)}_{\alpha_3})\ \mathfrak{D}^{\lambda_3\mu_3}_{\alpha_3',\alpha_3}(U)\}\ |\lambda_3\mu_3;\alpha_3\rangle_c$$

by definition, Eq. (2.19).

$$= \sum_{\alpha_3''} \chi^{(k)}_{\alpha_3''}\ |\lambda_3\mu_3;\alpha_3''\rangle_c$$

by assumption (3.8). Thus,

$$\chi^{(k)}_{\alpha_3''} = \sum_{\alpha_3} (T_U^{(1,2)}\chi^{(k)}_{\alpha_3})\ \mathfrak{D}^{\lambda_3\mu_3}_{\alpha_3'',\alpha_3}(U). \qquad (3.10)$$

Multiply (3.10) by $\overline{\mathfrak{D}^{\lambda_3\mu_3}_{\alpha_3'',\alpha_3}(U)}$ and sum over α_3''. From the unitarity of the representations,

$$T_U^{(1,2)}\chi^{(k)}_{\alpha_3} = \textstyle\sum_{\alpha_3'} \chi^{(k)}_{\alpha_3'}\ \mathfrak{D}^{\lambda_3\mu_3}_{\alpha_3',\alpha_3}(U). \qquad (3.11)$$

Consider the inner product of h_k:

$$(h_k, h_{k'}) = \textstyle\sum_{\alpha_3} (\chi^{(k)}_{\alpha_3}\ |\lambda_3\mu_3;\alpha_3\rangle_c, \chi^{(k')}_{\alpha_3'}\ |\lambda_3'\mu_3';\alpha_3'\rangle_c)$$
$$= \sum_{\alpha_3} (\chi^{(k)}_{\alpha_3}, \chi^{(k')}_{\alpha_3}),$$

since the vectors $|\lambda_3\mu_3;\ \alpha_3\rangle_c$ are orthonormal. The representation $\mathfrak{D}^{\lambda_3\mu_3}$ associated with $\chi^{(k)}_{\alpha_3}$ is unitary and irreducible, hence, according to Schur's lemma, [Eq. (2.15)]

$$(\chi^{(k)}_{\alpha_3}, \chi^{(k')}_{\alpha_3})$$

[31] The method of labeling orthogonal spaces $\mathfrak{Q}^{(k)}_{\lambda_1\mu_1\lambda_2\mu_2}$ is discussed in Sec. 3C.

is independent of the row label α_3. Thus,

$$(h_k, h_{k'}) = d(k, k'; \lambda_3\mu_3) \sum_{\alpha_3} \cdot 1 = N_3\, d(k, k'; \lambda_3\mu_3) = \delta_{kk'}, \tag{3.12}$$

since, by assumption, the invariants h_k are orthonormal. Thus, if h_k, $k = 0, 1, \cdots, g-1$, is orthonormal with respect to the index k, then by Eq. (3.12), so is $\chi^{(k)}_{\alpha_3}$. The base vectors $|\lambda_3\mu_3; \alpha_3\rangle_k$ have unit norm, so that

$$|\lambda_3\mu_3; \alpha_3\rangle_k = (N_3)^{\frac{1}{2}}\chi^{(k)}_{\alpha_3} \tag{3.13}$$

are the orthonormal functions in

$$\mathfrak{L}^{(k)}_{\lambda_1\mu_1\lambda_2\mu_2}$$

associated with the irreducible representation

$$\mathfrak{D}^{\lambda_3\mu_3}(U).$$

The $3(\lambda\mu)$ coefficients may be defined

$$h_k \equiv \sum_{\alpha_i} \begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{Bmatrix}_k \times |\lambda_1\mu_1; \alpha_1\rangle\, |\lambda_2\mu_2; \alpha_2\rangle\, |\lambda_3\mu_3; \alpha_3\rangle_c. \tag{3.14}$$

The $3(\lambda\mu)$ coefficients provide that linear combination of triple-product functions $|\lambda_1\mu_1; \alpha_1\rangle \mid \lambda_2\mu_2; \alpha_2\rangle$ $|\lambda_3\mu_3; \alpha_3\rangle_c$ which yield an invariant h_k in the triple-product space. From Eqs. (3.9) and (3.13),

$$|\lambda_3\mu_3; \alpha_3\rangle_k = (N_3)^{\frac{1}{2}} \times \sum_{\alpha_1\alpha_2} \begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{Bmatrix}_k |\lambda_1\mu_1; \alpha_1\rangle\, |\lambda_2\mu_2; \alpha_2\rangle. \tag{3.15}$$

The $3(\lambda\mu)$ coefficients, defined by Eq. (3.14), when multiplied by $(N_3)^{\frac{1}{2}}$, yield the standard coupling coefficients. If the invariants h_k are given, the explicit evaluation of the $3(\lambda\mu)$ symbol involves taking the inner product of h_k with the triple-product vector

$$\begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{Bmatrix}_k = (|\lambda_1\mu_1; \alpha_1\rangle\, |\lambda_2\mu_2; \alpha_2\rangle\, |\lambda_3\mu_3; \alpha_3\rangle_c, h_k). \tag{3.16}$$

Thus, the evaluation of the $3(\lambda\mu)$ coefficients is reduced to the construction of an invariant h_k in the triple-product space, and as we see in the next two sections, this is not a difficult task.

B. $3(\lambda\mu)$ symbol for the nondegenerate case[32]: $\mathfrak{D}^{\lambda_1 0} \otimes \mathfrak{D}^{\lambda_2\mu_2}$

Notation: Associate the variables (ζ_1, ζ_2), (ζ_3, ζ_4), (ζ_5, ζ_6), with the base vectors $|\lambda_1\mu_1; \alpha_1\rangle$, $|\lambda_2\mu_2; \alpha_2\rangle$, $|\lambda_3\mu_3; \alpha_3\rangle_c$, respectively. Label the 2×2 antisymmetric forms:

$$\delta_{ij} = (\delta^{(1)}_{ij}\; \delta^{(2)}_{ij}, \delta^{(3)}_{ij}) = (\eta_i\sigma_j - \eta_j\sigma_i, \sigma_i\xi_j - \sigma_j\xi_i, \xi_i\eta_j - \xi_j\eta_i). \tag{3.17}$$

The most general invariant h_k may be constructed from a linear combination of products of 3×3 determinants. The following conditions must be imposed on h_k:

$$\text{(i)}\quad T_{12}h_k = 0, \quad T_{34}h_k = 0, \quad T_{56}h_k = 0, \tag{3.18}$$

since h_k is of the form Eq. (3.14). By inspection of Eq. (2.10a), one sees that $|\lambda_1 0; \alpha_1\rangle$ is independent of δ_{12}. Hence h_k must not contain δ_{12}, and $T_{12}h_k = 0$ is satisfied automatically. h_k may then have the form

$$h_k = \sum_{k_i} \beta_k(k_i)[\zeta_1\cdot(\zeta_3 \times \zeta_5)]^{k_0} \times (\zeta_3\cdot\delta_{56})^{k_1}(\zeta_1\cdot\delta_{34})^{k_5}(\zeta_5\cdot\delta_{34})^{k_6}. \tag{3.19}$$

(ii) From the requirement that h_k be in the space

$$\mathfrak{L}_{\lambda_1 0\lambda_2\mu_2\mu_3\lambda_3},$$

the degree conditions follow

$$k_1 + k_2 = \lambda_3,\; k_0 + k_2 + k_5 = \lambda_1,\; k_i \geq 0, \quad k_0 + k_6 = \mu_3,\; k_0 + k_1 = \lambda_2,\; k_5 + k_6 = \mu_2. \tag{3.20}$$

This may be rewritten in the form

$$k_2 = P - (\lambda_2 + \mu_2),\; k_0 = P - (\lambda_3 + \mu_2), \quad k_5 = P - (\lambda_3 + \mu_3),\; k_1 + k_2 = \lambda_3,\; k_5 + k_6 = \mu_2, \tag{3.21a}$$

where

$$P = k_0 + k_1 + k_2 + k_5 + k_6 = \tfrac{1}{3}(\lambda_1 + \lambda_2 + 2\mu_2 + 2\lambda_3 + \mu_3). \tag{3.21b}$$

From Eq. (3.21), the partition numbers λ_1, $[\lambda_2\mu_2]$, $[\lambda_3\mu_3]$, uniquely specify the integers k_i, $g(\lambda_i\mu_i) = 1$. Redefine the coefficient $\beta_k(k_i)$. Eq. (3.19) becomes

$$h(k_i) = \Delta(k_i)\, \frac{[\zeta_1\cdot(\zeta_3 \times \zeta_5)]^{k_0}}{k_0!}\frac{(\zeta_3\cdot\delta_{56})^{k_1}}{k_1!}\frac{(\zeta_1\cdot\delta_{56})^{k_2}}{k_2!}\frac{(\zeta_1\cdot\delta_{34})^{k_5}}{k_5!}\frac{(\zeta_5\cdot\delta_{34})^{k_6}}{k_6!}, \tag{3.22}$$

[32] The general expression for $[\lambda,\mu] \otimes [k,0]$ has been derived by M. Moshinsky, Rev. Mod. Phys. **34**, 813 (1962) in terms of a finite series.

$\Delta(k_i)$ normalizes the inner product $[h(k_i),\ h(k_i)]$ to unity, and is evaluated in Appendix B. The result is

$$[h(k_i), h(k_i')] = [\Delta(k_i)]^2\,\delta_{k_ik'_i}\,\frac{(P+2)!\,(k_0+k_1+k_2+1)!\,(k_0+k_1+k_6+1)(k_0+k_5+k_6+1)!}{2k_0!\,k_1!\,k_2!\,k_5!\,k_6!\,(k_0+k_1+1)!\,(k_0+k_6+1)!}. \quad (3.23)$$

Since $h(k_i)$ is orthogonal, there exists N_3 orthonormal base vectors

$$|\lambda_3\mu_3;\alpha_3\rangle \quad \text{in} \quad \mathfrak{Q}_{\lambda_10\lambda_2\mu_2},$$

corresponding to each set of values k_i, provided that k_i satisfy conditions (3.21). To determine whether the decomposition is complete, sum over the number of reduced direct-product vectors consistent with the constraints on k_i:

$$\begin{aligned} n &= \sum_{\lambda_3} N_3 = \tfrac{1}{2}\sum(\lambda_3+1)(\mu_3+1)(\lambda_3+\mu_3+2) \\ &= \tfrac{1}{2}\sum_{k_2=0}^{\lambda_1}\sum_{k_3=0}^{\lambda_1-k_2}(\lambda_2+1+k_2-k_3) \\ &\qquad\times(\mu_2-\lambda_1+1+k_2+2k_3) \\ &\qquad\times(\lambda_2+\mu_2+2+2k_2+k_3) \\ &= \tfrac{1}{2}(\lambda_1+1)(\lambda_1+2)\tfrac{1}{2}(\lambda_2+1)(\mu_2+1)(\lambda_2+\mu_2+2) \\ &= N_1N_2. \end{aligned}$$

The number of base vectors $|\lambda_3\mu_3;\alpha_3\rangle$ in $\mathfrak{Q}_{\lambda_10\lambda_2\mu_2}$ is N_1N_2, and since this is the dimension of the space, the reduction is complete.

The inner product of $h(k_i)$, Eq. (3.22), and the triple-product vector yields the $3(\lambda\mu)$ symbol for the nondegenerate case. $h(k_i)$ must be expanded, constraints from the integration [see Eq. (1.2)] must be applied, and (using binomial identities) the sums must be contracted. The details are quite tedious and only the results are given here.[33]

$$\begin{aligned} \begin{Bmatrix}\lambda_10 & \lambda_2\mu_2 & \lambda_3\mu_3\\ \alpha_1 & \alpha_2 & \alpha_3'\end{Bmatrix} &= C\sum_{a,\ldots,e}\frac{(-1)^{a+b+e}(\lambda_3+1+k_0+b-c)!}{a!\,(k_5-a)!\,(\mu_3-q_3-b)!\,c!\,d!\,(p_3-d)!\,(\lambda_3-p_3-k_2+d)!} \\ &\times\frac{(\lambda_3+\mu_3+1-k_2+d)!}{[-(\mu_3-q_3)+\lambda_1-p_1-a+b]!\,(\mu_3+p_3+1-k_6+\mu_2-q_2-c-d)!\,[-(\mu_3-q_3)+k_6-k_2+\lambda_1-p_1-a+c+d]!} \\ &\times\frac{[\mu_3-q_3+k_2+k_5-(\lambda_1-p_1)-b-d]!}{(\mu_2-q_2-e)!\,[\lambda_3+\mu_3-q_3+1+k_0-(\lambda_1-p_1)+a-c]!\,[\mu_3-q_3+k_2-(\lambda_1-p_1)+a-b-d]!\,(b-c-e)!} \\ &\times\frac{(\mu_3+p_3-q_3-b-d)!}{e!\,[r-(\mu_2-q_2)+e]!\,(k_0+p_3-p_2-b-d+e)!\,(\mu_2+p_2-q_2-r-e)!}, \end{aligned} \quad (3.24)$$

where

$$\begin{aligned} r = r_1+r_2 &= k_0+\mu_2-q_2-(\mu_3-q_3) &&\text{from}\quad t_3=(t_0)_1+(t_0)_2, &&\alpha_3' : t_3=(t_0)_3, \\ p_1+p_2 &= 2k_0+\mu_2-q_2+p_3-(\mu_3-q_3) &&\text{from}\quad y_1+y_2=y_3, \end{aligned}$$

and

$$C = \frac{(-1)^{k_0+r_1}\,\Delta(k_1)}{N(\lambda_10;\alpha_1)N(\lambda_2\mu_2;\alpha_2)N(\lambda_3\mu_3;y_3t_3t_3)}\frac{r!}{r_1!\,r_2!} \times\frac{(\mu_3+p_3-q_3+1)!\,(\mu_2+p_2-q_2+1)!\,(\mu_2+p_2-q_2-r_2)!}{(\mu_3+p_3+1)!\,(\lambda_3+\mu_3-q_3+1)!\,(\mu_2+p_2-q_2)!}. \quad (3.25)$$

$N(\lambda_i\mu_i;\ \alpha_i)$ is given by Eq. (2.10b) and $\Delta(k_i)$ by Eq. (3.23). For $\mu_2 = 0$, Eq. (3.24) reduces to one sum and for the two special cases where α_3 is minimum or maximum,

$$\text{(i)}\quad \alpha_{3_m} = [y_{3\min},\ t_{3\min},\ (t_0)_3 = -t_{3\min}], \qquad p_3,\ q_3 = 0, \qquad r_3 = 2t_3,$$

$$\text{(ii)}\quad \alpha_{3_M} = [y_{3\max},\ t_{3\max},\ (t_0)_3 = t_{3\max}], \qquad p_3 = \lambda_3, \qquad q_3 = \mu_3, \qquad r_3 = 0,$$

the above expression reduces to a single factor. Let the phase convention for the $3(\lambda\mu)$ symbol be the

[33] Aside from the general expression for $[\lambda,\mu]\otimes[k,0]$ derived by Moshinsky (Ref. 32), other special cases appear in the literature, e.g., Hecht (Ref. 14) has coefficients for the special cases $[\lambda,\mu]\otimes[2,0]$, $[0,2]$, $[4,0]$, $[2,1]$, $[1,1]$ in terms of single factors. N. Mukunda and L. K. Pandit, J. Math. Phys. **6**, 1547 (1965) have closed expressions for the product $[\lambda,\mu]\otimes[3,0]$. Coefficients of use to high-energy physicists have been constructed by S. Sawada and M. Yonezawa, Progr. Theoret. Phys. (Kyoto) **23**, 662 (1960); A. R. Edmonds, Proc. Roy. Soc. (London) **A268**, 567 (1962); M. A. Rashid, Nuovo Cimento **26**, 118 (1962); and J. J. de Swart (Ref. 20), among others.

following[34]:

$$\begin{Bmatrix} \lambda_1 0 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_{1M} & \alpha_2' & \alpha_{3M} \end{Bmatrix} > 0. \tag{3.26}$$

Case (ii) then allows us to determine the correct phase factor:

$$\begin{Bmatrix} \lambda_1 0 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_1 & \alpha_2 & \alpha_{3M} \end{Bmatrix} = \frac{(-1)^{\lambda_1 - p_1 + r_1 + k_0}\,\Delta(k_i)}{N(\lambda_1 0;\alpha_1)N(\lambda_2\mu_2;\alpha_2)N(\lambda_3\mu_3;\alpha_{3M})}$$
$$\times \frac{(\mu_2+p_2-q_2+1)!\,(\mu_2+p_2-q_2-r_2)!\,r!}{(\mu_2-q_2)!\,(\mu_2+p_2-q_2)!\,k_0!\,k_2!\,k_6!\,(\lambda_1-p_1)!\,(\lambda_2-p_2)!\,(\lambda_2+\mu_2-q_2+1)!\,[k_1-(\lambda_2-p_2)]!\,r_1!\,r_2!}. \tag{3.27}$$

If $\alpha_1 = \alpha_{1M}$, it is clear the $3(\lambda\mu)$ symbol must be multiplied by $(-1)^{k_0}$ to satisfy the convention, Eq. (3.26).

C. $3(\lambda\mu)$ symbol for the degenerate case: $\mathfrak{D}^{\lambda_1\mu_1} \otimes \mathfrak{D}^{\lambda_2\mu_2}$

To determine the $3(\lambda\mu)$ coefficients uniquely, it is necessary to determine the invariants $h_k(k_i)$ uniquely. In Sec. (1) below, conditions are placed on the invariant, but, unlike the previous case, the invariant is still not uniquely determined. Linear combinations of the invariants are possible, and in Sec. 2, two methods are employed to orthogonalize and uniquely determine the invariants. The explicit expression and completeness of reduction then follow, as in Sec. 3B above.

1. *Form of invariant*

The following conditions must be imposed on the invariant $h_k(k_i)$:

(α) $T_{12}h_k = 0, \quad T_{34}h_k = 0, \quad T_{56}h_k = 0,$

since the invariant has the form Eq. (3.13). The most general invariant h_k must then have a summand of the form

$$\beta_k(k_i)[\zeta_1\cdot(\zeta_3\times\zeta_5)]^{k_0}F(k_i)[\delta_{12}\cdot(\delta_{34}\times\delta_{56})]^{k_0'}, \tag{3.28a}$$

where

$$F(k_i) = (\zeta_3\cdot\delta_{56})^{k_1}(\zeta_1\cdot\delta_{56})^{k_2}(\zeta_5\cdot\delta_{12})^{k_3} \times (\zeta_3\cdot\delta_{12})^{k_4}(\zeta_1\cdot\delta_{34})^{k_5}(\zeta_5\cdot\delta_{34})^{k_6}. \tag{3.28b}$$

(β) Since h_k is a member of the space

$$\mathfrak{L}^{(k)}_{\lambda_1\mu_1\lambda_2\mu_2\mu_3\lambda_3},$$

$$k_0+k_3+k_6=\mu_3, \quad k_0'+k_1+k_2=\lambda_3,$$
$$k_0+k_1+k_4=\lambda_2, \quad k_0+k_5+k_6=\mu_2, \quad k_i \geq 0,$$
$$k_0+k_2+k_5=\lambda_1, \quad k_0+k_3+k_4=\mu_1. \tag{3.29}$$

From Eq. (3.29), it may be noted that

$$k_0 - k_0' = P - (\mu_1+\mu_2+\lambda_3), \tag{3.30a}$$

where

$$P = k_0 + k_1 + \cdots + k_6 + 2k_0' = \tfrac{1}{3}(\lambda_1 + 2\mu_1 + \lambda_2 + 2\mu_2 + 2\lambda_3 + \mu_3). \tag{3.30b}$$

The requirement that the invariant h_k lies in the triple-product space

$$\mathfrak{L}^{(k)}_{\lambda_1\mu_1\lambda_2\mu_2\mu_3\lambda_3},$$

[Eqs. (3.29) and (3.30)] allows for a range of values of k_i instead of a unique set, as in the previous case, and this gives rise to the multiplicity problem.

The terms of the summation, Eq. (3.28), are not independent because of the identity

$$[\zeta_1\cdot(\zeta_3\times\zeta_5)][\delta_{12}\cdot(\delta_{34}\times\delta_{56})] = H_1 + H_2, \tag{3.31a}$$

where

$$H_1 = (\zeta_3\cdot\delta_{56})(\zeta_5\cdot\delta_{12})(\zeta_1\cdot\delta_{34}),$$
$$H_2 = (\zeta_1\cdot\delta_{56})(\zeta_3\cdot\delta_{12})(\zeta_5\cdot\delta_{34}). \tag{3.31b}$$

To require that the invariant h_k be a sum over linearly independent terms, set

$$\text{(i)}\ k_0' = 0, \quad \text{for} \quad k_0 - k_0' = P - (\mu_1+\mu_2+\lambda_3) \geq 0,$$
$$\text{(ii)}\ k_0 = 0, \quad \text{for} \quad k_0 - k_0' < 0. \tag{3.32}$$

If the largest common term, $F(\rho_i)$, is then factored from the sum, the invariant $h_k(\rho_i)$ may be put in the form

$$h_k(\rho_i) = [\zeta_1\cdot(\zeta_3\times\zeta_5)]^{k_0}F(\rho_i)\sum \beta_k(\rho_i;n_1,n_2)(H_1)^{n_1}H_2^{n_2}, \quad n_1+n_2=N \tag{3.33}$$

[34] This phase convention, the choice of α_{1M}, α_{3M}, agrees with most authors, but the definition of highest weight state differs, e.g., since Hecht (Ref. 14), Elliott (Ref. 12), and Elliott (Ref. 13) have $-Y$, they would be considering the minimum Y state, compared to the notation of this paper. de Swart (Ref. 20) and Kuriyan, *et al.* (Ref. 22) choose $I_z = I_{\max}$ (and associated Y) as the highest weight state.

for $k_0 - k_0' \geq 0$. For $k_0 - k_0' < 0$, the invariant h_k becomes

$$h_k(\rho_i) = [\delta_{12}\cdot(\delta_{34} \times \delta_{56})]^{k'_0} \times F(\rho_i) \sum \epsilon_k(\rho_i; n_1, n_2) H_1^{n_1}(H_2)^{n_2}, \quad n_1 + n_2 = N, \tag{3.34}$$

where $k = 0, 1, \cdots, N = g - 1$ and $k_i = \rho_i + n_1$, $i = 1, 3, 5$; $k_i = \rho_i + n_2$, $i = 2, 4, 6$ and with the conditions

$$\begin{aligned} k_0 + \rho_3 + \rho_6 + N &= \mu_3, & k_0' + \rho_1 + \rho_2 + N &= \lambda_3, \\ k_0 + \rho_1 + \rho_4 + N &= \lambda_2, & k_0' + \rho_5 + \rho_6 + N &= \mu_2, \\ k_0 + \rho_2 + \rho_5 + N &= \lambda_1, & k_0' + \rho_3 + \rho_4 + N &= \mu_1. \end{aligned}$$
$$\rho_i \geq 0, \tag{3.35}$$

According to Eq. (3.32), k_0 or k_0' equals zero in Eq. (3.35), depending on the sign of the difference.

The exact expression for $N + 1 = g$,[35] the number of times the irreducible representation $\mathfrak{D}^{\lambda_3\mu_3}$ is contained in the direct-product representation $\mathfrak{D}^{\lambda_1\mu_1} \otimes \mathfrak{D}^{\lambda_2\mu_2}$, is quite complicated if simple inequalities are assumed on the partition numbers $[\lambda_i\mu_i]$. Conversely, if complicated bounds are imposed on P, then the expression for N is simple. Write the conditions (3.35) in the alternate form

$$\begin{aligned} \rho_4 - \rho_6 &= P - (\mu_2 + \lambda_3 + \mu_3), \\ \rho_5 - \rho_3 &= P - (\mu_1 + \lambda_3 + \mu_3), \\ \rho_2 - \rho_4 &= P - (\mu_1 + \lambda_2 + \mu_2), \\ \rho_3 - \rho_1 &= P - (\lambda_2 + \mu_2 + \lambda_3), \\ \rho_6 - \rho_2 &= P - (\lambda_1 + \mu_1 + \lambda_3), \\ \rho_1 - \rho_5 &= P - (\lambda_1 + \mu_1 + \mu_2). \end{aligned} \tag{3.36}$$

Equations (3.36), (3.30), and one relation of Eq. (3.35) constitute the alternate set to Eqs. (3.35). $N = n_1 + n_2$ is chosen such that there is one unique set of values ρ_i. At least two ρ_i equal zero, one for $i = 1, 3, 5$, and one for $i = 2, 4, 6$, respectively, depending on the value of N. From Eq. (3.36) the expressions for g follow:

$$\begin{aligned} \rho_2 = 0, \quad \rho_4 &= \lambda_2 + \mu_2 + \mu_1 - P, \quad k_0 \geq 0, \\ \rho_6 &= P - (\lambda_1 + \mu_1 + \lambda_3), \quad \rho_i \geq 0 \end{aligned} \tag{3.37}$$

and Case I:

$$\begin{aligned} \rho_1 &= 0, \quad N = \lambda_3, \\ \rho_3 &= P - (\lambda_2 + \mu_2 + \lambda_3), \\ \rho_5 &= \lambda_1 + \mu_1 + \mu_2 - P, \end{aligned}$$

Case II:

$$\begin{aligned} \rho_3 &= 0, \quad N = P - (\lambda_2 + \mu_2), \\ \rho_1 &= \lambda_2 + \mu_2 + \lambda_3 - P, \\ \rho_5 &= P - (\mu_1 + \lambda_3 + \mu_3), \end{aligned}$$

or Case III:

$$\begin{aligned} \rho_5 &= 0, \\ \rho_1 &= P - (\lambda_1 + \mu_1 + \mu_2), \\ N &= \mu_1 + \lambda_1 + \mu_2 + \lambda_3 - P, \\ \rho_3 &= \mu_1 + \lambda_3 + \mu_3 - P. \end{aligned}$$

The other possible bounds on P, and expressions for N, are obtained by exchanging $(\lambda_1\mu_1) \leftrightarrow (\lambda_2\mu_2)$ and $(\lambda_1\mu_1) \leftrightarrow (\mu_3\lambda_3)$ along with the corresponding changes of ρ_i [see Sec. 3D]. Altogether there are nine possibilities. When $k_0' > 0$, the correct degeneracy expression, N, is obtained from Eq. (3.37) by exchanging $\lambda_i \leftrightarrow \mu_i$ and making the corresponding changes in ρ_i [see Eq. (3.66)].

2. Determination of $\beta_k(\rho_i; n_1, n_2)$

Given the invariants Eqs. (3.33) and (3.34), the problem still remains to uniquely specify the g^2 coefficients $\beta_k(\rho_i; n_1, n_2)$. Orthogonality of the invariants $h_k(\rho_i)$ provides $\frac{1}{2}g(g - 1)$ conditions, normalization g conditions, thus $\frac{1}{2}g(g + 1)$ conditions in all on the g^2 coefficients. Any set of coefficients $\beta_k(\rho_i; n_1, n_2)$ may be related to another set $\beta_k'(\rho_i; n_1, n_2)$ through the transformation

$$\beta_k'(\rho_i; n_1, n_2) = \sum_{\sigma=0}^{N} B_{k\sigma}\beta_\sigma(\rho_i; n_1, n_2) \tag{3.38}$$

or

$$h_k'(\rho_i) = \sum_{\sigma=0}^{N} B_{k\sigma}h_\sigma(\rho_i), \qquad k = 0, 1, \cdots, N.$$

This implies, from Eq. (3.14), that

$$\begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{Bmatrix}_k' = \sum B_{k\sigma} \begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{Bmatrix}_\sigma. \tag{3.39}$$

[35] B. Preziosi, A. Simoni, and B. Vitale, Nuovo Cimento **34**, 1101 (1964), have calculated g from a straightforward multiplication of Young tableaus. Other methods for determining the degeneracy g appear in the literature. Freudenthal's formula, an implicit formula in terms of a recursion relation, appears in N. Jacobson, *Lie Algebras* (Interscience Publishers, Inc., New York, 1962). The formula is derived by relating the weight and multiplicity structure of $SU(3)$ [also $SU(n)$]. J. P. Antoine and D. Speiser, J. Math. Phys. **5**, 1226 (1964); **5**, 1560 (1964), describe graphical methods for general simple compact Lie groups. S. Gasiorowicz, "A Simple Graphical Method in the Analysis of $SU(3)$," Argonne Report ANL-6729, is a review article of the Speiser method for $SU(3)$.

If the new set of invariants $h_k'(\rho_i)$ are orthonormalized, and if the $3(\lambda\mu)$ coefficients are chosen real, then B_{ij} is an orthogonal transformation.

Two methods are now discussed to uniquely specify the coefficients $\beta_k(\rho_i; n_1, n_2)$ and form orthogonal states $h_k(\rho_i)$.[36]

3. Moshinsky's operator X

Moshinsky has constructed an Hermitian operator[37] X

$$X = \sum T_{\alpha+2,\beta} T_{\beta\rho} T_{\rho,\alpha+2}, \quad \alpha, \beta, \rho = 1, 2, \tag{3.40}$$

which commutes with the operators

$$T_{12}, T_{34}, T_{56}, T_{ii}, \qquad i = 1, \cdots, 6,$$

and the generators M_{ij}, and which combines the variables of the separate spaces[38]

$$\mathfrak{D}_{\lambda_1\mu_1}, \mathfrak{D}_{\lambda_2\mu_2}.$$

The invariant $h_k(\rho_i)$ must then be diagonalized with respect to X

$$Xh_k(\rho_i) = x_k(\rho_i)h_k(\rho_i). \tag{3.41}$$

The eigenvalues $x_k(\rho_i)$ are, of course, real, and Moshinsky[37] has shown that they are distinct.

The operator X, operating on the summand of Eq. (3.33), yields

$$\begin{aligned} X\{[\zeta_1\cdot(\zeta_3 \times \zeta_5)]^{k_0}F(\rho_i)(H_1)^{n_1}H_2^{n_2}\} \\ = [\zeta_1\cdot(\zeta_3 \times \zeta_5)]^{k_0}F(\rho_i) \\ \times [f(\rho_i; n_1, n_2)H_1^{n_1}H_2^{n_2} + g(\rho_i; n_1, n_2) \\ \times (H_1)^{n_1-1}H_2^{n_2+1} + h(\rho_i; n_1, n_2)H_1^{n_1+1}H_2^{n_2-1}], \end{aligned} \tag{3.42}$$

where

$$\begin{aligned} f(\rho_i; n_1, n_2) \\ = (\mu_1 + 1)!\,[\mu_2(\mu_1 + 2) + (k_3 + 1)(k_0' + k_1)] \\ + (\lambda_1 + \mu_1 + 3)[k_1(k_2 + 1) + k_6(k_0' + k_2 + k_3 + 2)] \\ + k_1k_3(k_2 + 1) + k_4k_6(k_0 + k_2 + 1) \end{aligned} \tag{3.43a}$$

and

$$g(\rho_i; n_1, n_2) = -k_1k_3k_5, \qquad h(\rho_i; n_1, n_2) = k_2k_4k_6. \tag{3.43b}$$

Similarly, X on a summand of Eq. (3.34) has the same form as Eq. (3.42), but

$$\begin{aligned} f'(\rho_i; n_1, n_2) = f(\rho_i; n_1, n_2)\,|_{k_0=0} \\ + k_0'[k_0'k_2 + (\mu_1 + 2)(\mu_2 + 2) + k_1(k_2 + 1) \\ + (\lambda_1 + \mu_1 + 1)(k_1 + k_2 + k_3 + k_4 + 2) + 3k_6 \\ + 2k_1 + k_1k_3 + k_2k_6 + k_3k_6 + k_1k_5 - 2k_2] \end{aligned} \tag{3.44}$$

and $g(\rho_i; n_1, n_2)$, $h(\rho_i; n_1, n_2)$ are given by Eq. (3.43b).

A secular determinant, obtained from Eq. (3.41), must be solved for the eigenvalues $x_k(\rho_i)$ to determine the coefficients $\beta_k(\rho_i; n_1, n_2)$. Given the coefficients $\beta_k(\rho_i; n_1, n_2)$, the $3(\lambda\mu)$ coefficients are found by taking the inner product of $h_k(\rho_i)$ with the triple product vector [see Eq. (3.16)]. The general $3(\lambda\mu)$ coefficient, in terms of the $\beta_k(\rho_i; n_1, n_2)$ is then Eq. (3.48).

The symmetry relations of the $3(\lambda\mu)$ coefficients are not obvious from the form of the operator X, Eq. (3.40). If the simple case $g = 2$ is considered, e.g.,

$$[1, 1] \times [1, 1] = 2[1, 1] + \cdots,$$

the standard convention is to choose the $3(\lambda\mu)$ coefficients such that under the various symmetry operations, one coefficient is symmetric, the other antisymmetric. The $3(\lambda\mu)$ coefficients determined by X do not have this property.

4. Choice of $\beta_k(\rho_i; n_1, n_2)$

To require that the $3(\lambda\mu)$ coefficients have simple symmetry properties, let the $\beta_k(\rho_i; n_1, n_2)$ be chosen such that

$$\beta_k(\rho_i; n_1, n_2) = (-1)^k\beta_k(\rho_i; n_2, n_1) \tag{3.45}$$

for $k = 0, 1, \cdots, N$. The condition (3.45) separates the invariants $h_k(\rho_i)$ into states which are even or odd under the various permutation operations on the variables. The odd, even, states are then mutually orthogonal. To finally determine the coefficients $\beta_k(\rho_i; n_1, n_2)$ a Gram–Schmidt orthogonalization procedure may be employed. Divide the coefficients $\beta_k(\rho_i; i, j)$ by the normalization $\beta_k(\rho_i; N, 0)$,

$$a_{kj} = \beta_k(\rho_i; i, j)/\beta_k(\rho_i; N, 0), \qquad k = 0, 1, \cdots, N \tag{3.46}$$

[36] The row and state labeling problem has been discussed by G. E. Baird and L. C. Biedenharn in a series of papers: J. Math Phys. 4, 1449 (1963); 5, 1723 (1964); 5, 1730 (1964). In particular, they show the interesting result that the multiplicity structure of $SU(n)$ operators may be put into a one-to-one association with the multiplicity structure of the corresponding states.

[37] M. Moshinsky, J. Math. Phys. 4, 1128 (1963).

[38] L. O'Raifeartaigh and A. J. Macfarlane (to be published) have constructed an operator from the generators of $SU(3)$. Professor Moshinsky has pointed out in discussion that this operator differs from the operator X only by Casimir operators. Other operators satisfying the above conditions may, of course, also be constructed.

and choose the coefficients a_{ki} so that the invariants $h_k(\rho_i)$ have the form

$$h_k(\rho_i) = [\zeta_1\cdot(\zeta_3 \times \zeta_5)]^{k_0} \times F(\rho_i)\beta_k \sum_{\alpha=0}^{[k/2]} a_{k\alpha}(H_1H_2)^{\alpha}(H_1^{k-2\alpha} \pm (H_2)^{k-2\alpha}), \quad (3.47)$$

where $a_{k0} = 1$ and $\pm$ refer to k even, odd, respectively, e.g.,

$$h_0(\rho_i) = [\zeta_1\cdot(\zeta_3 \times \zeta_5)]^{k_0}F(\rho_i)\beta_0(H_1^N + H_2^N),$$

$$h_2(\rho_i) = [\zeta_1\cdot(\zeta_3 \times \zeta_5)]^{k_0}F(\rho_i)\beta_2 \times [(H_1^N + H_2^N) + a_{21}(H_1H_2)(H_1^{N-2} + H_2^{N-2})].$$

The factors β_k are normalizations and the coefficients a_{ki} are uniquely determined by integration, the necessary integrations being performed in Appendix B.

5. *Explicit expression*

The inner product of the invariant $h_k(\rho_i)$, Eq. (3.33), with the triple product vector [see Eq. (3.16)] yields the $3(\lambda\mu)$ coefficient in terms of the coefficients[39] $\beta_k(\rho_i; n_1, n_2)$:

$$\begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_1 & \alpha_2 & \alpha_3' \end{Bmatrix}_k = (-1)^{\phi} \sum \beta_k(\rho_i; n_1, n_2) \frac{\prod_i k_i!\, k_0!\, (-1)^{m_{03}+m_{04}+m_{06}}}{\prod_{ij} n_{ij}! \prod_s m_{0s}!} \times \prod_{i=1,2} S(\lambda_i\mu_i; \alpha_i) S(\mu_3\lambda_3; -\alpha_3'), \text{[40]} \quad (3.48)$$

$$\phi = \mu_3 + k_0 + q_1 + q_2, \qquad \alpha_3' : r_3 = 0,$$

where

$$S(\lambda_i\mu_i; \alpha_i) = M(\lambda_i\mu_i; \alpha_i) \sum_{u_i} \frac{(-1)^{u_i+n_i+m_i} m_i!\, n_i!\, (p_i - r_i + m_i)!}{(p_i + m_i + n_i + 1)!\, u_i!\, (p_i - n_i)!\, (p_i - r_i + m_i - n_i)!} \times \frac{[r_i - (\mu_i - q_i) + n_i]!}{(-p_i + r_i + u_i)!\, (\mu_i + p_i - q_i - r_i - u_i)!\, [n_i - (\mu_i - q_i) - p_i + r_i + u_i]!} \quad (3.49a)$$

and

$$M(\lambda_i\mu_i; \alpha_i) = \frac{(\mu_i + p_i - q_i + 1)!\, (\mu_i + p_i - q_i - r_i)!}{(\mu_i + p_i - q_i)!\, N(\lambda_i\mu_i; \alpha_i)}. \quad (3.49b)$$

The following constraints must be applied to Eq. (3.48):

$$n_1 = n_{31} + n_{41}, \quad n_2 = n_{51} + n_{61}, \quad i = 1, 3, 5,$$
$$n_3 = n_{11} + n_{21}, \quad k_i = \rho_i + n_1,$$
$$m_1 = n_{32} + n_{42}, \quad m_2 = n_{52} + n_{62}, \quad i = 2, 4, 6,$$
$$m_3 = n_{12} + n_{22}, \quad k_i = \rho_i + n_2,$$
$$\sum_{j=1}^{3} n_{ij} = k_i, \quad i = 1, \cdots, 6, \quad \sum_{s=1}^{6} m_{0s} = k_0,$$
$$r = r_1 + r_2 = k_0 + \mu_2 - q_2 + \mu_1 - q_1 - (\mu_3 - q_3),$$
$$p_1 + p_2 = 2k_0 - \mu_3 + p_3 + q_3 + \mu_1 - q_1 + \mu_2 - q_2,$$
$$p_3 + q_3 = m_{01} + m_{04} + n_{33} + n_{63} + m_3 + n_3,$$
$$\lambda_1 + \mu_1 - (p_1 + q_1) = m_{05} + m_{06} + n_{23} + n_{53} + m_1 + n_1,$$
$$\mu_3 - q_3 = m_{02} + m_{05} + m_{32} + n_{62} - n_3,$$
$$-\mu_1 + q_1 + r_1 = m_{03} + m_{04} + n_{22} + n_{52} - n_1,$$
$$-p_3 = m_{03} + m_{06} + n_{31} + n_{61} - m_3,$$
$$p_1 - r_1 = m_{01} + m_{02} + n_{21} + n_{51} - m_1,$$
$$n_{13} + n_{43} + m_{02} + m_{03} + m_2 + n_2 = \lambda_2 + \mu_2 - (p_2 + q_2),$$
$$m_{04} + m_{05} + n_{11} + n_{41} - m_2 = p_2 - r_2,$$
$$m_{01} + m_{06} + n_{12} + n_{42} - n_2 = -\mu_2 + q_2 + r_2, \quad (3.49c)$$

$N(\lambda_i\mu_i; \alpha_i)$ is given by Eq. (2.10b).

The phase convention is that of Eq. (3.26). The necessary coefficient for this case is

$$\begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_{1M} & \alpha_2' & \alpha_{3M} \end{Bmatrix}_k = C' \times \sum_{n_1+n_2=N} \frac{\beta_k(\rho_i; n_1, n_2)k_1!\, k_5!}{(\mu_2 + p_2 + 1 - k_6)!\, [-(\mu_2 - q_2) + k_5]!}, \quad (3.50a)$$

[39] Closed expressions for the coefficients for the direct product $[\lambda\mu] \otimes [1,1]$ have been given by Kuriyan *et al.* (Ref. 22) and Hecht (Ref. 14). T. A. Brody, M. Moshinsky, and I. Renero, J. Math. Phys. **6**, 1540 (1965), have recursion relations to determine coefficients for general direct product $[\lambda_1\mu_1] \otimes [\lambda_2\mu_2]$.

[40] The choice α's, $r_3 = 0$, is made because the $3(\lambda\mu)$ coefficient, Eq. (3.48), may then be divided by the factor $((t_1), (t_0)_1; t_2(t_0)_2|t_1t_2t_3(t_0)_3 = t_3)$, the CG coefficient for $SU(2)$, and multiplied by $(N_3)^{1/2}$, to obtain the isoscalar factor, as defined by Edmonds (Ref. 33).

where

$$C' = \frac{(-1)^{k_0}(\mu_2 + p_2 - q_2 + 1)!}{N(\lambda_1\mu_1; \alpha_{1M})N(\lambda_2\mu_2; \alpha_2')N(\mu_3\lambda_3; \alpha_{3M})(\mu_2 - q_2)!\,(\mu_2 + p_2 - q_2)!} \tag{3.50b}$$

α_2' is $y_2 = \lambda_3 + 2\mu_3 - (\lambda_1 + 2\mu_1)$, $(t_0)_2 = \frac{1}{2}(\lambda_3 - \lambda_1)$, with t_2 assuming, according to Biedenharn's results,[36] g values for given α_{1M}, α_{3M}. To define a proper phase convention, a nonvanishing $3(\lambda\mu)$ coefficient must be made positive. However, the phase of $\beta_k(\rho_i; n_1, n_2)$ is, in general, undetermined. For $k = 0$, when $n_1 = N$, $n_2 = 0$, and $n_2 = N$, $n_1 = 0$, it is clear from Eq. (3.50) that the $3(\lambda\mu)$ coefficient must be multiplied by $(-1)^{k_0}$ to satisfy Eq. (3.26). In general, the $3(\lambda\mu)$ coefficient must be multiplied by $(-1)^{\phi}$, where ϕ is a function of the parameters k_i.

6. *Completeness of the reduction*

Equations (3.29) plus the restriction, Eq. (3.32), give g independent terms, appropriate linear combinations of which yield g orthonormal invariants $h_k(\rho_i)$, and, from Eq. (3.13), g orthonormal direct-product base vectors associated with the irreducible representations $\mathfrak{D}^{\lambda_3\mu_3}$. To determine completeness, sum over the reduced direct-product states:

$$\begin{aligned} n &= \sum gN_3 \\ &= \tfrac{1}{2}\sum g(\lambda_3 + 1)(\mu_3 + 1)(\lambda_3 + \mu_3 + 2) \end{aligned} \tag{3.51}$$

$n = n_a + n_b$, where n_a is the case $k_0 \geq 0$, and n_b is the case $k_0 < 0$, or

$$n = n_a + n_b' - n_0, \tag{3.52}$$

where n_b' are the terms $k_0 \leq 0$ and n_0 are the terms $k_0 = 0$. Instead of Eq. (3.51), the numbers k_i, Eq. (3.29), may be substituted for $\lambda_3\mu_3$; the sum over all possible k_i, using Eq. (3.32), then includes the degeneracy g. Note that

$$n_b' = n_a \,|_{\lambda_1\leftrightarrow\mu_1,\ \lambda_2\leftrightarrow\mu_2}.$$

The result of the summation is

$$\begin{aligned} n_a &= \tfrac{1}{8}(\lambda_1 + 1)(\mu_1 + 1)(\lambda_1 + 2)(\lambda_2 + \mu_2 + 2) \\ &\quad\times [\mu_1(\lambda_2 - \mu_2 + 1) + 2(\lambda_2 + 1)(\mu_2 + 1)], \\ n_0 &= \tfrac{1}{4}(\mu_1 + 1)(\lambda_1 + 1)(\lambda_2 + \mu_2 + 2) \\ &\quad\times [\lambda_1 + \lambda_1\mu_1 + \mu_1 + (\lambda_1 - \mu_1)(\mu_2 - \lambda_2) \\ &\qquad + 2(\lambda_2 + 1)(\mu_2 + 1)]. \end{aligned} \tag{3.53}$$

Putting Eqs. (3.53) into Eq. (3.52), one can then see that

$$n = \sum gN_3 = N_1 \cdot N_2.$$

Thus, the reduction of the direct-product representation is complete; the gN_3 product base vectors $|\lambda_3\mu_3;\, \alpha_3\rangle_k$, $k = 0, 1, \cdots, g - 1$, associated with the irreducible representation $\mathfrak{D}^{\lambda_3\mu_3}$, span the sum of the spaces $\mathfrak{Q}^{(k)}_{\lambda_1\mu_1\lambda_2\mu_2}$.

D. Symmetry of the $3(\lambda\mu)$ coefficients[41]

Let the operator P_{12} exchange the coordinates $(\zeta_1, \zeta_2) \leftrightarrow (\zeta_3, \zeta_4)$. Then, from Eq. (3.33) ($k_0 \geq 0$),

$$\begin{aligned} P_{12}h_k(\rho_i) &= (-1)^{k_0}[\zeta_1\cdot(\zeta_3 \times \zeta_5)]^{k_0}[P_{12}F(\rho_i)] \\ &\quad\times \sum \beta_k(\rho_i; n_2, n_1)(H_1)^{n_1}H_2^{n_2}, \end{aligned}$$

using the fact that $P_{12}H_1 = H_2$. Let $\mathcal{P}_{12}$ be the operator which exchanges the parameters $\rho_1 \leftrightarrow \rho_2$, $\rho_3 \leftrightarrow \rho_6$, $\rho_4 \leftrightarrow \rho_5$. Using Eq. (3.28b), it may be seen that $\mathcal{P}_{12}P_{12}F(\rho_i) = F(\rho_i)$, and hence,

$$\begin{aligned} \mathcal{P}_{12}P_{12}h_k(\rho_i) &= (-1)^{k_0+k}[\zeta_1\cdot(\zeta_3 \times \zeta_5)]^{k_0}F(\rho_i) \\ &\quad\times \sum \beta_k(\rho_i'; n_1, n_2)(H_1)^{n_1}H_2^{n_2}. \end{aligned}$$

If it can be shown that

$$\beta_k(\rho_i'; n_1, n_2) = \beta_k(\rho_i; n_1, n_2), \tag{3.54}$$

then the result follows

$$\mathcal{P}_{12}P_{12}h_k(\rho_i) = (-1)^{k_0+k}h_k(\rho_i). \tag{3.55}$$

Using Eq. (3.14),

$$\mathcal{P}_{12}P_{12}h_k(\rho_i) = \sum \begin{Bmatrix} \lambda_2\mu_2 & \lambda_1\mu_1 & \lambda_3\mu_3 \\ \alpha_2 & \alpha_1 & \alpha_3 \end{Bmatrix}_k |\lambda_1\mu_1;\alpha_1\rangle\, |\lambda_2\mu_2;\alpha_2\rangle\, |\lambda_3\mu_3;\alpha_3\rangle_c.$$

The relation

$$\begin{Bmatrix} \lambda_2\mu_2 & \lambda_1\mu_1 & \lambda_3\mu_3 \\ \alpha_2 & \alpha_1 & \alpha_3 \end{Bmatrix}_k = (-1)^{k_0+k}\begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{Bmatrix}_k \tag{3.56}$$

then holds. A similar result follows for $k_0' > 0$.

Likewise, let P_{13} be the exchange $(\zeta_1, \zeta_2) \leftrightarrow (\zeta_5, \zeta_6)$. It may be shown that

$$\mathcal{P}_{13}P_{13}h_k(\rho_i) = (-1)^{k_0+k}h_k(\rho_i) \tag{3.57}$$

[41] de Swart (Ref. 20) and Hecht (Ref. 14) have discussed symmetries for the case $g = 2$. J. R. Derome and W. T. Sharp, J. Math. Phys. 6, 1584 (1965), have discussed symmetries for 3-j and 6-j symbols of a general group without specifying the phase or method of labeling degenerate states.

assuming Eq. (3.54), where $\mathcal{P}_{13}$ is the interchange of the labels

$$\rho_1 \leftrightarrow \rho_4, \qquad \rho_2 \leftrightarrow \rho_3, \qquad \rho_5 \leftrightarrow \rho_6. \tag{3.58}$$

From Eq. (3.57),

$$\begin{Bmatrix} \mu_3\lambda_3 & \lambda_2\mu_2 & \mu_1\lambda_1 \\ -\alpha_3 & \alpha_2 & -\alpha_1 \end{Bmatrix}_k = (-1)^{k_0+k} \begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{Bmatrix}_k. \tag{3.59}$$

Similarly, exchange of $(\zeta_3, \zeta_4) \leftrightarrow (\zeta_5, \zeta_6)$, P_{23}, and the interchange $\mathcal{P}_{23}$

$$\rho_1 \leftrightarrow \rho_6, \qquad \rho_2 \leftrightarrow \rho_5, \qquad \rho_3 \leftrightarrow \rho_4 \tag{3.60}$$

implies

$$\begin{Bmatrix} \lambda_1\mu_1 & \mu_3\lambda_3 & \mu_2\lambda_2 \\ \alpha_1 & -\alpha_3 & -\alpha_2 \end{Bmatrix}_k = (-1)^{k_0+k} \begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{Bmatrix}_k. \tag{3.61}$$

It is clear that Eq. (3.54) holds. The inner product $(h_0(\rho_i), h_2(\rho_i)) = 0$ determines $a_{21}(\rho_i, k_0)$.

$$a_{21}(\rho_i, k_0) = -\frac{[||^{k_0}F(\rho_i)(H_1^N + H_2^N), ||^{k_0}F(\rho_i)(H_1^N + H_2^N)]}{[||^{k_0}F(\rho_i)(H_1^N + H_2^N), ||^{k_0}F(\rho_i)(H_1H_2)(H_1^{N-2} + H_2^{N-2})]}. \tag{3.62}$$

Exchange the coordinates, then, Eq. (3.62) remains the same except $F(\rho_i)$ goes to $F'(\rho_i) = PF(\rho_i)$, where P stands for P_{12}, P_{13}, P_{23}. Next, exchange the parameters ρ_i, since

$$\mathcal{P}PF(\rho_i) = F(\rho_i), \qquad a_{21}(\rho_i, k_0) = a_{21}(\rho_i', k_0),$$

Eq. (3.54) then holds by induction. Assume that the coefficients $a_{kj}(\rho_i, k_0)$ (for k, N even), for $j < \frac{1}{2}k = (\frac{1}{2}N - 1), \cdots, 1, 0$, have the property that $a_{kj}(\rho_i, k_0) = a_{kj}(\rho_i', k_0)$. The inner product $(h_N(\rho_i), h_k(\rho_i)) = 0$, for $k = 0, 2, \cdots, N-2$ provides $\frac{1}{2}N$ inhomogeneous equations for $a_{Ni}(\rho_i, k_0)$:

$$b_{kN}(\rho_i; k_0) + \sum_{\alpha=1}^{N/2} b_{k\alpha}a_{N\alpha}(\rho_i; k_0) = 0,$$

$$k = 0, 2, \cdots, N-2.$$

Since, by assumption, $b_{jk}(\rho_i; k_0) = b_{jk}(\rho_i'; k_0)$,

$$a_{N\alpha}(\rho_i; k_0) = a_{N\alpha}(\rho_i'; k_0).$$

A similar argument applies to k odd.

The conjugation result follows similarly. Note first that

$$\mathcal{R}R\, |\lambda_1\mu_1; \alpha_1\rangle\, |\lambda_2\mu_2; \alpha_2\rangle\, |\lambda_3\mu_3; \alpha_3\rangle_c$$
$$= A\, |\lambda_1\mu_1; \alpha_1\rangle\, |\lambda_2\mu_2; \alpha_2\rangle\, |\lambda_3\mu_3; \alpha_3\rangle_c, \tag{3.63}$$

where R is the operation $\zeta_1 \leftrightarrow \delta_{12}$, $\zeta_3 \leftrightarrow \delta_{34}$, $\zeta_5 \leftrightarrow \delta_{56}$, [see Eq. (2.20)] and $\mathcal{R}$ changes the labels $\lambda_i \leftrightarrow \mu_i$, $\alpha_i \rightarrow -\alpha_i$ [see Eq. (2.22b)], and

$$A = \left\{\frac{(\mu_1+1)!\,(\mu_2+1)!\,(\lambda_3+1)!}{(\lambda_1+1)!\,(\lambda_2+1)!\,(\mu_3+1)!}\right\}^{\frac{1}{2}} \tag{3.64}$$

using Eqs. (2.20) and (2.21). From Eq. (3.14) and the fact that $(h_k(\rho_i), h_k'(\rho_i)) = \delta_{kk'}$

$$[\mathcal{R}Rh_k(\rho_i), \mathcal{R}Rh_k'(\rho_i)] = A^2\, \delta_{kk'} \tag{3.65}$$

using also Eq. (3.63). From Eq. (3.65),

$$a_{21}(\rho_i', k_0') = c_{21}(\rho_i; k_0'),$$

where $c_{kj}(\rho_i, k_0')$ are the coefficients of $h_k'(\rho_i)$, $k_0 - k_0' < 0$, with the normalization factored out [similar to Eq. (3.47)]. $\mathcal{R}$ is the exchange of parameters,

$$\rho_1 \leftrightarrow \rho_6, \qquad \rho_2 \leftrightarrow \rho_3, \qquad \rho_4 \leftrightarrow \rho_5. \tag{3.66}$$

In general then, $a_{ij}(\rho_i', k_0') = c_{ij}(\rho_i, k_0')$. Thus,

$$\mathcal{R}Rh_k(\rho_i) = (-1)^k h_k(\rho_i)'A \tag{3.67}$$

$$k_0 - k_0' \geq 0 \qquad k_0 - k_0' < 0$$

The factor A [Eq. (3.64)] renormalizes the $h_k(\rho_i)$.

$$\mathcal{R}Rh_k(\rho_i) = A \sum_{\alpha_i} \begin{Bmatrix} \mu_1\lambda_1 & \mu_2\lambda_2 & \mu_3\lambda_3 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 \end{Bmatrix}_k$$
$$\times\, |\lambda_1\mu_1; \alpha_1\rangle\, |\lambda_2\mu_2; \alpha_2\rangle\, |\lambda_3\mu_3; \alpha_3\rangle_c.$$

The result then follows:

$$\begin{Bmatrix} \mu_1\lambda_1 & \mu_2\lambda_2 & \mu_3\lambda_3 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 \end{Bmatrix}_k = (-1)^k \begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_3\mu_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{Bmatrix}_k. \tag{3.68}$$

ACKNOWLEDGMENTS

The author is greatly indebted to Professor K.M. Case for suggesting the use of the Bargmann method, as well as for many helpful discussions. He is also grateful to Professor K.T. Hecht and Professor A.C.T. Wu for several useful suggestions.

Part of this paper was written at the University of Mexico, and the author would like to thank the National Science Foundation for the support, and Professor M. Moshinsky for the hospitality extended him.

APPENDIX A. ORTHONORMALITY OF BASE VECTORS

Since Condon and Shortley[19] have already given the normalization for the $SU(2)$ lowering operator,

$$|\lambda\mu; ytt_0\rangle = \left\{\frac{(2t-r)!}{(2t)!\,r!}\right\}^{\frac{1}{2}} (T_-)^r\, |\lambda\mu; ytt\rangle,$$

it is only necessary to calculate the normalization for the base vector $|\lambda\mu;\, y,\, t,\, t\rangle$. Equation (2.10a) is, with $r = 0$,

$$\begin{aligned}|\lambda\mu; ytt\rangle &= N(\lambda\mu; ytt)(-1)^q(\xi_1)^p(\sigma_1)^{\lambda-p}(-\delta_{12}^{(2)})^{\mu-q}(\delta_{12}^{(3)})^q \\ &= N(\lambda\mu;\alpha')(-1)^q \\ &\times \sum \binom{q}{b}\binom{\mu-q}{a}(-1)^{a+b}\xi_1^{(1)}\eta_1^{(2)}\sigma_1^{(3)}\xi_2^{(4)}\eta_2^{(5)}\sigma_2^{(6)},\end{aligned} \quad \text{(A1)}$$

where

$$(1) = \mu + p - (a+b), \qquad (2) = b,$$
$$(3) = \lambda - p + q, \qquad (4) = a + b,$$
$$(5) = q - b, \qquad (6) = \mu - q - a.$$

Take the inner product $(|\lambda\mu;\, y,\, t,\, t\rangle, |\lambda'\mu';\, y',\, t',\, t'\rangle)$. Using Eq. (1.2),

$$(2) : b = b', \qquad (4) : a = a', \qquad (5) : q = q',$$
$$(6) : \mu = \mu', \qquad (1) : p = p', \qquad (3) : \lambda = \lambda',$$

showing orthogonality. The inner product of Eq. (A1) becomes

$$= N(\lambda\mu; ytt)^2(q!)^2[(\mu-q)!]^2 \sum_{a,b} \frac{p!\, b!\, \lambda!\, a!\, q!\, \mu!}{(a!)^2(b!)^2}$$

and this may be summed (using binomial identities) to be

$$= N(\lambda\mu; ytt)^2 \times \frac{p!\, q!\, (\mu-q)!\, (\lambda-p)!\, (\lambda+\mu-q+1)!\, (\mu+p+1)!}{(\mu+p-q+1)!\, (\lambda+1)!}. \quad \text{(A2)}$$

$N(\lambda\mu;\, ytt)$, as given in Eq. (A2), times the factor $\{(2t-r)!/(2t)!r!\}^{\frac{1}{2}}$, is $N(\lambda\mu;\alpha)$, Eq. (2.10b).

APPENDIX B. EVALUATION OF INTEGRALS

A. Normalization of $h(k_i)$ for the nondegenerate case $[\lambda_1 0] \otimes [\lambda_2 \mu_2]$

$$h(k_i) = \Delta(k_i) \times \frac{[\zeta_1\cdot(\zeta_3\times\zeta_5)]^{k_0}}{k_0!}\frac{(\zeta_3\cdot\delta_{56})^{k_1}}{k_1!}\frac{(\zeta_1\cdot\delta_{56})^{k_2}}{k_2!}\frac{(\zeta_5\cdot\delta_{34})^{k_5}}{k_5!}\frac{(\zeta_1\cdot\delta_{34})^{k_6}}{k_6!}.$$

Divide $h(k_i)$ by $\Delta(k_i)$:

$$f(k_i) \equiv h(k_i)/\Delta(k_i). \quad \text{(B1)}$$

Multiply $f(k_i)$ by $\tau_0^{k_0}\tau_1^{k_1}\tau_2^{k_2}\tau_5^{k_5}\tau_6^{k_6}$ and sum k_i

$$\begin{aligned}\Phi &\equiv \sum_{k_i} f(k_i)\prod_i \tau_i^{k_i} \\ &= \exp\{\tau_0[\zeta_1\cdot(\zeta_3\times\zeta_5)] + \tau_1(\zeta_3\cdot\delta_{56}) \\ &\quad + \tau_2(\zeta_1\cdot\delta_{56}) + \tau_5(\zeta_5\cdot\delta_{34}) + \tau_6(\zeta_1\cdot\delta_{34})\},\end{aligned} \quad \text{(B2)}$$

$$[\Phi(\tau_i), \Phi(\tau_i')] = \sum_{k_i,k'_i}[f(k_i), f(k_i')]\prod_i \bar{\tau}_i^{k_i}\prod_j (\tau_j')^{k'_j}. \quad \text{(B3)}$$

Integrate Eq. (B3) with respect to ζ_4, ζ_6:

$$\begin{aligned}(\Phi,\Phi') = \int \exp\{&\overline{(\tau_1\,\delta_{35} + \tau_2\,\delta_{15})} \\ &\cdot(\tau_1'\,\delta_{35} + \tau_2'\,\delta_{15}) + \tau_0'[\zeta_1\cdot(\zeta_3\times\zeta_5)] \\ &+ \overline{(\tau_5\,\delta_{53} + \tau_6\,\delta_{13})}\cdot(\tau_5'\,\delta_{53} + \tau_6'\,\delta_{13}) \\ &+ \overline{\tau_0[\zeta_1\cdot(\zeta_3\times\zeta_5)]}\}\, d\mu_9,\end{aligned} \quad \text{(B4)}$$

where use has been made of Bargmann's result[1]

$$[e_a(z), f(z)] = f(a), \qquad e_a(z) = \exp(\bar{a}\cdot z). \quad \text{(B5)}$$

Perform a change of variables

$$\tau_2'\zeta_1' = \tau_2'\zeta_1 + \tau_1'\zeta_3, \qquad \tau_5'\zeta_5' = \tau_5'\zeta_5 + \tau_6'\zeta_1,$$

and a similar change for the complex-conjugate variables. The exponential of Eq. (B4) has the general form

$$\exp[-\zeta_3\cdot(b1 - c\mathsf{A})\zeta_3 + \bar{a}_1\cdot\zeta_3 + a_2\cdot\bar{\zeta}_3 + C]. \quad \text{(B6)}$$

By a translation of ζ_3, Eq. (B6) may be put in the form

$$\exp[-\zeta_3'\cdot(b1 - c\mathsf{A})\zeta_3' + D], \quad \text{(B7)}$$

where $b = 1 + \tau_1'\bar{\tau}_1/\tau_2'\bar{\tau}_2$, $c = \tau_5'\bar{\tau}_5$. (B8)

$A_{ij} = (\zeta_5\cdot\bar{\zeta}_5)\delta_{ij} - (\bar{\zeta}_5)_i(\zeta_5)_j$, suppressing primes, and D is independent of ζ_3', $\bar{\zeta}_3'$. Equation (B7) may be integrated with respect to ζ_3' using Bargmann's result,

$$\int \exp(\bar{z}\cdot\mathsf{A}z)\, d\mu_3(z) = [\det(1-\mathsf{A})]^{-1}, \quad \text{(B9)}$$

where $\bar{z}\cdot\mathsf{A}z = \sum_{ij}\bar{z}_iA_{ij}z_j$, and the matrix $1 - \mathsf{A}$ is assumed to have a positive-definite Hermitian part. The integral of (B7) becomes

$$(\Phi,\Phi') = \int \frac{\exp(D)\, d\mu_6(\zeta_1,\zeta_5)}{\det(b1 - c\mathsf{A})}, \quad \text{(B10)}$$

$$\det(b1 - c\mathsf{A}) = b(b - c(\zeta_5\cdot\bar{\zeta}_5))^2.$$

The denominator of (B10) is only a function of $\zeta_5\cdot\bar{\zeta}_5$, so (B10) may be integrated with respect to ζ_1; $\exp(D)$ may be put in the form

$$\exp[-\bar{\zeta}_1'\cdot(m1 - \mathsf{H})\zeta_1' + E'], \quad \text{(B11)}$$

where

$$m = 1 + \frac{\tau_6'\bar{\tau}_6}{\tau_5'\bar{\tau}_5} - \frac{\tau_1'\bar{\tau}_1}{\tau_2'\bar{\tau}_2}[b - c(\zeta_5\cdot\bar{\zeta}_5)]^{-1}$$

and

$$H_{ij} = -h(\zeta_5)_i(\bar{\zeta}_5)_j + k[(\zeta_5\cdot\bar{\zeta}_5)\,\delta_{ij} - (\zeta_5)_i(\bar{\zeta}_5)_j],$$

$$h = c/b\,\frac{\tau_1'\bar{\tau}_1}{\tau_2'\bar{\tau}_2}\,[b - c(\zeta_5\cdot\bar{\zeta}_5)]^{-1},$$

$$k = \tau_2'\bar{\tau}_2 + \tau_0'\bar{\tau}_0[b - c(\zeta_5\cdot\bar{\zeta}_5)]^{-1}.$$

The result of integrating (B11) with respect to ζ_1 is

$$(\Phi, \Phi') = \int \frac{\exp(E)\, d\mu_3(\zeta_5)}{b(b - c(\zeta_5\cdot\bar{\zeta}_5))^2 \det(m1 - \mathsf{H})},$$

If

$$(\zeta_5\cdot\bar{\zeta}_5)\left[1 + \frac{\tau_6'\bar{\tau}_6}{\tau_5'\bar{\tau}_5}\,b\right] = \zeta_5'\cdot\bar{\zeta}_5',$$

then (Φ, Φ') may be put in the simple form

$$(\Phi, \Phi') = \int \frac{d\mu_3(\zeta_5')}{\{1 - [(\tau'\cdot\bar{\tau})(\zeta_5'\cdot\bar{\zeta}_5') - d(\zeta_5'\cdot\bar{\zeta}_5')^2]\}^2}, \tag{B12}$$

where

$$\tau'\cdot\bar{\tau} = \tau_0'\bar{\tau}_0 + \tau_1'\bar{\tau}_1 + \tau_2'\bar{\tau}_2 + \tau_5'\bar{\tau}_5 + \tau_6'\bar{\tau}_6,$$

$$d = \tau_2'\bar{\tau}_2\tau_5'\bar{\tau}_5 + \tau_2'\bar{\tau}_2\tau_6'\bar{\tau}_6 + \tau_1'\bar{\tau}_1\tau_6'\bar{\tau}_6.$$

Expand the denominator of (B12):

$$(\Phi, \Phi') = \sum \frac{(n+1)!\,(-1)^m(\tau'\cdot\bar{\tau})^{n-m}\,d^m}{m!\,(n-m)!}\,I(n+m), \tag{B13}$$

$$I(n+m) = \int d\mu_3(\zeta_5')(\zeta_5'\cdot\bar{\zeta}_5')^{n+m} = \tfrac{1}{2}(n+m+2)! \tag{B14}$$

using (1.2). Thus,

$$(\Phi, \Phi') = \frac{1}{2}\sum \frac{(P+2)!\,(\bar{\tau}_0\tau_0')^{k_0}\cdots(\bar{\tau}_6\tau_6')^{k_6}}{k_0!}\,S,$$

where $P = k_0 + k_1 + k_2 + k_5 + k_6$ and

$$S = \sum \frac{(P+1-m)!\,(-1)^m}{(k_1 - m + y_1 + y_2)!\,(k_2 - y_1 - y_2)!\,(k_5 - y_1)!\,(k_6 - m + y_1)!\,y_1!\,y_2!\,(m - y_1 - y_2)!}.$$

S may be summed using binomial identities, and the result is

$$(\Phi, \Phi') = \frac{1}{2}\sum \frac{(\tau_0'\bar{\tau}_0)^{k_0}\cdots(\tau_6'\bar{\tau}_6)^{k_6}(P+2)!}{k_0!\,k_1!\,k_2!\,k_5!\,k_6!} \times \frac{(k_0+k_5+k_6+1)!\,(k_0+k_1+k_2+1)!\,(k_0+k_1+k_4+1)!}{(k_0+k_1+1)!\,(k_0+k_5+1)!}. \tag{B15}$$

Comparison of (B15) and (B3) provides Eq. (3.23).

The expansion of the denominator of (B12), and similar expansions in this Appendix, may be justified in the following manner. In Eq. (B3), let $J(k_i, k_j') = [f(k_i), f(k_j')]$ be written $J(A; k_i, k_j')$, where the integration of ζ_5 is not taken over the entire plane, but only over a finite portion.

$$J(k_i, k_j') = \lim_{A\to\infty} J(A; k_i, k_j').$$

The integral (B12) would be written

$$(\Phi, \Phi') = \int_A \frac{d\mu_3(\zeta_5)}{\{1 - [\tau'\cdot\bar{\tau})(\zeta_5\cdot\bar{\zeta}_5) - d(\zeta_5\cdot\bar{\zeta}_5)^2]\}^2}.$$

This integral may be expanded, since the τ's may be made sufficiently small. The coefficient of $\prod(\bar{\tau}_i)^{k_i}\prod\tau_j'^{k_j}$ is $J(A; k_i, k_j')$, or

$$(\Phi, \Phi') = \sum \frac{(n+1)!\,(-1)^m(\tau'\cdot\bar{\tau})^{n-m}\,d^m}{m!\,(n-m)!} \times I(A, n+m),$$

$$I(A, n+m) = \int_A d\mu_3(\zeta_5)(\zeta_5\cdot\bar{\zeta}_5)^{n+m}.$$

In this integral, the limit may be taken,

$$I(n+m) \equiv \lim_{A\to\infty} I(A, n+m) = \tfrac{1}{2}(n+m+2)!$$

The integral (B12), and similar types, are well defined if they are understood in the above sense.

B. Normalization for $h_k(\rho_i)$

In order to calculate the normalization for $h_k(\rho_i)$,

$$h_k(\rho_j) = \sum_{m_1+m_2=N} \beta_k(\rho_i; n_1, n_2)H(k_i),$$

where

$$H(k_i) = [\zeta_1\cdot(\zeta_3\times\zeta_5)]^{k_0}F(\rho_i)H_1^{n_1}(H_2)^{n_2}$$

and $F(\rho_i)$ is given by Eq. (3.28b), with

$$k_i = \rho_i + n_1,\qquad k_i' = \rho_i' + n_1',\qquad i = 1, 3, 5,$$

$$k_i = \rho_i + n_2,\qquad k_i' = \rho_i' + n_2',\qquad i = 2, 4, 6,$$

it is necessary to evaluate the inner product $[H(k_i), H(k_i')]$. Multiply $H(k_i)$ by $\prod(\tau_i^{k_i}/k_i'!)$ and sum over k_i:

$$(\Phi, \Phi') = \sum_{k_i, k_j'} [H(k_i), H(k_j')] \times \prod \left(\frac{(\bar{\tau}_i)^{k_i}}{k_i!}\right) \prod \left(\frac{(\tau_j')^{k'_j}}{k_j'!}\right) \tag{B16}$$

$$= \int \exp [\Phi(\bar{\tau}_i, \bar{\zeta}) + \Phi(\tau_j', \zeta)] \, d\mu_{18}(\zeta). \tag{B17}$$

The integration may first be performed with respect to ζ_2, ζ_4, ζ_6, using Eq. (B5), and then ζ_5, using Eq. (B9). The result may then be expanded and integrated with respect to the remaining variables. The sums may be contracted by means of binomial identities. The following result is obtained:

$$(\Phi, \Phi') = \sum (\bar{\tau}_0 \tau_0')^{k_0} \beta^{\alpha_3} (-d)^{\alpha_2} [(4) + (6)]^{\alpha_1} [(3)(4)]^{z_3} \times [(2) + (3)]^{z_4} [(5)(6)]^{z_5} [(1) + (5)]^{z_6} \cdot S, \tag{B18}$$

where $(i) = \bar{\tau}_i \tau_i'$ (no summation) and S is the factor

$$S = \frac{(m_1 + k_0 + \alpha_2 + 1)! \, (2m_1 + 2k_0 + 2\alpha_2 + \alpha_3 - \alpha_1 + z_4 + z_6 + 3)!}{k_0! \, 2z_3! \, z_4! \, z_5! \, z_6! \, \alpha_1! \, (m_1 + k_0 + \alpha_2 - \alpha_1 + 1)! \, \alpha_2! \, \alpha_3!} \times \frac{(2m_1 + k_0 + 2\alpha_2 - \alpha_1 + z_4 + z_6 + 2)! \, (m_1 + k_0 + \alpha_2 - \alpha_1 + z_4 + z_6 + 1)!}{(2m_1 + 2k_0 + 2\alpha_2 - \alpha_1 + z_4 + z_6 + 3)!}, \tag{B19}$$

$$\begin{aligned} m_1 &= z_3 + z_5 + \alpha_1 + \alpha_3, \\ d &= (1)(3) + (2)(5) + (3)(5) \\ &= (\overline{\tau_1 \tau_3 \tau_5 - \tau_2 \tau_4 \tau_6})(\tau_1' \tau_3' \tau_5' - \tau_2' \tau_4' \tau_6'). \end{aligned} \tag{B20}$$

Expression (B18) must now be expanded out. The terms may be collected in the form:

$$(0)^{k_0} (1)^{\rho_1} \cdots (6)^{\rho_6} (\overline{\tau_1 \tau_3 \tau_5})^{x_1} (\overline{\tau_2 \tau_4 \tau_6})^{x_2} (\tau_1' \tau_3' \tau_5')^{y_1} (\tau_2' \tau_4' \tau_6')^{y_2},$$

where $\qquad x_1 + x_2 = N = y_1 + y_2$

with the appropriate coefficient giving the result $[H(k_i), H(k_j')]$.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 1 JANUARY 1967

Recoupling Coefficients for the Group $SU(3)$*

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(Received 21 February 1966)

The Hilbert space method, employed in the previous article to obtain the coupling coefficients of $SU(3)$, is used here to obtain the recoupling, or $6(\lambda\mu)$, coefficients of $SU(3)$. The coefficients are formulated in terms of a generating function involving an integral, and an explicit expression is integrated out for the general nondegenerate case. The symmetries of the $6(\lambda\mu)$ coefficients are discussed.

1. INTRODUCTION

The $6(\lambda\mu)$ coefficient of $SU(3)$, which relates the alternate ways three representations $[\lambda_i, \mu_i]$, $i = 1, 2, 3$, may be coupled, can be written in the form[1]

$$\begin{bmatrix} \lambda\mu; k, k' & \lambda_2\mu_2 & \lambda_{12}\mu_{12}; k_{12} \\ \lambda_1\mu_1 & \lambda_3\mu_3 & \lambda_{13}\mu_{13}; k_{13} \end{bmatrix} = \sum_{\alpha_i} \begin{Bmatrix} \lambda_1\mu_1 & \lambda_2\mu_2 & \lambda_{12}\mu_{12} \\ \alpha_1 & \alpha_2 & \alpha_{12} \end{Bmatrix}_{k_{12}} \begin{Bmatrix} \lambda_{12}\mu_{12} & \lambda_3\mu_3 & \lambda\mu \\ \alpha_{12} & \alpha_3 & \alpha \end{Bmatrix}_{k} \times \begin{Bmatrix} \mu_{13}\lambda_{13} & \mu_2\lambda_2 & \mu\lambda \\ -\alpha_{13} & -\alpha_2 & -\alpha \end{Bmatrix}_{k'} \begin{Bmatrix} \mu_1\lambda_1 & \mu_3\lambda_3 & \mu_{13}\lambda_{13} \\ -\alpha_1 & -\alpha_3 & -\alpha_{13} \end{Bmatrix}_{k_{13}}, \tag{1.1}$$

where use has been made of the orthogonal properties[2] and the symmetry properties of the $3(\lambda\mu)$ coefficients derived in the previous paper.[3]

* Work supported in part by the National Science Foundation (Grant No. GP-1536). This paper is based on a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Michigan, Ann Arbor, Michigan.

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[1] A form similar to this has been derived by J. J. de Swart, Nuovo Cimento **31**, 420 (1964). Equation (1.1) is the recoupling coefficient multiplied by the factor $(-1)^{k'+k_{13}}(N_{12}N_{13})^{-1/2}$, where N_{12} and N_{13} are the dimensions of the spaces

$$\mathfrak{H}_{\lambda_{12}\mu_{12}}, \mathfrak{H}_{\lambda_{13}\mu_{13}}$$

(see Ref. 3 below).

[2] J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963).

[3] M. Resnikoff, preceding paper, J. Math. Phys. **8**, 63 (1967). This article is hereafter referred to as (I).