BRIEF COMMUNICATIONS

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One-dimensional unsteady fluid motion between two infinite walls

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We considered an incompressible fluid motion driven by space-dependent body force. For a one-dimensional case, the problem was solved analytically, with the arbitrary choice of body force coordinate dependence. It was shown that unsteady fluid flow can be represented as a series of separate modes, each with its own characteristic response time. [DOI: 10.1063/1.1481743]

The problem of fluid motion induced by a variable body force has many applications. There are several ways to apply this force to the fluid. One is the nonuniform distribution of particles settling in the container. These particles interact with nearby fluid, initiating its motion. On the other hand, moving fluid exerts a force on the particles and influences their motion.

The literature on the dynamics of suspensions is very extensive and due to the limited scope of this study, it is impossible to cover the topic completely. We reviewed a sampling of studies closely related to the subject of this paper.

There are many studies of sedimentation of spheres in a container, which assume that particles are distributed uniformly in the area no closer than the radius of a particle to the container wall. Close to the wall (closer than a), the concentration of particles is zero, because no particle can be closer to the wall, than its own radius.

Bruneau *et al.*[1] stated this problem in a simple way: There are two infinitely high walls with particles distributed between them (no closer than a). They discussed only a steady-state problem, with no-slip boundary conditions, and introduced a zero net-flux condition (no net fluid flux through the cross section of the container) $\int_{\text{cross section}} w\,dS = 0$.

One reason to introduce this condition is that the purpose of the analytical solution is to investigate fluid motion in a finite real container, where the zero net-flux condition is obviously satisfied. Introducing this condition makes it possible to solve a system of hydrodynamic equations. The authors managed to solve the problem exactly and found that the intrinsic convection is a combination of Poiseuille flow and a slip velocity near walls. They obtained the expressions for the fluid velocity on the edge of the wall layer and in the center of the container. These velocities are found to be of the order of $V_0 s$, where V_0 is the Stokes velocity of settling particles, and s is a volume fraction of solid particles.

In another work made by Bruneau *et al.*,[2] they discussed the same steady-state problem in the geometry of an infinitely high rectangular container. As before, they looked for solution as a superposition of constant stress flow and Poiseuille flow. An expression was obtained for fluid velocity inside the container. It was shown that if the length of one side of the container tends towards infinity, then the solution approaches the one for one-dimensional case. The dependence of an intrinsic convection on a suspension concentration was estimated. An accurate description of the intrinsic convection, at moderate and high concentrations, would require an analysis of multi-particle hydrodynamic interactions in the presence of walls.

Beenakker and Mazur[3] studied a possible dependence of sedimentation on container shape. The phenomenon of intrinsic convection was predicted: an intrinsic, microscopic

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density inhomogeneity at the container wall causes a macroscopic vortex motion of the mean volume velocity.

The above authors agreed on the following: In order for fluid bulk motion in the container to exist, particles should not be distributed uniformly throughout the container. The difference of particle concentration in different parts of the container results in pressure gradient in the fluid, causing fluid motion.

In this work, we focus on the unsteady large-scale fluid dynamics, and discuss arbitrary spatial distribution of the body force throughout the container. As an example, we considered a problem of unsteady intrinsic convection between the two walls.

We consider a fluid flow between two infinitely high walls separated by a distance L_0. The fluid experiences body force $\mathbf{f}$, which depends on coordinate x and acts in the z direction. We do not specify for now the origin of this body force, which acts on the fluid and arouses a motion. It is obvious that fluid motion will occur only in the z direction and governing equations are one-dimensional. We assume the fluid to be Newtonian and use nondimensional conservation of momentum equation

$$\frac{\partial w(x,t)}{\partial t} = -\pi(t) + \frac{\partial^2 w(x,t)}{\partial x^2} + G(x), \tag{1}$$

with the following initial and boundary conditions

$$w(x,\,0)=0, \quad w(0,\,t)=w(1,\,t)=0,$$
$$\int_0^1 w(x,\,t)\,dx = 0. \tag{2}$$

The governing equation (1) is nondimensionalized with respect to the following quantities

$$w_0 \sim f L_0^2/\mu, \quad \tau \sim \rho_f L_0^2/\mu,$$
$$x \sim L_0, \quad p = f\pi(t)z, \tag{3}$$

where w is z component of fluid velocity, ρ_f is the fluid density, p is the pressure, $\pi(t)$ is an unknown nondimensional function of time. Also, f is the scale of the body force, $G(x)$ is the nondimensional function of order of unity.

In the steady-state case $\pi(t)$ approaches some limit π_{st} and governing equation reduces to

$$\frac{\partial^2 w_{\mathrm{st}}}{\partial x^2} = \pi_{\mathrm{st}} - G(x), \tag{4}$$

$$w_{\mathrm{st}}(0)=0, \quad w_{\mathrm{st}}(1)=0,$$
$$\int_0^1 w_{\mathrm{st}}(x)\,dx = 0. \tag{5}$$

After some algebraic manipulations we obtain the solution as Fourier series

$$w_{\mathrm{st}}(x) = -\sum_{k=1}^{\infty} \frac{2\pi_{\mathrm{st}}(1-(-1)^k)\dfrac{1}{\pi k} - G_k}{(\pi k)^2}\sin(\pi k x), \tag{6}$$

TABLE I. First ten eigenvalues found from Eq. (11).

$\lambda_1 = 6.2832$	$\lambda_6 = 21.8083$
$\lambda_2 = 8.9868$	$\lambda_7 = 25.1327$
$\lambda_3 = 12.5664$	$\lambda_8 = 28.1324$
$\lambda_4 = 15.4505$	$\lambda_9 = 31.4159$
$\lambda_5 = 18.8496$	$\lambda_{10} = 34.4415$

$$\pi_{\mathrm{st}} = \sum_{k=1}^{\infty} \frac{G_k(1-(-1)^k)}{(\pi k)^3} \Bigg/ \sum_{k=1}^{\infty} \frac{2(1-(-1)^k)^2}{(\pi k)^4}$$

and

$$G_k = 2\int_0^1 G(x)\sin(\pi k x)\,dx, \quad k=1,\ldots,\infty. \tag{7}$$

The reason why we prefer series solution, is because it would be also applicable in two-dimensional (2D) case. To solve unsteady problem we introduce new variables, $\pi(t) = \pi_{\mathrm{st}} + \pi^*(t)$, $w = w_{\mathrm{st}} + w^*$, obtaining

$$\frac{\partial w^*}{\partial t} = -\pi^*(t) + \frac{\partial^2 w^*}{\partial x^2}, \tag{8}$$

$$w^*(x,0) = -w_{\mathrm{st}}(x), \quad w^*(0,\,t)=0,$$
$$w^*(1,\,t)=0, \quad \int_0^1 w^*(x,t)\,dx = 0, \tag{9}$$

where $w_{\mathrm{st}}(x)$ is a steady-state solution found in (6). We seek solution of (8) in the form

$$w^* = \sum_{n=1}^{\infty} \exp(-\lambda_n^2 t)(A_n\cos(\lambda_n x) + B_n\sin(\lambda_n x)) + \int_t^{\infty} \pi^*(t')\,dt', \tag{10}$$

where eigenvalues λ_n are still unknown. After applying conditions (9) and extensive mathematical manipulations we come to an equation for λ_n

$$\lambda_n \sin(\lambda_n) = 2(1-\cos(\lambda_n)). \tag{11}$$

This equation gives us two sets of solutions

$$\sin(\lambda/2)=0 \quad (\text{first set})$$

and

$$\tan(\lambda/2) = \lambda/2 \quad (\text{second set}). \tag{12}$$

The first ten solutions λ_n of Eq. (11) are shown in a Table I. Odd solutions of Eq. (11) are from the first set of solutions, even ones are from the second set of solutions. Finally, we obtain eigenfunctions $f_n(x)$

$$f_n(x) = \begin{cases} \sin(\lambda_n x), & \text{if } n \text{ is odd} \\ \dfrac{2}{\lambda_n}(\cos(\lambda_n x)-1) + \sin(\lambda_n x), & \text{if } n \text{ is even.} \end{cases} \tag{13}$$

Functions $f_n(x)$ are found to be mutually orthogonal

$$\int_0^1 f_n(x) f_m(x)\,dx = \begin{cases} 0, & \text{if } n \neq m \\ 1/2, & \text{if } n = m. \end{cases} \tag{14}$$

TABLE II. Response time parameters.

$\tau_{res,1}=0.025\,330$	$\tau_{res,6}=0.002\,102$
$\tau_{res,2}=0.012\,382$	$\tau_{res,7}=0.001\,583$
$\tau_{res,3}=0.006\,332$	$\tau_{res,8}=0.001\,263$
$\tau_{res,4}=0.004\,189$	$\tau_{res,9}=0.001\,013$
$\tau_{res,5}=0.002\,814$	$\tau_{res,10}=0.000\,843$

Solution to the unsteady problem (8) and (9) can be expressed as

$$w^*=\sum_{n=1}^{\infty} C_n \exp(-\lambda_n^2 t) f_n(x),$$
$$C_n=-2\int_0^1 w_{st}(x) f_n(x)\,dx. \tag{15}$$

We can also introduce nondimensional response time parameters $\tau_{res,n}$ as $\tau_{res,n}=1/\lambda_n^2$. This represents the time scales for the corresponding modes $f_n(x)$ exponential decay. First ten response times are shown in Table II. Finally, the analytical solution of unsteady problem (1) and (2) is given as

$$w(x,t)=w_{st}(x)+\sum_{n=1}^{\infty} C_n \exp(-\lambda_n^2 t) f_n(x). \tag{16}$$

As an example, we solve a problem of unsteady intrinsic convection between two infinite walls. In this case, body force distribution function is constant everywhere, except it is zero in the vicinity of walls (closer than ξ). Here, ξ is a nondimensional parameter, a ratio of a sphere radius a to the distance between walls L_0. We arbitrarily set $\xi=0.1$. Using (15), we calculate the weights C_n of each mode. As soon as this distribution function is symmetric with respect to the middle of the container, then $C_n=0$, $n=1,3,5,7...$, i.e., we have only even modes in the expansion (16). In our example, the slowest decaying mode is the second one $(n=2)$. Therefore, $\tau_{res,2}=1/\lambda_2^2\approx 0.012\,38$ represents the characteristic time, when a steady-state flow develops. In Fig. 1, we plotted flow profiles at several consecutive time moments.

The solution can easily be extended into a nonzero initial condition case. In Eq. (9) we would write $w^*(x,0)=w_{initial}(x)-w_{st}(x)$. In other words we could substitute $w_{st}(x)-w_{initial}(x)$ in place of $w_{st}(x)$ in Eq. (15). Eigenvalues and characteristic times, however, would remain unchanged.

We investigated the fluid motion driven by an x coordinate dependent body force in a one-dimensional case. The solution of the problem (the flow field) was obtained for unsteady and steady-state cases. As for time dependence of fluid flow, the fluid velocity was found to be the superposition of specific modes, each exponentially decaying with its own characteristic time. These times were found by solving a transcendental equation. As an example, the unsteady problem of intrinsic convection is solved.

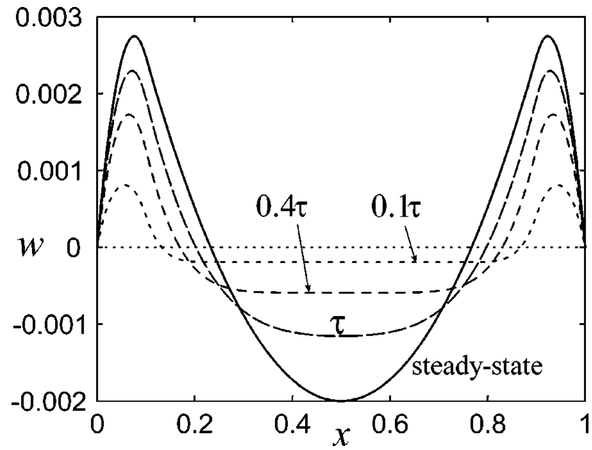

FIG. 1. Development of fluid flow between infinite walls at several consecutive time moments.

Applicability of this theory represents a separate issue. It should be remembered that the unsteady motion studied in this paper can be hidden under other effects that occur simultaneously. For example, in the problem of intrinsic convection, in order to achieve initial conditions, we need to mix fluid and particles and wait until the fluid settles. This waiting time can easily be longer than the characteristic time $\tau_{res,2}$. Also, in a real container, there are three-dimensional (3D) fluctuations that can occur. All these effects can make it difficult to observe this unsteady motion.

The force exerted on the fluid does not necessarily have to be gravity. Gravity can be replaced by any force that can be switched on after mixing. For example, in the case of magnetic particles, magnetic forces can be used. Applications in biology are also possible.

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