Transport and modulation of relativistic electron beams by periodic ion channels

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A theoretical investigation of an intense relativistic electron beam propagating in a periodic ion channel is presented. Analytic expressions for the electric field are found for the case of a cosine modulation of the channel ion density. Two different types of channels are considered: (i) periodic beam-induced ionization channels, and (ii) periodic channels created by an external source. Analytical conditions are derived for the matched radius of the beam and for approximate envelope motion using the "smooth" approximation. Numerical solutions to the envelope equation show that by changing the period or the space-charge neutralization fraction of the channel the beam can be made to focus and diverge or to undergo stable, modulated transport. It is shown that the results are qualitatively similar to previous work concerning periodic magnetic and electrostatic focusing.

I. INTRODUCTION

Intense relativistic electron beams have important applications to free-electron lasers[1] and fusion plasma heating.[2] An important area of research concerns the stable transport of these high-current beams in gas or plasma backgrounds. Recently, significant progress has been made using laser-ionized channels to guide electron beams in the ion focus regime.[3–5] Alternatively, while periodic magnetic focusing has been used for many years,[6] the use of periodic foil arrays or plasmas has been suggested only more recently.[5,7–9]

In this article we solve the Poisson equation and the envelope equation for an intense relativistic electron beam propagating in a periodic ion channel.[5] Analytical conditions are derived for the matched radius of the beam and for approximate envelope motion using the "smooth" approximation. We also develop stability criteria that can be compared to previous analyses of periodic magnetic or electrostatic focusing. Two separate types of periodic ion channels are considered. The first type of periodic ion channel could be generated by beam-induced ionization of axially periodic gas puffs.[7] A second type of periodic ion channel could be generated by a large number of laser beamlets incident perpendicular to the electron beam or with a surface flashover plasma source coupled to the beam pulsed power source.[7] As demonstrated in recent experiments,[3–5] a low-pressure background gas (such as diethylaniline or benzene) is readily ionized by UV excimer laser radiation.

II. BEAM-INDUCED PERIODIC SPACE-CHARGE NEUTRALIZED CHANNELS

The configuration of electron beam-induced ionization of periodic gas puff channels is depicted in Fig. 1. It is assumed that the intense relativistic electron beam instantaneously ejects the channel plasma electrons radially. The channel ion density is assumed to vary sinusoidally in the axial direction with period L. This is incorporated into the envelope equation by assuming an axially periodic space-charge neutralization factor f_e that varies as

$$f_e(z) = \alpha/2[1 + \cos(2\pi z/L)], \tag{1}$$

where α is an adjustable parameter.

The starting point of the analysis is the paraxial envelope equation for a zero emittance beam that we write in standard form as[10]

$$\ddot{r}_b + \frac{\dot{r}_b\dot{\gamma}}{\gamma} + \left(\frac{\Omega}{2\gamma}\right)^2 r_b = \frac{-e}{\gamma m}(E_r - v_z B_\theta) + \frac{(P_\theta/m)^2}{\gamma^2 r_b^3}, \tag{2}$$

where r_b is the beam radius, $\gamma = (1-\beta^2)^{-1/2}$ is the relativistic mass factor with $\beta = v_z/c$, P_θ/m is the canonical angular momentum, $\Omega = |eB|/m$ is the electron cyclotron frequency, and the relativistic particle energy changes according to $d/dt(\gamma mc^2) = -ev_zE_z$. In general, introducing a uniform charge neutralization fraction leads to electric and magnetic fields of the form $E_r = -ner(1-f_e)/(2\epsilon_0)$, $E_z = 0$, and $B_\theta = -nerv_z/(2\epsilon_0 c^2)$, where n is the beam density. For our specific choice of $f_e(z)$ the above expres-

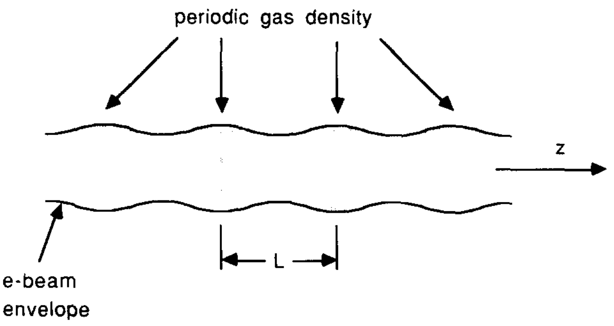

FIG. 1. Schematic of an intense relativistic electron beam propagating in a periodic ion channel produced by beam-induced ionization.

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sions do not necessarily hold and we must solve the electrostatic problem to determine their exact form.

The electric field must satisfy Poisson's equation $\nabla\cdot\mathbf{E}(r,z) = \rho(z)/\epsilon_0$, where the charge density $\rho(z) = -ne[1 - f_e(z)]$ for $r \leqslant r_b$ ($\rho = 0$ for $r > r_b$) follows from $f_e(z)$ defined by Eq. (1). Our solution (see the Appendix for details) yields for an electron at the edge of the beam

$$E_r = \frac{ner_b}{2\epsilon_0}\left[-1 + \frac{\alpha}{2} + \alpha K_1\left(\frac{2\pi r_b}{L}\right)I_1\left(\frac{2\pi r_b}{L}\right)\cos\left(\frac{2\pi z}{L}\right)\right], \tag{3a}$$

$$E_z = \frac{L}{2\pi}\frac{ne\alpha}{2\epsilon_0}\left[1 - \frac{2\pi r_b}{L}K_1\left(\frac{2\pi r_b}{L}\right)I_0\left(\frac{2\pi r_b}{L}\right)\right]\sin\left(\frac{2\pi z}{L}\right). \tag{3b}$$

Since substitution of the field expressions given in Eqs. (3a) and (3b) result in an intractable form of the envelope equation we solve the problem for an appropriate simplified case. Defining the nondimensional variable $\chi = 2\pi r_b/L$ and the electric field normalization parameter $E_0 = neL/(4\pi\epsilon_0)$ we can rewrite Eqs. (3a) and (3b) in nondimensional form as

$$E_r/E_0 = [-1 + \alpha/2 + \alpha K_1(\chi)I_1(\chi)\cos(2\pi z/L)]\chi, \tag{4a}$$

$$E_z/E_0 = [1 - \chi K_1(\chi)I_0(\chi)]\alpha\sin(2\pi z/L). \tag{4b}$$

For the beam and channel parameters of interest ($r_b = 0.5$–1 cm and $L = 1$–100 cm) χ will take on values in the range $0 < \chi \leqslant 10$. Clearly for $K_1(\chi)I_1(\chi) \cong 0.5$, $\chi K_1(\chi)I_0(\chi) \cong 1$ and Eqs. (4a) and (4b) reduce to

$$\frac{E_r}{E_0} \cong \left[-1 + \frac{\alpha}{2} + \frac{\alpha}{2}\cos\left(\frac{2\pi z}{L}\right)\right]\chi, \tag{5a}$$

$$E_z/E_0 \cong 0. \tag{5b}$$

If we choose as an acceptable range $K_1(\chi)I_1(\chi) = 0.4$–0.5, we find that this condition is satisfied for χ values in the range $0 < \chi \leqslant 0.65$. For example, an upper-limit condition would be a 1 cm radius beam in a channel of period 10 cm giving $\chi = 0.628$.

Using Eqs. (5a) and (5b) as approximations to the fields in the periodic plasma channel, the z motion of the electron is given by $z = \beta ct$ with $\beta = v_z/c =$ const. Under these approximations and since we are considering propagation with no external (guide) magnetic field, Eq. (2) reduces to

$$\frac{d^2 r_b}{dz^2} - \frac{2I}{I_0\beta^3\gamma r_b}[1 - f_e(z) - \beta^2] = 0, \tag{6}$$

where $I = nev_z\pi r_b^2$ is the total beam current (constant) inside the circular envelope of radius r_b, $I_0 = 4\pi\epsilon_0 mc^3/e$ is the limiting current, and $f_e(z)$ is given by Eq. (1). Introducing the nondimensional variables $r' = r_b/r_{b0}$, where r_{b0} is the initial radius at $z = 0$, and $z' = z/L$, we can rewrite Eq. (6) in the form

$$\frac{d^2 r'}{dz'^2} - (a - b\cos 2\pi z')\frac{1}{r'} = 0, \tag{7}$$

where

$$a = \left(\frac{L}{r_{b0}}\right)^2 \frac{2I}{I_0\beta^3\gamma}\left(1 - \beta^2 - \frac{\alpha}{2}\right), \tag{8}$$

$$b = \left(\frac{L}{r_{b0}}\right)^2 \frac{2I}{I_0\beta^3\gamma}\frac{\alpha}{2}. \tag{9}$$

Equation (7) is solved numerically using a standard Runge–Kutta method. The equation is solved subject to the initial conditions $r'(0) = 1$ and $(d/dz')r'(0) = 0$.

From Eq. (6) with f_e uniform the equilibrium condition is defined by $f_e = 1 - \beta^2$. For periodic $f_e(z)$ one might expect that the focusing should be equivalent if we set the average value of f_e over one period of the channel to $1 - \beta^2$, i.e., $\alpha/2 = 1 - \beta^2$. Numerical results for the solution of Eq. (7) under these conditions are presented in Fig. 2 with the beam-channel system characterized by the parameter b. Figure 2 represents the solution for a beam current of typically 700 A. It is apparent from this figure that the stability conditions for a periodic ion channel are more complicated than for the uniform case.

From the above result it is seen that for reasonable beam and channel parameters, relatively small currents are required to keep the envelope uniform and stable. To allow focusing and transport of higher currents we propose $\alpha/2 > 1 - \beta^2$ in Eq. (7). Typically for periodic magnetic focusing systems when the beam is "matched," the maxima and minima will have the same magnitude along the channel, i.e., $r'(z) = r'(z + L)$, etc. For an unmatched beam, there will be a slow variation of the envelope function with a wavelength that is generally large compared to the period L. Superimposed on this slow variation is a "ripple" of short wavelength that exhibits the periodic structure of the channel.[11]

In the smooth approximation we let δ represent a fast amplitude function with the period L of the channel and write

$$r'(z') = \bar{r}'(z')[1 + \delta(z')], \tag{10}$$

where $\bar{r}'(z')$ is the mean value of $r'(z')$ and it is assumed that $\bar{r}'$ and its derivatives vary slowly enough that they may be considered as constant within one period L. The mean value of $\delta(z')$ and its derivatives over the period L are zero and $|\delta(z')| \ll 1$.

Substituting Eq. (10) into Eq. (7) and expanding yields the differential equation

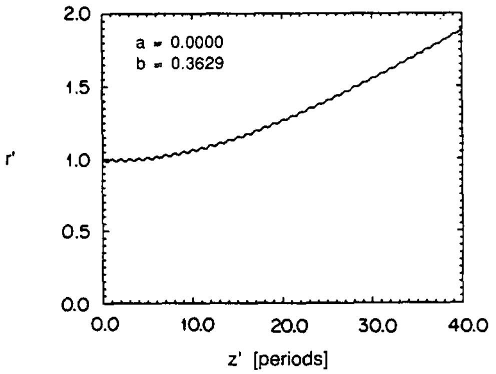

FIG. 2. Numerical solution for the beam envelope of Eq. (7) with $\alpha/2 = 1 - \beta^2$ for a beam defined by Eq. (9) with typically $\gamma = 3$, $r_{b0} = 1$ cm, $L = 10$ cm, and $I = 700$ A.

$$(1+\delta)\frac{d^2\bar{r}'}{dz'^2}+2\frac{d\bar{r}'}{dz'}\frac{d\delta}{dz'}+\bar{r}'\frac{d^2\delta}{dz'^2}$$
$$-(a-b\cos 2\pi z')(1-\delta)1/\bar{r}'=0. \quad (11)$$

This equation involves both oscillatory and smooth terms. For the oscillating terms, neglecting terms of order δ and $d\delta/dz'$ in comparison to $d^2\delta/dz'^2$, we may write

$$\frac{d^2\delta}{dz'^2}=-\frac{(b\cos 2\pi z')}{\bar{r}'^2}. \quad (12)$$

Integrating this expression twice and applying the boundary conditions that $\delta=0$ and $d\delta/dz'=0$ at $z'=0$ we get the following expression for $\delta(z')$.

$$\delta(z')=-(b/4\pi^2\bar{r}'^2)(1-\cos 2\pi z'). \quad (13)$$

Averaging Eq. (11) over one period, the remaining terms are

$$\frac{d^2\bar{r}'}{dz'^2}-\frac{a}{\bar{r}'}-\frac{b}{\bar{r}'}\int_{z'}^{z'+1}\delta\cos 2\pi z''\,dz''=0. \quad (14)$$

Substituting Eq. (13) into Eq. (14) and performing the integration yields the equation for $\bar{r}'$ as

$$\frac{d^2\bar{r}'}{dz'^2}-\frac{a}{\bar{r}'}-\frac{b^2}{8\pi^2\bar{r}'^3}=0. \quad (15)$$

For the case when $a=0$, Eq. (15) may be integrated directly. By applying the boundary conditions $\bar{r}'=1$ and $d\bar{r}'/dz'=0$ at $z'=0$, and using Eq. (10), the radius of the beam envelope is given by

$$r'(z')=\left(\frac{b^2z'^2}{8\pi^2}+1\right)^{1/2}$$
$$\times\left[1-\frac{b}{4\pi^2}\left(\frac{b^2}{8\pi^2}z'^2+1\right)^{-1}(1-\cos 2\pi z')\right]. \quad (16)$$

Comparison with Fig. 2 shows Eq. (16) to give an almost exact fit.

The matched radius condition may be determined from Eq. (15) by letting $\bar{r}'=1$ and $d^2\bar{r}'/dz'^2=0$ yielding

$$a=-b^2/8\pi^2. \quad (17)$$

The matched radius of the beam is then given by Eqs. (13) and (10) and may be written

$$r'(z')=1-(b/4\pi^2)(1-\cos 2\pi z'), \quad (18)$$

where Eqs. (17) and (18) are subject to the restriction $|\delta(z')|\ll 1$, which may be written

$$b/8\pi^2\ll 1. \quad (19)$$

For combinations of a and b satisfying Eq. (17) we should thus expect to see stable electron beam envelopes. This relationship should permit determination of the channel parameters required to obtain a stable envelope for a specified beam as long as the smooth approximation remains valid.

Figure 3 shows the beam envelope for a current of 1 kA with $\gamma=3$, $L=10$ cm, and α determined from Eq. (17) to be 0.2237. Although the long-period modulation is not totally eliminated the envelope appears stable and periodic over 40 periods. In Fig. 4(a) we present results for $I=10$ kA, $\gamma=3$, $L=10$ cm, and α determined from Eq. (17) to be 0.2391. Here the envelope appears stable and periodic, however, the long-period modulation has increased in amplitude. This amplitude can be minimized by increasing α

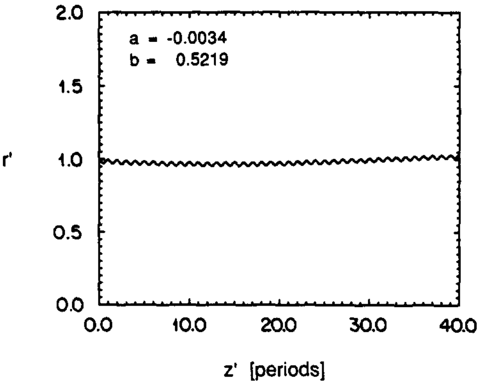

FIG. 3. Numerical solution for the beam envelope of Eq. (7) with typically $\gamma=3$, $I=1$ kA, $r_{b0}=1$ cm, $L=10$ cm, and α determined from Eq. (17) to be 0.2237.

slightly from the value predicted by Eq. (17). Figure 4(b) shows the same beam parameters and channel period with α increased to 0.252. Clearly for small a the result of Eq. (17) is almost free of the long-period modulation, while for larger values of a, the long-period modulation is minimized by further adjustment of α. This is probably caused by violation of the smooth approximation since, as shown in Fig. 4, the ripples have become quite large and Eq. (19) is no longer strictly satisfied.

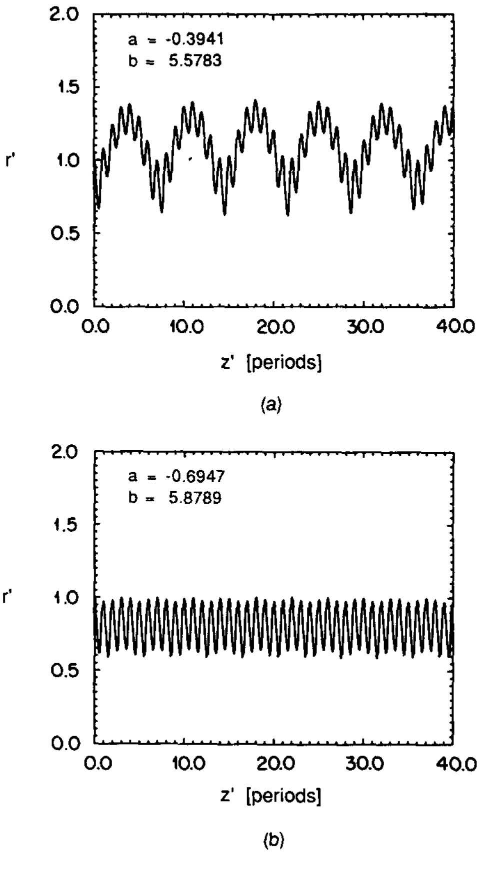

FIG. 4. Numerical solution for the beam envelope of Eq. (7) with typically $\gamma=3$, $I=10$ kA, $r_{b0}=1$ cm, $L=10$ cm, and (a) α determined from Eq. (17) to be 0.2391 and (b) $\alpha=0.252$.

We can derive an approximate expression for the betatron period of the long-wavelength oscillations shown in Figs. 3 and 4 from Eq. (15). For small deviations from the matching condition, Eq. (17), we may write $\bar{r}' = 1 + \epsilon$, where $|\epsilon| \ll 1$. Substituting this into Eq. (15) and expanding yields the differential equation

$$\frac{d^2\epsilon}{dz'^2} + \left(a + \frac{3b^2}{8\pi^2}\right)\epsilon = a + \frac{b^2}{8\pi^2}. \tag{20}$$

Applying the boundary conditions $\epsilon = 0$ and $d\epsilon/dz' = 0$ at $z' = 0$, we may write the solution for $\bar{r}'$ as

$$\bar{r}'(z') = 1 - [(a + b^2/8\pi^2)/(a + 3b^2/8\pi^2)] \times [1 - \cos(a + 3b^2/8\pi^2)^{1/2}z']. \tag{21}$$

When $a = -b^2/8\pi^2$ in this approximation the mean value of the envelope radius is a straight line as expected from the matched radius condition. For small deviations from the matched radius condition we get oscillations in the beam of period

$$l = 2\pi/(a + 3b^2/8\pi^2)^{1/2}. \tag{22}$$

This relationship gives the period of the betatron oscillation shown in Figs. 3 and 4(a) almost exactly. The agreement is not as good for Fig. 4(b) which is not unexpected; as mentioned before, Eq. (19) is no longer strictly satisfied for the parameters of Fig. 4(b) and the analytical results from the smooth approximation must be applied cautiously.

It has also been determined from the numerical results that relatively stable, periodic envelopes are obtained from Eq. (17) when $a \leqslant -0.4$. Clearly the stability of the original nonlinear equation is complicated and not governed entirely by the results of the linearized analysis.

III. PERIODIC EXTERNALLY IONIZED CHANNELS

The configuration of this problem is depicted in Fig. 5. Again it is assumed that the intense relativistic electron beam instantaneously ejects the channel plasma electrons radially and the channel ion density varies sinusoidally in the axial direction with period L. Such a channel could be produced by a large number of laser beamlets incident perpendicular to the electron beam or by a surface flashover plasma source powered by coupling to the beam pulsed power source.[7]

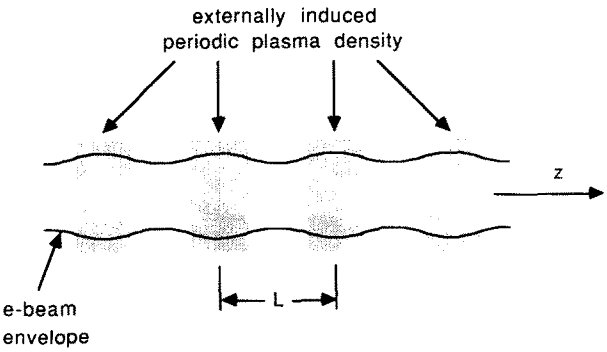

FIG. 5. Schematic of an intense relativistic electron beam propagating in a periodic channel produced by an external source.

In this case the focusing on the beam caused by the ions increases as the beam radius increases since more ions are enclosed by the beam (the ions are assumed to be stationary over the beam pulse length). We treat the ions as having a density that is uniform in radius, which leads us to choose the following radially dependent form for the space-charge neutralization fraction

$$f_e(z) = \frac{\kappa}{2}\frac{r_b^2}{r_{b0}^2}\left[1 + \cos\left(\frac{2\pi z}{L}\right)\right], \tag{23}$$

where κ is an adjustable parameter. Determination of κ for the matched radius condition would then allow a determination of the ion density required in the channel. The introduction of the r_b^2/r_{b0}^2 term allows for the fact that the number of ions (and hence degree of space-charge neutralization) increases with the volume of the beam.

Since we wish to find an equilibrium at the injected beam radius, for the matched beam we require $r_b \approx r_{b0}$, so the function $f_e(r,z)$ will be only weakly dependent upon r. In this case the previous solutions for E_r and E_z [Eqs. (3)–(5)] are appropriate and we need only substitute Eq. (23) into Eq. (6) to obtain the new envelope equation. In our nondimensional variables it is written as

$$\frac{d^2r'}{dz'^2} - \frac{a'}{r'} + b'[1 + \cos(2\pi z')]r' = 0, \tag{24}$$

where

$$a' = \left(\frac{L}{r_{b0}}\right)^2 \frac{2I}{I_0\beta^3\gamma^3}, \tag{25}$$

$$b' = \left(\frac{L}{r_{b0}}\right)^2 \frac{2I}{I_0\beta^3\gamma}\frac{\kappa}{2}. \tag{26}$$

Equation (24) has been well studied since it is the exact form obtained for a space-charge dominated beam in a periodic magnetic focusing system.[12,13]

In this case we may write down the results directly following a similar linearization and smooth approximation technique as used before. From Clogston and Heffner[12] the condition for the matched radius of the beam may be written in our notation as

$$b' = a'[1 + (3/8\pi^2)a'] \tag{27}$$

provided that a' is a small quantity and nearly equal to b'. The final equation for the orbit of the outermost electron can be written

$$r' = 1 - (a'/4\pi^2)(1 - \cos 2\pi z') \tag{28}$$

provided Eq. (27) is satisfied and $3a'/8\pi^2 \ll 1$.

For small departures from Eq. (27) the equation for the beam envelope may be approximated

$$r' = 1 + [(a' - b')/(a' + b')] \times \{1 - \cos[(a' + b')^{1/2}z']\}. \tag{29}$$

When $a' = b'$ in this rough approximation (corresponding to $\kappa/2 = 1 - \beta^2$ as we intuitively expected earlier) the envelope is a straight line at $r_b = r_{b0}$. When $a' \neq b'$ the solutions are well known with scallops developing in the beam. These scallops are the long-wavelength (compared to the period L of the channel) oscillations seen previously. From Eq. (29) the betatron period l of these oscillations is given by

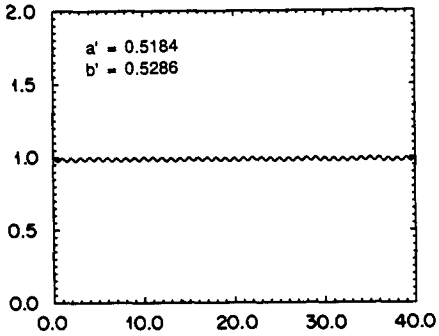

FIG. 6. Numerical solution for the beam envelope of Eq. (24) with typically $\gamma = 3, I = 1$ kA, $r_{b0} = 1$ cm, $L = 10$ cm, and κ determined from Eq. (27) to be 0.2266.

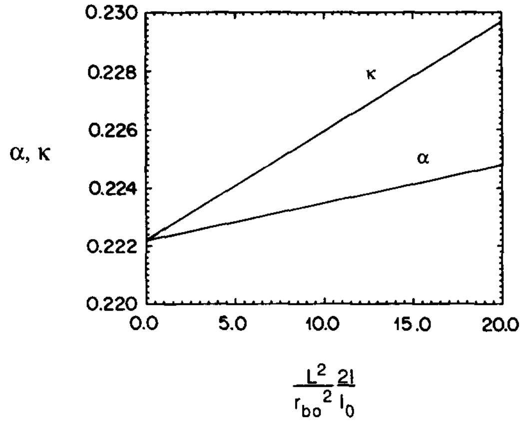

FIG. 8. Maximum magnitudes of the space-charge neutralization fraction, α and κ vs $(L/r_{b0})^2(2I/I_0)$ for a $\gamma = 3$ beam for the matched radius conditions given by Eqs. (17) and (27).

$$l = 2\pi/(a' + b')^{1/2}. \tag{30}$$

Equation (24) is also solved numerically using a standard Runge–Kutta method subject to the initial conditions $r'(0) = 1$ and $(d/dz')r'(0) = 0$.

Figure 6 shows the beam envelope for a current of typically 1 kA with $\gamma = 3$, $L = 10$ cm, and κ determined from Eq. (27) to be 0.2266. Clearly the matched radius condition predicts the envelope behavior quite well for a' values in this range. Figure 7 shows the beam envelope for a current of typically 10 kA with $\gamma = 3$, $L = 10$ cm, and κ determined from Eq. (27) to be 0.266. As was seen in the previous section, for these large values of a' the ripples have increased in magnitude and the long-period oscillations have returned. Notable in comparison with Fig. 4(a) is the effect on the beam envelope of the much smaller betatron period for the long-wavelength oscillations of Eq. (30) as compared to Eq. (22).

IV. DISCUSSION

From the results presented in the previous two sections it has been shown that the results from the smooth approximation analysis of Eqs. (7) and (24) are good representations of the results obtained by direct numerical solution provided in general the conditions for validity of the smooth approximation are satisfied. For the matched radius condition given by Eqs. (17) and (27), $r_b \approx r_{b0}$ and we can compare the maximum magnitudes of the periodic space-charge neutralization fractions for the two different types of channels. Figure 8 shows a plot of the magnitudes α and κ vs $(L/r_{b0})^2(2I/I_0)$ for a $\gamma = 3$ beam. As expected for the matched radius condition there is not much difference in the magnitudes of α and κ. Figure 8 also indicates that α and κ are not very sensitive functions of current.

Finally it should be noted that periodic ion channels could be particularly useful for beam control or as a possible wiggler for free-electron laser applications since we have shown that the beam envelope can be made to propagate as a modulated, stable beam or focus and diverge depending on the period of the channel or the magnitude of the space-charge neutralization fraction.

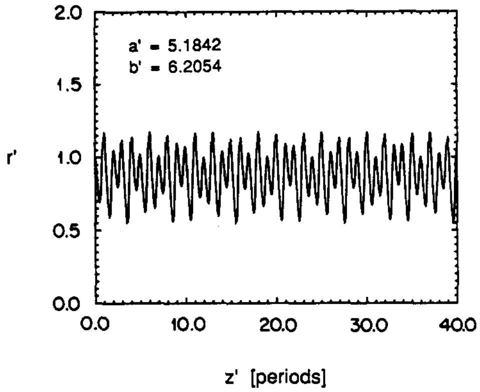

FIG. 7. Numerical solution for the beam envelope of Eq. (24) with typically $\gamma = 3, I = 10$ kA, $r_{b0} = 1$ cm, $L = 10$ cm, and κ determined from Eq. (27) to be 0.266.

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APPENDIX: ELECTROSTATIC FIELDS OF THE BEAM-CHANNEL SYSTEM

The charge density $\rho(z) = -ne[1 - f_e(z)]$ for $r \leqslant r_b$ ($\rho = 0$ for $r > r_b$) with $f_e(z)$ defined by Eq. (1) may be written as the sum of a constant term and a term that varies with z, i.e., $\rho = \rho_1 + \rho_2(z)$ with $\rho_1 = -ne$ and $\rho_2 = ne(\alpha/2)[1 + \cos(2\pi z/L)]$. Since we are considering an electrostatic problem, the solution of Poisson's equation is just a superposition of the two solutions for ρ_1 and ρ_2.

The solution for ρ_1 is well known and given by

$$\mathbf{E} = E_r = -ner/2\epsilon_0, \quad 0 < r \leqslant r_b, \tag{A1a}$$

$$\mathbf{E} = E_r = -ner_b^2/2\epsilon_0 r, \quad r > r_b. \tag{A1b}$$

For $\rho_2 = \rho(z)$ the solution is more difficult. Using $\mathbf{E} = -\nabla\Phi$ we expand $\rho(z)$ and $\Phi(r,z)$ in a Fourier series

$$\Phi(r,z) = \sum_{n=0}^{\infty} \phi_n(r) \cos\frac{2\pi n z}{L}, \tag{A2a}$$

$$\rho(r,z) = \sum_{n=0}^{\infty} \rho_n(r) \cos\frac{2\pi n z}{L}. \tag{A2b}$$

Using Eq. (1) however $\rho(z) = ne(\alpha/2)[1 + \cos(2\pi z/L)]$. From Eq. (A2b) we have $\rho_0 = \rho_1 = ne\alpha/2$ for $0 < r \leqslant r_b$ and we will assume $\rho = 0$ for $r > r_b$. All other ρ_n's are zero. We thus need only determine the potentials $\phi_0(r)$ and $\phi_1(r)$ since the rest are zero. Applying the boundary conditions $\phi_n(r)$ must be continuous at $r = r_b$ and $\partial\phi_n(r)/\partial r$ is bounded and continuous at $r = r_b$, we readily obtain for the $n = 0$ term

$$\phi_0(r) = (ne\alpha/8\epsilon_0)(r_b^2 - r^2), \quad 0 < r \leqslant r_b, \tag{A3a}$$

$$\phi_0(r) = (-ne\alpha/4\epsilon_0) r_b^2 \ln(r/r_b), \quad r > r_b. \tag{A3b}$$

For the $n = 1$ term we obtain the solution in terms of modified Bessel functions as

$$\phi_1(r) = \left(\frac{L}{2\pi}\right)^2 \frac{ne\alpha}{2\epsilon_0} \times \left[\frac{-2\pi r_b}{L} K_1\left(\frac{2\pi r_b}{L}\right) I_0\left(\frac{2\pi r}{L}\right) + 1\right], \quad 0 < r \leqslant r_b, \tag{A4a}$$

$$\phi_1(r) = \left(\frac{L}{2\pi}\right)^2 \frac{ne\alpha}{2\epsilon_0} \frac{2\pi r_b}{L} I_1\left(\frac{2\pi r_b}{L}\right) K_0\left(\frac{2\pi r}{L}\right), \quad r > r_b. \tag{A4b}$$

The complete solution is obtained by substituting the $n = 0$ and $n = 1$ terms into Eq. (A2a). Calculating the fields from $\mathbf{E} = -\nabla\Phi$ the complete solution of the electrostatic problem is the superposition of Eqs. (A1) and (A2a) and is written

$$\mathbf{E}(r,z) = \left[\frac{-ner}{2\epsilon_0} + \frac{ne\alpha r}{4\epsilon_0} + \frac{ne\alpha}{2\epsilon_0} r_b K_1\left(\frac{2\pi r_b}{L}\right) I_1\left(\frac{2\pi r}{L}\right) \cos\left(\frac{2\pi z}{L}\right)\right]\hat{r} + \left\{\frac{L}{2\pi}\frac{ne\alpha}{2\epsilon_0}\left[1 - \frac{2\pi r_b}{L} K_1\left(\frac{2\pi r_b}{L}\right) I_0\left(\frac{2\pi r}{L}\right)\right] \sin\left(\frac{2\pi z}{L}\right)\right\}\hat{z}, \quad 0 < r \leqslant r_b, \tag{A5a}$$

$$\mathbf{E}(r,z) = \left[\frac{-ner_b^2}{2\epsilon_0 r} + \frac{ne\alpha r_b^2}{4\epsilon_0 r} + \frac{ne\alpha r_b}{2\epsilon_0} I_1\left(\frac{2\pi r_b}{L}\right) K_1\left(\frac{2\pi r}{L}\right) \cos\left(\frac{2\pi z}{L}\right)\right]\hat{r} + \left[\frac{L}{2\pi}\frac{ne\alpha}{2\epsilon_0}\frac{2\pi r_b}{L} I_1\left(\frac{2\pi r_b}{L}\right) K_0\left(\frac{2\pi r}{L}\right) \sin\left(\frac{2\pi z}{L}\right)\right]\hat{z}, \quad r > r_b. \tag{A5b}$$

For the envelope equation treatment we require $\mathbf{E}(r,z)$ at $r = r_b$ so Eq. (A5a) reduces to

$$E_r = \frac{ner_b}{2\epsilon_0}\left[-1 + \frac{\alpha}{2} + \alpha K_1\left(\frac{2\pi r_b}{L}\right) I_1\left(\frac{2\pi r_b}{L}\right) \cos\left(\frac{2\pi z}{L}\right)\right], \tag{A6a}$$

$$E_z = \frac{L}{2\pi}\frac{ne\alpha}{2\epsilon_0}\left[1 - \frac{2\pi r_b}{L} K_1\left(\frac{2\pi r_b}{L}\right) I_0\left(\frac{2\pi r_b}{L}\right)\right] \sin\left(\frac{2\pi z}{L}\right). \tag{A6b}$$

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