Expansion Theorem for the Linearized Fokker–Planck Equation

JORDAN D. LEWIS*

Department of Nuclear Engineering, University of Michigan, Ann Arbor, Michigan

The linearized Fokker–Planck kinetic equation for each component of a homogeneous, nondegenerate, fully ionized plasma is separated by means of a spherical harmonic expansion into an infinite set of singular integro–differential equations. Each equation is shown to generate a continuous set of eigenfunctions, for which asymptotic high-speed forms are found. By extending the theory of singular differential equations an expansion formula is developed, which is shown to be complete with respect to functions square integrable in velocity space.

INTRODUCTION

THE Fokker–Planck kinetic equation may be used to describe the temporal behavior of a homogeneous, nondegenerate, fully ionized plasma.[1–3] The Fokker–Planck equation has been derived by several authors (see e.g., the discussion in Robinson and Bernstein[3]), starting with various levels of sophistication in kinetic theory. Whatever the starting point, the results are essentially the same.[2,3] For our purposes, it is convenient to write the Fokker–Planck equation in the form[2,3]

$$\frac{\partial F_a}{\partial t} = \sum_b \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3 v' \left(F_b' \frac{\partial F_a}{\partial \mathbf{v}} - \frac{m_a}{m_b} F_a \frac{\partial F_b'}{\partial \mathbf{v}'} \right) \cdot \mathbf{Q}_{ab}(\mathbf{v}, \mathbf{v}'). \tag{1}$$

Here $F_a(\mathbf{v}, t) \, d^3 v$ is the expected number of particles of kind a in the velocity space volume element $d^3 v$ about $\mathbf{v}$ at time t,

$$\mathbf{Q}_{ab} = \Gamma_{ab} g^{-3} (\mathbf{1} - \mathbf{gg}),$$

where Γ_{ab} is a positive constant, $\mathbf{g} = \mathbf{v} - \mathbf{v}'$, and $\mathbf{1}$ is the unit dyadic. We denote by F_j' the function $F_j(\mathbf{v}', t)$.

It is not difficult to show[2] that $\partial F_a / \partial t = 0$ if and only if F_a and F_b are the equilibrium, or Maxwell, distributions F_{a0}, F_{b0}. In the vicinity of equilibrium we may write $F_j = F_{j0}[1 + f_j(\mathbf{v}, t)]$. Neglecting terms quadratic in f_j we then obtain from (1) the linear equation

$$F_{a0} \frac{\partial f_a}{\partial t} = \sum_b \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3 v' F_{b0}' F_{a0} \left(\frac{\partial f_a}{\partial \mathbf{v}} - \frac{m_a}{m_b} \frac{\partial f_b'}{\partial \mathbf{v}'} \right) \cdot \mathbf{Q}_{ab}. \tag{2}$$

We refer to the quantity $F_{j0} f_j$ as the perturbation from equilibrium.

Equation (2) as it stands is in fact a pair of coupled equations for f_a and f_b. Due to the quite small value of the electron–ion mass ratio, the equations however are only very weakly coupled. Thus, for example, the effect of the ion perturbation on the electron perturbation is small when compared with the effect of the ions and electrons in the unperturbed equilibrium distributions.

In the following we consider the equation for the ions. The treatment of the electron equation is quite similar, and the modifications necessary for this case are indicated later (Sec. V). In the approximation $m_i \gg m_e$ it can be shown[3] that the ions act like a single component gas. In this case Eq. (2) becomes, for the ions,

$$F_{i0} \frac{\partial f_i}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3 v' F_{i0}' F_{i0} \left(\frac{\partial f_i}{\partial \mathbf{v}} - \frac{\partial f_i'}{\partial \mathbf{v}'} \right) \cdot \mathbf{Q}_{ii}. \tag{3}$$

Our purpose here is to develop an expansion theorem based on (3). We follow the standard method of assuming solutions to (3) of the form $f(\mathbf{v}, t) = g_\lambda(\mathbf{v}) \exp(-\lambda t)$. This reduces (3) to the form $L(\mathbf{v}) g_\lambda(\mathbf{v}) = -\lambda g_\lambda(\mathbf{v})$, where L is a three-dimensional integro–differential operator. In Sec. I we show that (3) requires $\text{Re } \lambda > 0$ and $\text{Im } \lambda = 0$ as we would expect physically. In Sec. II, we introduce a spherical harmonic expansion which replaces the three-dimensional equation by an infinite set of uncoupled equations, $L_{lm} g_{lm} = -\lambda_{lm} g_{lm}$, where $L_{lm}(v)$ is a singular integro–differential operator. These are cast into a self-adjoint form in Sec. III by introducing a suitable algebraic transformation on the functions $g_{lm}(v, \lambda_{lm})$. With boundary conditions obtained by combining (3) with the conservation laws, we proceed to find the eigenvalue spectrum which is continuous and for $l = 0, 1$ consists of all $\lambda \geq 0$ and for $l \geq 2$ consists of all $\lambda > 0$.

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* Present address: The Battelle Development Corporation, Columbus, Ohio.
[1] M. Rosenbluth, W. M. MacDonald and D. L. Judd, Phys. Rev. 107, 1 (1957).
[2] D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill Book Company, Inc., New York, 1964), Chaps. 2 and 3.
[3] B. B. Robinson and I. B. Bernstein, Ann. Phys. (N.Y.) 18, 110 (1962).

The expansion theorem is developed in Sec. IV. (Although the spectral resolution theorem implies the existence of an expansion theorem for self-adjoint operators, there remains the nontrivial task of constructing the expansion explicitly.) Since L_{lm} is singular at $v = 0$ and $v \to \infty$, we temporarily replace the interval $0 \le v < \infty$ by the interval $0 < v_1 \le v \le v_2 < \infty$ and show that L_{lm} generates a complete orthonormal set on this interval. To return to the original interval and thus obtain the desired expansion theorem, we use the above completeness property together with an extension of the theory of singular differential equations. This finally yields a set of functions $\{\Psi_{lm}(\mathbf{v}, \lambda_{lm})\}$ that is complete with respect to functions square integrable in velocity space. Since the spectrum is continuous, the expansion has the form of a generalized Fourier integral. For $v \gg (3kT/m_i)^{\frac{1}{2}}$ we have found asymptotic forms of the expansion functions.

I. PROPERTIES OF THE KINETIC EQUATION

For convenience we drop the subscript "i" from F_{0i} and f_i. If f satisfies the conditions

$$\lim_{v \to 0} vf = 0, \quad \lim_{v \to \infty} F_0 \frac{\partial f}{\partial v} = 0, \tag{4}$$

it is possible to show that the linear kinetic equation (3) conserves number, momentum, and kinetic energy densities.

Introducing $f(\mathbf{v}, t) = g_\lambda(\mathbf{v}) \exp(-\lambda t)$ in (3), we find

$$-\lambda F_0 g_\lambda = \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' F_0 F_0' \left(\frac{\partial g_\lambda}{\partial \mathbf{v}} - \frac{\partial g_\lambda'}{\partial \mathbf{v}'} \right) \cdot \mathbf{Q}. \tag{5}$$

Multiplying (5) by g_λ^*, where * denotes complex conjugate, and then integrating over $\mathbf{v}$ we have, after integrating by parts,

$$\lambda \int d^3v F_0 |g_\lambda|^2 = \int d^3v \frac{\partial g_\lambda^*}{\partial \mathbf{v}} \cdot \int d^3v' F_0 F_0' \left(\frac{\partial g_\lambda}{\partial \mathbf{v}} - \frac{\partial g_\lambda'}{\partial \mathbf{v}'} \right) \cdot \mathbf{Q}$$

$$- \int d^3v \frac{\partial}{\partial \mathbf{v}} \cdot g_\lambda^* \int d^3v' F_0 F_0' \left(\frac{\partial g_\lambda}{\partial \mathbf{v}} - \frac{\partial g_\lambda'}{\partial \mathbf{v}'} \right) \cdot \mathbf{Q}. \tag{6}$$

The second term on the right in (6) vanishes provided g_λ satisfies the second of conditions (4) and

$$\lim_{v \to 0} v^{\frac{1}{2}} g_\lambda(\mathbf{v}) = 0. \tag{7}$$

Assuming these conditions hold, we exchange v and v' in (6), noting that $\mathbf{Q}(\mathbf{v}, \mathbf{v}') = \mathbf{Q}(\mathbf{v}', \mathbf{v})$. We add the result to (6), obtaining

$$2\lambda \int d^3v F_0 |g_\lambda|^2$$

$$= \iint d^3v \, d^3v' F_0 F_0' \left(\frac{\partial g_\lambda^*}{\partial \mathbf{v}} - \frac{\partial g_\lambda'^*}{\partial \mathbf{v}'} \right) \cdot \mathbf{Q} \cdot \left(\frac{\partial g_\lambda}{\partial \mathbf{v}} - \frac{\partial g_\lambda'}{\partial \mathbf{v}'} \right). \tag{8}$$

Since $\mathbf{Q}$ is a positive quadratic form, it follows that the right side of (8) is positive or zero. Hence $\operatorname{Re} \lambda \ge 0$, and clearly $\operatorname{Im} \lambda = 0$.

Employing standard methods[4] we can find from (8) the most general form of g_λ when $\lambda = 0$;

$$g_0(\mathbf{v}) = a_1 v^2 + \mathbf{a}_2 \cdot \mathbf{v} + a_3, \tag{9}$$

with a_1, $\mathbf{a}_2$, and a_3 arbitrary constants.

II. EXPANSION IN SPHERICAL HARMONICS

From (5) we have

$$-\lambda F_0 g_\lambda = \frac{\partial}{\partial \mathbf{v}} \cdot \left(F_0 \frac{\partial g_\lambda}{\partial \mathbf{v}} \cdot \int d^3v' F_0' \mathbf{Q} \right)$$

$$- \frac{\partial}{\partial \mathbf{v}} \cdot \left(F_0 \int d^3v' F_0' \frac{\partial g_\lambda'}{\partial \mathbf{v}'} \cdot \mathbf{Q} \right). \tag{5}$$

To perform a parts integration on the second term, consider the quantity

$$\frac{\partial}{\partial \mathbf{v}'} \cdot (F_0' g_\lambda' \mathbf{Q}) = F_0' g_\lambda' \frac{\partial}{\partial \mathbf{v}'} \cdot \mathbf{Q}$$

$$+ F_0' \frac{\partial g_\lambda'}{\partial \mathbf{v}'} \cdot \mathbf{Q} + g_\lambda' \frac{\partial F_0'}{\partial \mathbf{v}'} \cdot \mathbf{Q}. \tag{10}$$

With $F_0 = N_0 (\alpha_i/\pi)^{\frac{3}{2}} \exp(-\alpha_i v^2)$, $\alpha_i = m_i/2kT$, we have $\partial F_0'/\partial \mathbf{v}' = -2\alpha_i \mathbf{v}' F_0'$. Also $\mathbf{v}' \cdot \mathbf{Q} = \mathbf{v} \cdot \mathbf{Q}$, so (10) gives

$$\int d^3v' F_0' \frac{\partial g_\lambda'}{\partial \mathbf{v}'} \cdot \mathbf{Q} = \int d^3v' \frac{\partial}{\partial \mathbf{v}'} \cdot (F_0' g_\lambda' \mathbf{Q})$$

$$- \int d^3v' F_0' g_\lambda' \frac{\partial}{\partial \mathbf{v}'} \cdot \mathbf{Q} + 2 \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' F_0' g_\lambda' \mathbf{Q}. \tag{11}$$

The first term on the right in (11) vanishes if g_λ satisfies (4). Using the relations

$$\mathbf{Q} = \Gamma_{ii} \frac{\partial^2 g}{\partial \mathbf{v} \, \partial \mathbf{v}}, \quad \frac{\partial}{\partial \mathbf{v}'} \cdot \mathbf{Q} = -2\Gamma_{ii} \frac{\partial}{\partial \mathbf{v}} \left(\frac{1}{g} \right),$$

and introducing a dimensionless time τ and dimensionless velocity $\mathbf{c}$ in (3),

$$\tau(= \tau_i) = 4\pi N_0 \Gamma_{ii} (\alpha_i/\pi)^{\frac{3}{2}} t, \quad \mathbf{c} = \alpha_i^{\frac{1}{2}} \mathbf{v},$$

the kinetic equation takes the form

$$-4\pi\lambda e^{-c^2} g_\lambda(\mathbf{c})$$

$$= \frac{\partial}{\partial \mathbf{c}} \cdot \left(e^{-c^2} \frac{\partial g_\lambda}{\partial \mathbf{c}} \cdot \frac{\partial^2}{\partial \mathbf{c} \partial \mathbf{c}} \int d^3c' e^{-c'^2} |\mathbf{c} - \mathbf{c}'| \right)$$

$$- 2 \frac{\partial}{\partial \mathbf{c}} \cdot \left(e^{-c^2} \frac{\partial}{\partial \mathbf{c}} \int d^3c' e^{-c'^2} g_\lambda' |\mathbf{c} - \mathbf{c}'|^{-1} \right)$$

$$- 2 \frac{\partial}{\partial \mathbf{c}} \cdot \left(\mathbf{c} e^{-c^2} \cdot \frac{\partial^2}{\partial \mathbf{c} \partial \mathbf{c}} \int d^3c' e^{-c'^2} g_\lambda' |\mathbf{c} - \mathbf{c}'| \right). \tag{12}$$

⁴ Reference 2, p. 85.

The time τ is measured in units of the "Spitzer self-collision time,"[5] and $c = |\mathbf{c}|$ is in units of the rms thermal speed.

Equation (12), in three dimensions, may be replaced by a set of uncoupled equations in one dimension by introducing the spherical harmonic expansion

$$g_\lambda(\mathbf{c}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{lm}(c, \lambda_{lm}) Y_l^m(\theta, \phi).$$

We find (see Appendix A)

$$\int d^3c' e^{-c'^2} g'_\lambda |\mathbf{c} - \mathbf{c}'|^{-1} = \sum_{l,m} \frac{4\pi}{2l + 1} Y_l^m R_{lm}, \quad (13)$$

$$\int d^3c' e^{-c'^2} g'_\lambda |\mathbf{c} - \mathbf{c}'| = \sum_{l,m} \frac{4\pi}{4l^2 - 1} Y_l^m S_{lm}, \quad (14)$$

$$\int d^3c' e^{-c'^2} |\mathbf{c} - \mathbf{c}'| = 4\pi T, \quad (15)$$

where

$$R_{lm}(c) = \int_0^c dc' c' \left(\frac{c'}{c}\right)^{l+1} e^{-c'^2} g'_{lm} + \int_c^\infty dc' c' \left(\frac{c}{c'}\right)^l e^{-c'^2} g'_{lm},$$

$$S_{lm}(c) = \int_0^c dc' c'^3 \left(\frac{c'}{c}\right)^{l-1} \left[\frac{2l-1}{2l+3}\left(\frac{c'}{c}\right)^2 - 1\right] e^{-c'^2} g'_{lm}$$

$$+ \int_c^\infty dc' c'^3 \left(\frac{c}{c'}\right)^l \left[\frac{2l-1}{2l+3}\left(\frac{c}{c'}\right)^2 - 1\right] e^{-c'^2} g'_{lm},$$

$$T(c) = \tfrac{1}{4} e^{-c^2} + \left(c + \frac{1}{2c}\right) \frac{\pi^{\frac{1}{2}}}{4} \operatorname{erf}(c),$$

with

$$\operatorname{erf}(c) \equiv 2\pi^{-\frac{1}{2}} \int_0^c dx \exp(-x^2).$$

The expressions for R_{lm} and S_{lm} were found by Rosenbluth et al.,[1] using a different method, for the axially symmetric case $m = 0$.

Combining (12)–(15) we find

$$-\lambda e^{-c^2} \sum_{l,m} Y_l^m g_{lm}$$

$$= \sum_{l,m} \frac{\partial}{\partial \mathbf{c}} \cdot \left(e^{-c^2} \frac{\partial}{\partial \mathbf{c}} (Y_l^m g_{lm}) \cdot \frac{\partial^2 T}{\partial \mathbf{c} \partial \mathbf{c}}\right)$$

$$- \sum_{l,m} \frac{2}{2l+1} \frac{\partial}{\partial \mathbf{c}} \cdot \left(e^{-c^2} \frac{\partial}{\partial \mathbf{c}} (Y_l^m R_{lm})\right)$$

$$- \sum_{l,m} \frac{2}{4l^2 - 1} \frac{\partial}{\partial \mathbf{c}} \cdot \left(\mathbf{c} e^{-c^2} \cdot \frac{\partial^2}{\partial \mathbf{c} \partial \mathbf{c}} (Y_l^m S_{lm})\right). \quad (16)$$

Since $T(c)$ is isotropic we can write

$$\partial^2 T / \partial \mathbf{c} \partial \mathbf{c} = \hat{\mathbf{e}}_c \hat{\mathbf{e}}_c T'' + (\mathbf{I} - \hat{\mathbf{e}}_c \hat{\mathbf{e}}_c) c^{-1} T',$$

where primes now denote total differentiation with respect to c. Performing the indicated angular differentiations in (16) and then employing the orthogonality

[5] L. Spitzer, Jr., Physics of Fully Ionized Gases (Interscience Publishers, Inc., New York, 1962), 2nd ed., pp. 132–136.

property of the spherical harmonics, we obtain the "radial" equation

$$-\lambda_{lm} g_{lm} = T'' \frac{d^2 g_{lm}}{dc^2} + \left[T''' + \left(\frac{2}{c} - 2c\right) T''\right]$$

$$\times \frac{dg_{lm}}{dc} - \frac{l(l+1)}{c^3} T' g_{lm}$$

$$- \frac{2}{2l+1} \left[R''_{lm} + \left(\frac{2}{c} - 2c\right) R'_{lm} - \frac{l(l+1)}{c^2} R_{lm}\right]$$

$$- \frac{2}{4l^2 - 1} \left(c S'''_{lm} + (3 - 2c^2) S''_{lm}\right.$$

$$\left. - \frac{l(l+1)}{c^2} (c S'_{lm} - S_{lm})\right). \quad (17)$$

The index m is clearly superfluous and is deleted in the following. Performing the primed differentiations in (17) we find

$$-\lambda_l g_l = \left(\frac{\pi^{\frac{1}{2}}}{4c^3} \operatorname{erf}(c) - \frac{1}{2c^2} e^{-c^2}\right) \frac{d^2 g_l}{dc^2}$$

$$+ \left[\left(\frac{1}{2c^3} + \frac{2}{c}\right) e^{-c^2} - \left(\frac{1}{c^4} + \frac{2}{c^2}\right) \frac{\pi^{\frac{1}{2}}}{4} \operatorname{erf}(c)\right] \frac{dg_l}{dc}$$

$$+ \left[2 e^{-c^2} - \frac{l(l+1)}{c^3}\right.$$

$$\times \left\langle \frac{1}{4c} e^{-c^2} + \left(1 - \frac{1}{2c^2}\right) \frac{\pi^{\frac{1}{2}}}{4} \operatorname{erf}(c)\right\rangle\right] g_l$$

$$+ \frac{4c^3(l+1)(l+2)}{(2l+1)(2l+3)} \int_0^c dc' e^{-c'^2} \left(\frac{c'}{c}\right)^{l+4} g_l(c')$$

$$- \frac{4c}{2l+1}\left(1 + \frac{c^2 l(l-1)}{2l-1}\right)$$

$$\times \int_0^c dc' e^{-c'^2} \left(\frac{c'}{c}\right)^{l+2} g_l(c')$$

$$- \frac{4c^3 l(l-1)}{4l^2 - 1} \int_c^\infty dc' e^{-c'^2} \left(\frac{c}{c'}\right)^{l-3} g_l(c')$$

$$- \frac{4c}{2l+1}\left(1 - \frac{c^2(l+1)(l+2)}{2l+3}\right)$$

$$\times \int_c^\infty dc' e^{-c'^2} \left(\frac{c}{c'}\right)^{l-1} g_l(c'). \quad (18)$$

For boundary conditions we use conditions (4), which were obtained from the conservation laws. Although (7) is stronger than the first of conditions (4), we see below that the solutions of (18) which satisfy (4) also satisfy (7).

III. THE SPECTRUM

If we introduce the transformation

$$g_l(c, \lambda_l) = c^{-1} e^{c^2/2} \psi_l(c, \lambda_l), \quad (19)$$

we obtain from (18) the formally self-adjoint equation

$$\frac{d}{dc}\left(P\frac{d\psi_l}{dc}\right) + (Q_l + \lambda_l)\psi_l + \int_0^\infty dc'K_l(c, c')\psi_l(c') = 0,$$
(20)

with

$$P(c) = \frac{\pi^{\frac{1}{2}}}{4c^3}\,\text{erf}\,(c) - \frac{1}{2c^2}\,e^{-c^2},$$

$$Q_l(c) = \left(\frac{3}{c^5} - \frac{1}{c}\right)\frac{\pi^{\frac{1}{2}}}{4}\,\text{erf}\,(c) + \left(\frac{7}{2} - \frac{3}{2c^4} - \frac{1}{c^2}\right)e^{-c^2}$$
$$- \frac{l(l+1)}{c^3}\left[\frac{1}{4c}\,e^{-c^2} + \left(1 - \frac{1}{2c^2}\right)\frac{\pi^{\frac{1}{2}}}{4}\,\text{erf}\,(c)\right],$$

$$K_l(c, c') = \frac{4}{2l+1}\,e^{-\frac{1}{2}(c^2+c'^2)}\left[\frac{(l+1)(l+2)}{2l+3}\left(\frac{c^3}{c'^3}\right)\right.$$
$$\left. - \left(\frac{c}{c'}\right) - \frac{l(l+1)}{2l-1}\left(\frac{cc'^2}{c'c^2}\right)\right]\left[\frac{(c/c')^l}{(c'/c)^l}\right]\frac{(c \le c')}{(c' \le c)}.$$

Transforming (4) via (19), we have

$$\lim_{c\to\infty} e^{-c^2/2}\frac{d\psi_l}{dc} = 0,$$
(21)

$$\lim_{c\to 0}\psi_l = 0.$$
(22)

We later show (Appendix B) that the problem (20)–(22) is self-adjoint.

The spectrum of (20) is that set of numbers $\{\lambda_l\}$ such that (20) has nontrivial solutions which satisfy (21) and (22). In view of the self-adjoint property the λ_l are real. We have already seen that λ_l must be positive for $l \ge 2$, and positive or zero for $l = 0, 1$.

We can find the spectrum of (20) by first considering the related problem

$$(d/dc)[P(dy_l/dc)] + (Q_l + \lambda_l)y_l = 0$$
(23)

with conditions on the functions $y_l(c)$ identical to (21) and (22). Clearly P, dP/dc, and Q_l are bounded and continuous for all finite c except possibly near $c = 0$. For $c \ll 1$ we have

$$P(c) = \tfrac{1}{3} - \tfrac{1}{5}c^2 + O(c^4),$$

$$Q_l(c) = \tfrac{17}{5} + O(c^2) - l(l+1)\left(\frac{1}{3c^2} - \frac{1}{15} + O(c^2)\right).$$

In general we can write

$$P(c) = \frac{1}{2c^3}\int_0^c dx(e^{-x^2} - e^{-c^2})$$

and thus $P(c) > 0$ for all $c < \infty$. It follows that for $l \ne 0$, (23) has a regular singular point at $c = 0$.

For c small, (23) has the asymptotic solutions

$$y_l(c) \simeq c^{l+1}, c^{-l} \quad (c \ll 1).$$
(24)

The first of these satisfies (22) for all l. It also satisfies

the stronger condition obtained from (7),

$$\lim (c \to 0)c^{-\frac{1}{2}}y_l(c) = 0.$$

The second solution satisfies neither condition. This is clear for $l \ne 0$. For $l = 0$, the second solution is a constant which cannot be zero since the solutions (24) are linearly independent.

For c sufficiently large and for $\lambda_l \ne 0$, (23) takes the asymptotic form

$$\frac{d^2y_l}{dc^2} - \frac{3}{c}\frac{dy_l}{dc} + \frac{4\lambda_l c^3}{\pi^{\frac{1}{2}}}y_l = 0 \quad (c \text{ large}, \lambda_l \ne 0).$$

We find $(\lambda_l > 0)$

$$y_l(c) \simeq A_l c^{\frac{3}{4}}\cos(\nu_l c^{\frac{5}{2}} - \gamma_l),$$
(25)

with $\nu_l \equiv 4\lambda_l^{\frac{1}{2}}/5\pi^{\frac{1}{4}}$.

Given l and λ_l, Eq. (23) has only one solution which satisfies the condition at $c = 0$. This solution thus contains only one arbitrary constant and it follows that A_l and γ_l in (25) are not independent. Whatever the relation between A_l and λ_l is, (25) satisfies (21) for all positive γ_l. Hence the spectrum of (20) contains all positive λ_l.

To determine the spectrum of the problem in Eqs. (20)–(22), we first note that the solutions of the problem in Eqs. (21)–(23) do not exist in Hilbert space; hence we must introduce a transformation in order to apply a Hilbert space theorem. We recognize the symmetric differential and integral operators in (20) as, respectively, closed and completely continuous operators. It is possible to introduce an algebraic transformation on (20) such that the respective closed and completely continuous properties are retained, while the solutions of the transformed problem in Eqs (21)–(23) now do exist in Hilbert space. Since the transformation has destroyed the symmetry of the problem, we employ a recent extension[6] of Weyl's perturbation theorem which applies to asymmetric linear operators. This extension states that[6] the addition of a completely continuous operator to a closed operator does not change the limit points of the spectrum of the closed operator. With our knowledge of the spectral properties of Eqs. (21)–(23) and in view of the symmetry of Eq. (20), it now follows that the spectrum of each l- component of the linearized Fokker–Planck equation contains all positive real λ_l; we showed earlier that the spectrum is empty for $\lambda_l < 0$.

For $l = 0$ and $l = 1$ we found, respectively, two-fold and three-fold degenerate zero eigenvalues, corresponding to a shift to an equilibrium different from

[6] I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators* (Daniel Davey & Co., New York, 1966).

that originally postulated. For $l \geq 2$ the eigenvalues are positive and real and are $(2l + 1)$-fold degenerate, corresponding to the $2l + 1$ different spherical harmonics of order l.

IV. THE EXPANSION THEOREM

With the spectral properties of the kinetic equation in hand, we can now proceed to the development of an expansion theorem based on (20). Our method is essentially an extension of the theory of singular ordinary differential equations, based on a theory due to Weyl and Levinson,[7] to include singular integro-differential equations with Hilbert–Schmidt kernels.

Since the eigenfunctions of the kinetic equation are bounded and continuous on every finite interval, it is natural to pursue an expansion formula for functions $u(c)$ square integrable on the interval $\Delta: 0 \leq c < \infty$. As with the earlier theory we first establish an expansion formula on a finite subinterval δ of Δ, $\delta: a \leq c \leq b$, $0 < a$, $b < \infty$, so that the singularities of the linear operator are external to δ. The expansion formula we seek is then obtained by taking $\delta \to \Delta$ in a suitable manner. In the following we mean by L_Δ the integro-differential operator in (20) and by L_δ the operator obtained when the lower and upper limits of the integral in (20) are replaced by a and b, respectively. In the following, the index l is retained only where it is necessary to avoid confusion.

We have already seen that P, P', and Q_l are continuous on δ, and that $K_l(c, c')$ is bounded and integrable on the square $a \leq c \leq b$, $a \leq c' \leq b$. Tamarkin[8] has shown that, subject to these conditions, the solutions of $L_\delta \phi = -\lambda \phi$ which satisfy homogeneous boundary conditions at $c = a$ and $c = b$ form a complete orthogonal and normalizable set of eigenfunctions $(h_{\delta n})$ on δ, with an associated denumerable sequence of real eigenvalues $(\lambda_{\delta n})$. Assuming the $h_{\delta n}$ to be normalized, the expansion formula on δ is thus

$$u(c) = \sum_{n=0}^{\infty} h_{\delta n} \int_a^b dc u(c) h_{\delta n}^*(c), \qquad (26)$$

where $u(c)$ is any function square integrable on δ.

We now use the Weyl–Levinson theory to take $\delta \to \Delta$. Since the subsequent development of the expansion theorem is in every respect a duplication of the earlier theory, we display only the salient features.

[7] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill Book Company, Inc., New York, 1955), Chap. 9.
[8] J. D. Tamarkin, Trans. Am. Math. Soc. **29**, 755 (1927).

Given λ, the most general solution of $L_\delta \phi = -\lambda \phi$ is a linear combination of the two linearly independent solutions, say ϕ_1, ϕ_2. Thus, we can write

$$h_{\delta n}(c, \lambda_{\delta n}) = r_{\delta n 1} \phi_1(c, \lambda_{\delta n}) + r_{\delta n 2} \phi_2(c, \lambda_{\delta n}), \quad (27)$$

where $r_{\delta n 1}$ and $r_{\delta n 2}$ are complex constants. With (27), (26) becomes

$$u(c) = \sum_{n=0}^{\infty} \sum_{j,k=1}^{2} r_{\delta n j} r_{\delta n k}^* \phi_j \int_a^b dc u(c) \phi_k^*. \qquad (28)$$

Following Levinson[7] we define an Hermitian, positive semidefinite matrix ρ_δ, called the spectral matrix, with elements $\rho_{\delta j k}$ which consist of step functions with jumps at the eigenvalues $\lambda_{\delta n}$ given by

$$\rho_{\delta j k}(\lambda_{\delta n} + 0) - \rho_{\delta j k}(\lambda_{\delta n} - 0) = r_{\delta n j} r_{\delta n k}^*.$$

Let $\rho_\delta(\lambda + 0) = \rho_\delta(\lambda)$, and let $\rho_\delta(0)$ be the zero matrix. We use the spectral matrix to replace the infinite series in (28) by a Lebesgue–Stieltjes integral

$$u(c) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \phi_j(c, \lambda) \tilde{u}_k(\lambda) \, d\rho_{\delta j k}(\lambda), \qquad (29)$$

where

$$\tilde{u}_k = \int_a^b dc u(c) \phi_k^*(c, \lambda).$$

As $\delta \to \Delta$ (that is, $a \to 0$, $b \to \infty$), ρ_δ approaches a limit matrix ρ_Δ. To find ρ_Δ, let $\lambda = \mu + i\omega$, $\omega > 0$, and let $\chi_a = \phi_1 + m_a(\lambda)\phi_2$ be a solution of $L_\delta \phi = -\lambda \phi$ satisfying the homogeneous boundary condition

$$\cos \alpha \chi(a) + \sin \alpha P(a) \chi'(a) = 0,$$

and similarly let $\chi_b = \phi_1 + m_b(\lambda)\phi_2$ be a solution of the same equation satisfying

$$\cos \beta \chi(b) + \sin \beta P(b) \chi'(b) = 0.$$

Clearly $m_a = r_{\delta n 2}(a)/r_{\delta n 1}(a)$ and similarly for m_b. As $a \to 0$ and $b \to \infty$, m_a and m_b approach limiting values in the complex m plane denoted, respectively, by $m_0(\lambda)$ and $m_\infty(\lambda)$. These limiting values are clearly determined by the behavior of ϕ_1 and ϕ_2 for small and large c, for λ complex.

For ϕ_1 and ϕ_2 to be linearly independent, it is necessary and sufficient that their Wronskian equal a nonzero constant, say one;

$$P(\phi_1 \phi_2' - \phi_2 \phi_1') = 1. \qquad (30)$$

This last is satisfied if ϕ_1 and ϕ_2 satisfy the conditions

$$\phi_1(s, \lambda) = \sin \sigma, \qquad \phi_2(s, \lambda) = \cos \sigma,$$

$$P(s)\phi_1'(s, \lambda) = -\cos \sigma, \quad P(s)\phi_2'(s, \lambda) = \sin \sigma,$$

where s is an interior point of δ and $0 \leq \sigma < \pi$. These conditions are also sufficient to ensure that ϕ_1, ϕ_2 are entire functions of λ for each fixed c on δ (this follows from Tamarkin[8]).

With these properties secured, we can find[7] the limit values m_0 and m_∞ and hence the limit matrix ρ_Δ, whose elements are given by

$$\rho_{\Delta jk}(\lambda) - \rho_{\Delta jk}(\eta) = \lim_{\omega \to +0} \frac{1}{\pi} \int_\eta^\lambda \text{Im } M_{jk}(\mu + i\omega) \, d\mu,$$
(31)

where

$$M_{11}(\lambda) = \frac{1}{m_0 - m_\infty},$$

$$M_{12}(\lambda) = M_{21}(\lambda) = \frac{1}{2} \frac{m_0 + m_\infty}{m_0 - m_\infty},$$

$$M_{22}(\lambda) = \frac{m_0 m_\infty}{m_0 - m_\infty}.$$

To find the M_{jk} we need asymptotic forms of ϕ_1 and ϕ_2 for large and small c. These are given by (24) and (25), as may be verified by direct substitution. Taking ϕ_1, ϕ_2 to be asymptotic, respectively, to c^{-l}, c^{l+1} for c small, we apply the homogeneous boundary condition to χ_a and then take $a \to 0$ to find

$$m_0 = \infty \qquad (l \neq 0),$$

$$m_0 = -\cot \alpha \quad (l = 0). \tag{32}$$

Thus for $l \neq 0$ only M_{22} can have a nonzero imaginary part and, consequently, only ϕ_2 contributes to the expansion formula (29). When $l = 0$ both solutions are regular at $c = 0$ and the limit matrix is not determined until we specify α. The boundary condition (22) dictates the choice $\alpha = 0$.

For c large we take

$$\phi_2 \simeq A_l c^{\frac{3}{4}} \cos (\nu_l c^{\frac{5}{4}} - \gamma_l) \quad (\nu_l = 4\lambda_l^{\frac{1}{2}}/5\pi^{\frac{1}{4}}, \, \lambda_l > 0)$$

and find ϕ_1 by integrating (30);

$$\phi_1 \simeq -\frac{8c^{\frac{3}{4}} \sin [\nu_l(c^{\frac{5}{4}} - c_0^{\frac{5}{4}})]}{5\pi^{\frac{1}{2}} A_l \nu_l \cos (\nu_l c_0^{\frac{5}{4}} - \gamma_l)},$$

where c_0 is a constant of integration. Applying the homogeneous boundary condition to $\chi_b \equiv \phi_1 + m_b(\lambda)\phi_2$ and then taking $b \to \infty$ with Im $\lambda > 0$ we find

$$m_\infty = \frac{8i \exp [i(\nu_l c_0^{\frac{5}{4}} - \gamma_l)]}{5\pi^{\frac{1}{2}} A_l^2 \nu_l \cos (\nu_l c_0^{\frac{5}{4}} - \gamma_l)}. \tag{33}$$

Combining (31)–(33) we have finally

$$d\rho(\lambda_l) \equiv d\rho_{\Delta 22}(\lambda_l) = \frac{2 \, d\lambda}{\pi^{\frac{5}{2}} A_l^2(\lambda_l)\lambda_l^{\frac{1}{2}}} \quad (\lambda_l > 0). \tag{34}$$

Since the spectrum is empty for $\lambda_l < 0$, $\rho(\lambda_l)$ is constant on this range. The expansion formula (29) becomes

$$u(c) = \int_0^\infty \phi_2(c, \lambda_l)\tilde{u}(\lambda_l) \, d\rho(\lambda_l) \tag{35}$$

with

$$\tilde{u} = \int_0^\infty u(c)\phi_2^*(c, \lambda_l) \, dc.$$

The expansion converges in the mean for all functions $u(c)$ square integrable on $(0, \infty)$. If the spectral function ρ is not continuous at $\lambda_l = 0$, this point will contribute to the integral in (35).

We return to the description of perturbations from equilibrium. If $\exp (-c^2)f(\mathbf{c}, 0)$ is square integrable in velocity space, then from (19) and (35) we have

$$e^{-c^2}f(\mathbf{c}, \tau) = \sum_{l,m} \int_0^\infty d\rho(\lambda_l)\Psi_{lm}\tilde{F}_{lm}e^{-\lambda_l \tau} \tag{36}$$

with

$$\tilde{F}_{lm}(\lambda_l) = \frac{1}{2}(2l + 1)\int d^3c \, \Psi_{lm}^* f(\mathbf{c}, 0)$$

and

$$\Psi_{lm}(\mathbf{c}, \lambda_l) = c^{-1}e^{-c^2/2}\psi_l(c, \lambda_l)Y_l^m(\theta, \phi).$$

The functions ψ_l correspond to ϕ_2 in (35) and are the solutions of (20) satisfying (22).

We have defined the density, mean velocity, and kinetic temperature of the ion gas as being proportional respectively to the first three moments of the equilibrium distribution F_0. If ρ is continuous at $\lambda_l = 0$ this point does not contribute to the expansion formula (36) and the eigenfunctions (9) for $\lambda_l = 0$ are not contained in the expansion. By virtue of the conservation laws, the functions (9) are then orthogonal to (36). Thus (36) is complete only if ρ has a jump at $\lambda_l = 0$ for $l = 0, 1$. It follows that the exclusion of (9) from (36) yields an expansion which is complete with respect to all square integrable perturbations conserving N_0, $\langle \mathbf{v} \rangle$, and kT.

V. THE ELECTRON KINETIC EQUATION

We have developed an expansion theorem based on the uncoupled kinetic equation (3) for the ions. The extension to the electron kinetic equation is straightforward and requires only a little algebra.

As indicated earlier, we decouple the electron kinetic equation from the ion equation by dropping the term $(m_e/m_i)\partial f_i'/\partial \mathbf{v}'$ in (2). This amounts to neglecting the effect of the ion perturbation on the electron perturbation, but retains the effects of encounters with ions in the thermal distribution.

The uncoupled equation conserves electron number density provided (4) holds, but does not conserve momentum or kinetic energy in the electron gas. This is as it should be, since a substantial portion of the electron momentum, and a small amount of the energy, is lost to the ions.

Applying the methods of Sec. I we find as before $\text{Re } \lambda \geq 0, \text{Im } \lambda = 0$, and for $\lambda = 0$ we find $g_0 = \text{const}$, corresponding to (9). The remainder of the development proceeds as before. A spherical harmonic expansion yields a set of singular integro–differential equations, and the transformation (19) brings these into self-adjoint form. As before, the expansion formula has the form of a Riemann–Stieltjes, or generalized Fourier, integral.

If, for example, the ions are protons, then we can take $\Gamma_{ee} = \Gamma_{ei}$. If we use α_e in place of α_i in the definition of τ and c, then the electron equations may be obtained from the ion equations by replacing $T(c)$ in (15) by $T(c) + \alpha^{\frac{1}{2}} T(\alpha^{-\frac{1}{2}}c)$, where $\alpha \equiv \alpha_e/\alpha_i = m_e/m_i$.

VI. DISCUSSION

We have used boundary conditions obtained by requiring the solutions of the kinetic equation to be consistent with the conservation laws. The Hilbert space then 'emerged as a natural function space for the framework of the mathematical development. The question persists (see, e.g., the discussion in Uhlenbeck and Ford[9]) as to whether square integrability should be a requirement on the distribution functions from the beginning. In the light of the present work, this condition does not appear to be necessary, and for our purposes, it would not have been sufficient. To see this we note the condition $\int d^3v \, |F_0 f|^2 < \infty$ leads to

$$\lim_{v \to 0} v^{\frac{3}{2}} f = 0, \qquad (37)$$

which is weaker than the corresponding condition (4). Since both solutions of (20) satisfy (37) for $l = 0$, it would be possible to have an expansion theorem for solutions of the kinetic equation which are square integrable but do not satisfy the conservation laws.

[9] G. E. Uhlenbeck and G. W. Ford, *Lectures in Statistical Mechanics* (American Mathematical Society, Providence, Rhode Island, 1963), p. 88.

APPENDIX A. INTEGRALS

Let Θ be the angle between $\mathbf{c}$ and $\mathbf{c}'$. From the generating function relation for Legendre polynomials, we have

$$|\mathbf{c} - \mathbf{c}'|^{-1} = (c^2 + c'^2 - 2cc' \cos \Theta)^{-\frac{1}{2}}$$
$$= c^{-1} \sum_{l=0}^{\infty} \left(\frac{c'}{c}\right)^l P_l(\cos \Theta) \quad (c' \leq c), \quad (A1)$$

and similarly for $c \leq c'$. Writing

$$g_\lambda(\mathbf{c}) = \sum g_{lm} Y_l^m$$

and employing the addition theorem for Legendre polynomials, we obtain Eq. (13).

To find Eq. (14) we use the relation

$$(1 + x^2 - 2xy)^{\frac{1}{2}} = \int dx x (1 + x^2 - 2xy)^{-\frac{1}{2}}$$
$$- y \int dx (1 + x^2 - 2xy)^{-\frac{1}{2}}. \quad (A2)$$

Combining (A1) and (A2) we have

$$|\mathbf{c} - \mathbf{c}'| = (c^2 + c'^2 - 2cc' \cos \Theta)^{\frac{1}{2}}$$
$$= c^{-1} \sum_{l=0}^{\infty} \left(\frac{(c'/c)^{l+2}}{l+2} - \frac{(c'/c)^{l+1}}{l+1} \cos \Theta \right) P_l(\cos \Theta)$$
$$(c' \leq c), \quad (A3)$$

and similarly for $c \leq c'$. From the pure recurrence relation for Legendre polynomials, we have

$$\cos \Theta P_l(\cos \Theta) = \frac{l}{2l+1} P_{l-1}(\cos \Theta)$$
$$+ \frac{l+1}{2l+1} P_{l+1}(\cos \Theta). \quad (A4)$$

Combining (A3) and (A4) and then using the addition theorem as before we find Eq. (14).

Finally, Eq. (15) is obtained from Eq. (14) with $g_{lm} = \delta_{l0} \delta_{m0}$ with δ_{jk} the Kronecker delta.

APPENDIX B. THE SELF-ADJOINT PROPERTY

Let L_δ be either (i) the differential operator in Eq. (20) or (ii) the integro–differential operator in Eq. (20) defined as in Sec. IV on the closed interval δ: $a \leq c \leq b, 0 < a, b < \infty$. Let L_Δ be similarly defined on $(0, \infty)$. Then in either case (i) or case (ii) there exists a complete orthonormal set of functions $\{h_{\delta n}\}$ on δ, generated by $L_\delta \phi = -\lambda \phi$ with homogeneous boundary conditions at a and b.[7,8]

Let f and g be any nonzero functions square integrable on $(0, \infty)$, and consider the inhomogeneous

problems

$$L_\delta u + \Lambda u = f, \quad L_\delta v + \Lambda^* v = g \qquad \text{(B1)}$$

with the same homogeneous boundary conditions at a and b as in the above homogeneous problem. Let Im $\Lambda \neq 0$ so that Λ will not belong to the spectrum of the set $\{h_{\delta n}\}$. Then the problems (B1) have nontrivial solutions[8]

$$u(c) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \frac{\phi_j(c,\lambda)\tilde{u}_k(\lambda)\, d\rho_{\delta jk}(\lambda)}{\Lambda - \lambda},$$

$$v(c) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \frac{\phi_j(c,\lambda)\tilde{v}_k(\lambda)\, d\rho_{\delta jk}(\lambda)}{\Lambda^* - \lambda},$$

with

$$\tilde{u}_k = \int_a^b f(c)\phi_k^*(c,\lambda)\, dc, \quad \tilde{v}_k = \int_a^b g(c)\phi_k^*(c,\lambda)\, dc. \qquad \text{(B2)}$$

Now, let

$$u_\mu = \int_{-\mu}^{\mu} \sum_{j,k=1}^{2} \frac{\phi_j \tilde{u}_k\, d\rho_{\delta jk}}{\Lambda - \lambda},$$

$$v_\mu = \int_{-\mu}^{\mu} \sum_{j,k=1}^{2\cdot} \frac{\phi_j \tilde{v}_k\, d\rho_{\delta jk}}{\Lambda^* - \lambda},$$

with $\tilde{u}_k$ and $\tilde{v}_k$ as in (B2). Multiplying u_μ by g^* and v_μ^*

by f and then integrating over δ we have

$$\int_a^b g^*(c)u_\mu(c)\, dc$$
$$= \int_{-\mu}^{\mu} \sum_{j,k=1}^{2} \frac{\tilde{u}_k(\lambda)\, d\rho_{\delta jk}(\lambda)}{\Lambda - \lambda} \int_a^b g^*(c)\phi_j(c,\lambda)\, dc$$
$$= \int_{-\mu}^{\mu} \sum_{j,k=1}^{2} \frac{\tilde{u}_k(\lambda)\tilde{v}_j^*(\lambda)\, d\rho_{\delta jk}(\lambda)}{\Lambda - \lambda}, \qquad \text{(B3)}$$

and similarly

$$\int_a^b f(c)v_\mu^*(c)\, dc = \int_{-\mu}^{\mu} \sum_{j,k=1}^{2} \frac{\tilde{v}_k^*(\lambda)\tilde{u}_j(\lambda)\, d\rho_{\delta jk}(\lambda)}{\Lambda - \lambda}. \qquad \text{(B4)}$$

Since (24) and (25) are asymptotic solutions for both the differential and the integro–differential equations, the limit matrix ρ_Λ is the same in either case. Taking $\delta \to \Delta$ in (B3) and (B4) we have $\rho_{\delta jk} \to \rho_{\Delta 22}$ and thus

$$\int_0^{\infty} g^*(c)u_\mu(c)\, dc = \int_{-\mu}^{\mu} \frac{\tilde{u}_2\tilde{v}_2^*\, d\rho_{\Delta 22}}{\Lambda - \lambda} = \int_0^{\infty} f(c)v_\mu^*(c)\, dc.$$
$$\text{(B5)}$$

After taking $\mu \to \infty$ and employing (B1), (B5) becomes

$$\int_0^{\infty} (L_\Delta v)^* u\, dc = \int_0^{\infty} (L_\Delta u)v^*\, dc,$$

which is the desired result.

Real Spinor Fields

DAVID HESTENES*
Palmer Laboratory, Princeton University, Princeton, New Jersey

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The Dirac equation is expressed entirely in terms of geometrical quantities by providing a geometrical interpretation for the $(-1)^{\frac{1}{2}}$ which appears explicitly in the Dirac equation. In the modification of the Dirac electron theory which ensues, the $(-1)^{\frac{1}{2}}$ appears as the generator of rotations in the spacelike plane orthogonal to the plane containing the electron current and spin vectors. This amounts to a further "relativistic" constraint on the spinor theory and so may be expected to have physical consequences. It does not, however, conflict with well-substantiated features of the Dirac theory.

INTRODUCTION

IN 1928, Dirac proposed a relativistically invariant first-order wave equation for the electron.[1] Dirac's theory has led to a complex of physical explanations and predictions at once so surprising and convincing that it has gained general acceptance among physicists today. The previously perplexing phenomena of

electron spin was not only accounted for, but fine details of the hydrogen spectra and an accurate value of the electron magnetic moment were calculated without arbitrary assumptions. Moreover, after some theoretical trauma, it was realized that Dirac's equation entails the existence of a positively charged electron—at just about the same time that such a particle was discovered experimentally. We do well to understand precisely what features of the Dirac equation entail these remarkable results.

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* Present address: Physics Department, Arizona State University, Tempe, Arizona.
[1] P. A. M. Dirac, Proc. Roy. Soc. (London) 117, 610 (1928).