Pressure-induced flows at low Rossby numbers

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The flow induced by a pressure gradient along a pipe in a rapidly rotating system is examined for small but nonzero Rossby numbers. The regions of the flow field where inertial terms are important are identified and the flow determined for pipes of rectangular and circular cross section.

I. INTRODUCTION

Flow along a pipe which is rotating about an axis normal to the axis of the pipe has been examined theoretically by Benton and Boyer¹ and by Hocking,² under the assumption that the fluid velocity relative to the pipe is much smaller than the rotation velocity. Inertial forces can then be neglected and the flow is governed by linear equations describing the balance between the imposed pressure gradient and Coriolis and viscous forces. If the Ekman number E is small, viscous forces are only significant in boundary layers of the Ekman and Stewartson types. For a pipe of rectangular cross section, the flow consists of a uniform cross flow, which returns in Ekman layers on the top and bottom boundaries, and a uniform axial velocity. There are Stewartson layers on the side walls in which the velocities are reduced to zero. For a circular pipe, the axial flow is not uniform and the Ekman and Stewartson layers merge at the ends of the horizontal pipe diameter (the rotation vector is assumed to be vertical).

The neglect of inertia in these solutions is equivalent to setting the Rossby number R equal to zero. In general, inertial forces first become significant in rapidly rotating systems when the Rossby number is increased to be $O(E^{1/2})$; see, for example, Barcilon³ and Stewartson and Walker.⁴ In some axisymmetric flows, however, modification of the flow by inertia is delayed until the Rossby number becomes $O(E^{1/4})$. Examples are the source-sink flow discussed by Hide⁵ and the split-disk configuration of Stewartson.⁶ The inertial effects in all these flows are manifested by a change in the Stewartson $E^{1/4}$ layer. When the inertial effects appear at $R = O(E^{1/2})$, the linear Ekman condition can still be employed at the top and bottom of the vertical layer, but nonlinear terms appear in the equations for the Ekman layers themselves, as well as in the vertical layer, when $R = O(E^{1/4})$. In order to obtain the inertial modification to the source-sink and split-disk flows, Bennetts and Hocking⁷ devised a method for obtaining the nonlinear Ekman condition, based on a local similarity assumption. The same method, with some modifications is applied to pressure-induced flows in pipes of rectangular and circular cross sections.

In addition to presenting the linear solution for the circular pipe, Benton and Boyer¹ described the results of an experiment to measure the axial flow distribution. Their linear theory predicted a symmetrical distribution on either side of the vertical diameter of the pipe section, but their experiments showed a marked asymmetry. The nonlinear theory also shows an asymmetry of roughly the same extent

as that observed experimentally, although the velocity distributions are somewhat different.

II. RECTANGULAR PIPE

To avoid a possible complication from centrifugal forces and to facilitate comparison with the axisymmetric configurations of Bennetts and Hocking, we suppose that the pipe axis is slightly curved and that the whole system is rotating about the center of curvature of the pipe axis (Fig. 1). In terms of cylindrical coordinates (ar, θ, ay) , where r is measured from the center of curvature, y is vertical, and a is the height of the pipe, the pipe axis is the circle, or part of the circle, $r = d/a$, $y = 0$. The velocity components (u, v, w) are made nondimensional by a velocity scale U and are independent of θ . They can be written in terms of two nondimensional functions ψ and χ as

$$u = \frac{d}{ar} \frac{\partial \psi}{\partial y}, \quad v = \frac{d\chi}{ar}, \quad w = -\frac{d}{ar} \frac{\partial \psi}{\partial r},$$

so that ψ is a stream function for the flow in the meridional planes $\theta = \text{const}$. If we suppose that $d/a \gg 1$, we can write

$$r = d/a + x,$$

with $x \ll d/a$, and the Navier-Stokes equations in the rotating frame then reduce to equations for ψ and χ of the form

$$-\frac{\partial \chi}{\partial y} = \frac{1}{2} E \nabla^4 \psi + R \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)}, \quad (1)$$

$$\frac{\partial \psi}{\partial y} - E^{1/2} = \frac{1}{2} E \nabla^2 \chi + R \frac{\partial(\psi, \chi)}{\partial(x, y)}, \quad (2)$$

where the pressure has been eliminated and terms $O(a/d)$ times those retained have been omitted. In these equations,

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2,$$

and the Ekman and Rossby numbers are defined by

$$E = \nu/\Omega a^2, \quad R = U/2\Omega a,$$

where ν is the kinematic viscosity and Ω is the angular velocity of the system. The constant term in (2) is proportional to the imposed axial pressure gradient $\partial p/\partial \theta$, which

is related to the velocity scale U (and so to the Rossby number) by

$$\partial p / \partial \theta = -2\rho\Omega d E^{1/2} U, \quad (3)$$

where ρ is the fluid density.

The conditions ψ and χ must satisfy on the pipe boundary are

$$\begin{aligned} \psi = \partial\psi/\partial y = \chi = 0 & \quad \text{on } y = \pm \frac{1}{2}, \\ \psi = \partial\psi/\partial x = \chi = 0 & \quad \text{on } x = \pm b/2, \end{aligned}$$

where the width of the pipe is ab .

In the interior of the pipe, i.e., outside the boundary layers, the terms with coefficients E and R in (1) and (2) can be neglected, and we obtain

$$\chi = f_1(x), \quad \psi = E^{1/2}y + f_2(x).$$

The linear Ekman conditions can be applied at $y = \pm 1/2$ since $R \ll 1$. In terms of the functions ψ and χ , these conditions reduce to

$$\chi = \pm 2E^{-1/2}\psi \quad \text{at } y = \pm 1/2,$$

and, in consequence, the interior solution is

$$\chi = 1, \quad \psi = E^{1/2}y.$$

The particular definition of the pressure gradient (3) was chosen in order to produce this simple result.

In the vicinity of the side walls, but outside the Ekman layers, $\partial/\partial x \gg \partial/\partial y$ and with the Rossby terms retained the appropriate forms of (1) and (2) are

$$\frac{\partial\chi}{\partial y} = 0, \quad \frac{\partial\psi}{\partial y} - E^{1/2} = \frac{1}{2}E \frac{\partial^2\chi}{\partial x^2} - R \frac{\partial\psi}{\partial y} \frac{\partial\chi}{\partial x}. \quad (4)$$

If $R = 0$, these are the equations for the Stewartson $E^{1/4}$ layer. It is clear that the nonlinear terms must be included when $R = O(E^{1/4})$.

For the layer on the wall $x = -b/2$, we write

$$\begin{aligned} x &= -b/2 + R\xi, \\ \chi &= F(\xi), \quad \psi = -2E^{1/2}yG(\xi), \end{aligned} \quad (5)$$

and (4) becomes

$$G \left(1 + \frac{dF}{d\xi} \right) = -\frac{1}{2} - \mu \frac{d^2F}{d\xi^2}, \quad (6)$$

where

$$\mu = \frac{1}{4}R^{-2}E^{1/2}.$$

The boundary conditions are that χ and ψ should vanish at the wall and the solution should also match the interior solutions as $\xi \rightarrow \infty$. Thus,

$$F(0) = G(0) = 0, \quad F(\infty) = 1, \quad G(\infty) = -\frac{1}{2}.$$

(7)

FIG. 1. Cross section of pipe.

The layer on the other wall $x = b/2$ is also governed by (6) if we replace (5) by

$$x = b/2 + R\xi$$

and the boundary conditions (7) by

$$F(0) = G(0) = 0, \quad F(-\infty) = 1, \quad G(-\infty) = -\frac{1}{2}.$$

The boundary condition at the walls which requires the vertical velocity component $\partial\psi/\partial x$ to vanish there cannot be satisfied by the $E^{1/4}$ layer in the linear theory and an inner layer of thickness $O(E^{1/3})$ is required to complete the problem. It was shown in Ref. 7 that for a range of values of R the inner layer still retained its linear form, although the outer layer had been modified by inertia.

The two functions F and G must be related by a second equation, which is obtained from a consideration of the Ekman layers on the boundaries $y = \pm 1/2$. In these layers $\partial/\partial y \gg \partial/\partial x$, but for the range of values of R under consideration the inertial terms are not negligible. The symmetry of the problem allows us to consider only the layer on the wall $y = -1/2$, and we write

$$\begin{aligned} y &= -\frac{1}{2} + E^{1/2}\eta, & x &= \mp b/2 + R\xi, \\ \chi(x, y) &= \bar{\chi}(\xi, \eta), & \psi(x, y) &= E^{1/2}\bar{\psi}(\xi, \eta). \end{aligned}$$

With these substitutions (1) and (2) become

$$-\frac{\partial\bar{\chi}}{\partial\eta} = \frac{1}{2}\frac{\partial^4\bar{\psi}}{\partial\eta^4} + \frac{\partial\bar{\psi}}{\partial\xi}\frac{\partial^3\bar{\psi}}{\partial\eta^3} - \frac{\partial\bar{\psi}}{\partial\eta}\frac{\partial^3\bar{\psi}}{\partial\xi\partial\eta^2}, \quad (8)$$

$$\frac{\partial\bar{\psi}}{\partial\eta} = \frac{1}{2}\frac{\partial^2\bar{\chi}}{\partial\eta^2} + \frac{\partial\bar{\psi}}{\partial\xi}\frac{\partial\bar{\chi}}{\partial\eta} - \frac{\partial\bar{\psi}}{\partial\eta}\frac{\partial\bar{\chi}}{\partial\xi}, \quad (9)$$

and the boundary conditions are

$$\begin{aligned} \bar{\chi}(\xi, 0) &= \bar{\psi}(\xi, 0) = \partial\bar{\psi}(\xi, 0)/\partial\eta = 0, \\ \bar{\chi}(\xi, \infty) &= F(\xi), \quad \bar{\psi}(\xi, \infty) = G(\xi). \end{aligned}$$

The method we used to obtain a solution of these equations, and its rationale, are described fully in Ref. 7. Briefly, we write

$$\bar{\chi} = F(\xi)f(\eta), \quad \bar{\psi} = G(\xi)\bar{g}(\eta) \quad (10)$$

FIG. 2. Rectangular pipe: left-hand wall.

and determine f and $\bar{g}$ from (8) and (9), at a fixed value of ξ , say $\xi = \xi_0$. An additional scaling,

$$\bar{g}(\eta) = F(\xi_0)g(\eta)/G(\xi_0),$$

reduces the problem to the solution of the coupled ordinary differential equations

$$\frac{1}{2} \frac{d^3 g}{d\eta^3} + \beta \left[g \frac{d^2 g}{d\eta^2} - \left(\frac{dg}{d\eta} \right)^2 \right] = 1 - f, \quad (11)$$

$$\frac{1}{2} \frac{d^2 f}{d\eta^2} + \beta g \frac{df}{d\eta} = (1 + \alpha f) \frac{dg}{d\eta}, \quad (12)$$

in which

$$\alpha = F'(\xi_0), \quad \beta = F(\xi_0)G'(\xi_0)/G(\xi_0),$$

and the boundary conditions (10) are now

$$\begin{aligned} f(0) &= g(0) = dg(0)/d\eta = 0, \\ f(\infty) &= 1, \quad dg(\infty)/d\eta = 0. \end{aligned} \quad (13)$$

The value of $g(\infty)$ cannot be chosen arbitrarily but can be found from the solution of this system of equations. If we write this numerically determined value as

$$g(\infty) = h(\alpha, \beta),$$

we then have a nonlinear Ekman condition

$$G = Fh(F', FG'/G). \quad (14)$$

The linear Ekman condition corresponds to $\alpha = \beta = 0$, $h = -\frac{1}{2}$.

To find the flow in the layer, we must solve (6) and (14) simultaneously. The method used is also described fully in Ref. 7. A finite-difference approximation to (6) was combined with a locally plane approximation to h , determined by interpolation between previously calculated and stored values of h . Values of F and G were then calculated alternately, and the process continued until the iteration converged, with under-relaxation if necessary.

FIG. 3. Solution of the Ekman layer equations (11) and (12) for $\alpha = 0$.

The results of this procedure for the layer on the wall $x = -b/2$ are shown in Fig. 2 for a range of values of μ . The linear solution corresponds to $\mu \gg 1$ under the present scaling. The solution for the layer with $h = -\frac{1}{2}$ is

$$F = 1 - \exp[-\xi/(2\mu)^{1/2}]. \quad (15)$$

The layer has a thickness $O(\mu^{1/2})$ in ξ or $O(E^{1/4})$ in x . When $\mu = O(1)$, the layer has a thickness $O(R) = O(E^{1/4})$, but its structure differs from (15). When $\mu \ll 1$, the layer can be divided into two parts. In the outer part, viscosity can be neglected and (6) gives

$$G = -\frac{1}{2}(1 + F')^{-1}. \quad (16)$$

As $\xi \rightarrow 0$, F' increases and $G \rightarrow 0$. The asymptotic form of (14) for large F' was shown in Ref. 7 to be

$$G = -0.56F(F')^{-3/4} \quad (17)$$

and the solution of (16) and (17) is

$$F = 1.27\xi^{1/5}, \quad G = -1.97\xi^{4/5},$$

which satisfy the conditions at $\xi = 0$ but do not give the correct form since viscosity must be important near the wall. The neglected viscous term in (6) has the same magnitude as the other terms when

$$\mu\xi^{-9/5} = O(1),$$

so that there is an inner viscous layer of thickness $\mu^{5/9}$ in which $F = O(\mu^{1/9})$. Hence, when $R \gg E^{1/4}$, there is an inner viscous layer of thickness $O(R^{-1/9}E^{5/18})$ and an outer layer of thickness $O(R)$ in which there is a balance between Coriolis, inertial, and pressure forces. Similar nonviscous regions were found in some of the configurations described in Ref. 7. The thickness of the inner viscous layer can be written as $E^{1/4}(RE^{-1/4})^{-1/9}$, so that it decreases from the linear value $E^{1/4}$ quite slowly as $RE^{-1/4}$ increases.

A difficulty in the calculation of $h(\alpha, \beta)$ was mentioned in Ref. 7, namely, that no solution of (11) and (12) satisfying (13) could be found for fixed α and sufficiently large β , the boundary being approximately the line $\beta = \alpha + 1$. The reason for this barrier is not clear, but it limits the obtainable solutions in the present case to $\mu > 0.03$, since for

smaller values of μ the numerical solution near the wall required values of h from the wrong side of the barrier. It was suggested in Ref. 7 that the barrier might indicate the existence of two solutions for some $\beta < \alpha + 1$. Further numerical work confirmed this suggestion. For $\alpha = 0$, the values of h are shown in Fig. 3. No attempt has been made to use these results to extend the range of μ to values less than 0.03. The locally plane approximation to h would be quite inadequate to describe the behavior of h near the barrier. However, there appears to be no reason to doubt the existence of a solution for these lower values of μ ; as ξ increases from 0 to ∞ a continuous value of h can exist, although β may not vary monotonically through the layer.

The solution for the layer on the opposite wall $x = b/2$ is shown in Fig. 4. The linear solution is identical to (15) apart from a change in sign of ξ . Since F' is now negative, there can be no nonviscous region when $\mu \ll 1$, for (6) shows that G would increase as the wall is approached in such a layer. As Fig. 4 indicates, the value of F' decreases as μ decreases and for $\mu \simeq 0.6$, F' reaches the value of -1 at the wall. Smaller values of μ would require lower values of F' , but, as shown in Ref. 7, the value of $h(\alpha, \beta)$ cannot be determined if $\alpha < -1$. The reason for this failure is that the asymptotic approach of f and g to their values at infinity is exponential. If we suppose that

$$f \sim 1 + ce^{-\delta \eta}$$

and substitute in (11) and (12), we obtain four values of δ and the two roots with negative real parts can be discarded. However, if $\alpha < -1$ and β is negative, only one root can be discarded and the extra freedom means that a solution of the Ekman equations can be obtained for arbitrary F and G , i.e., there is no condition imposed by the Ekman layer on the flow in the vertical layer. In the examples treated in Ref. 7, a nonviscous region near the boundary in which $\alpha = -1$ developed, so that values of h for $\alpha < -1$ were not needed. As already pointed out, such a remedy is not available here. Thus for values of $\mu < 0.6$ or equivalently $R > 0.65E^{1/4}$ the controlling influence of the Ekman layers on the interior breaks down. Presumably, a solution could only be found by considering an initial-value problem, and there is no guarantee that a steady solution will exist.

III. CIRCULAR PIPE

The circular pipe flow differs from the rectangular pipe because the Ekman layers are no longer normal to the rotation vector. The linear solution ($R = 0$) for a pipe with boundary $x^2 + y^2 = \frac{1}{4}$ was found by Benton and Boyer¹ to be

$$\chi = (1 - 4x^2)^{3/4}, \quad \psi = E^{1/2}y. \quad (18)$$

The Ekman condition is

$$\chi = \pm 2E^{-1/2}(\cos \phi)^{1/2}\psi,$$

where ϕ is the angle between the normal to the layer and the vertical. Near the ends of the horizontal diameter $x = \pm \frac{1}{2}$, $y = 0$, there is a viscous region of extent $O(E^{2/5})$ in the x direction and $O(E^{1/5})$ in the y direction which takes the place of the $E^{1/4}$ and $E^{1/3}$ layers in the rectangular pipe.

FIG. 4. Rectangular pipe: right-hand wall.

When R is increased, it is the regions at the ends of the horizontal diameter which are first modified by inertia. If we consider a layer of thickness $O(\sigma)$ near $x = \pm \frac{1}{2}$, we have $x = O(\sigma)$ and $y = O(\sigma^{1/2})$ and the linear interior solution (18) shows that

$$\chi = O(\sigma^{3/4}), \quad \psi = O(E^{1/2}\sigma^{1/2}).$$

An examination of the magnitudes of the terms in (1) and (2) then shows that the inertial terms can be neglected if $R \ll E^{1/10}$ (compare the smaller value $E^{1/4}$ for the rectangular pipe). If $R = O(E^{1/10})$, there is a region of width $O(E^{2/5})$ at both ends of the horizontal diameter where viscous, Coriolis, and inertial forces are all important. If $R \gg E^{1/10}$, viscous terms become unimportant and there is a layer of thickness $O(R^4)$ which is governed by the equations

$$\frac{\partial \chi}{\partial y} = 0, \quad \frac{\partial \psi}{\partial y} \left(1 + R \frac{\partial \chi}{\partial x} \right) = E^{1/2} \quad (19)$$

and in which

$$\chi = O(R^3), \quad \psi = O(R^2 E^{1/2}).$$

In this layer, the imposed pressure gradient is balanced by the Coriolis acceleration, with the basic rotation increased by the rotation associated with the vertical component of relative vorticity within the layer. The layer is similar to the nonviscous layers of thickness $O(R)$ discussed in Bennetts and Hocking.⁷ An inner viscous layer will also be present, as the solution of (19) cannot satisfy all the boundary conditions. If the solution of (19) has nonzero values of χ and ψ at one of the boundaries, the inner viscous region has a thickness $O(R^{-6}E)$ and the Coriolis term is not important in such a region.

Although the condition $R \gg E^{1/10}$ may appear restrictive, it should be realized that it is not necessary to insist that $R \ll 1$ as well. If $R = O(1)$ the layers on either side of the pipe merge and a solution of (19) can be obtained for the whole of the interior of the pipe, apart from the Ekman layers and the viscous regions at the ends of the pipe diameter.

The nonlinear Ekman condition is needed before (19) can be solved, but it must be modified to allow for the inclination of the Ekman layers. In terms of coordinates

(n, s) normal and tangential to the boundary, the Ekman layer equations are

$$-\frac{\partial \chi}{\partial n} \cos \phi = \frac{1}{2} E \frac{\partial^4 \psi}{\partial n^4} + R \frac{\partial(\psi, \partial^2 \psi / \partial n^2)}{\partial(s, n)},$$

$$\frac{\partial \psi}{\partial n} \cos \phi = \frac{1}{2} E \frac{\partial^2 \chi}{\partial n^2} + R \frac{\partial(\psi, \chi)}{\partial(s, n)}.$$

If we proceed as before to write

$$\xi = R^{-1} s, \quad \eta = E^{-1/2} \cos^{1/2} \phi n,$$

$$\chi(n, s) = F(\xi) f(\eta), \quad \psi(n, s) = \frac{E^{1/2} F(\xi_0)}{\cos^{1/2} \phi G(\xi_0)} G(\xi) g(\eta),$$

we obtain

$$\frac{1}{2} \frac{d^3 g}{d\eta^3} + \beta \left[g \frac{d^2 g}{d\eta^2} - \left(\frac{dg}{d\eta} \right)^2 \right] - \gamma \left(\frac{dg}{d\eta} \right)^2 = 1 - f,$$

$$\frac{1}{2} \frac{d^2 f}{d\eta^2} + \beta g \frac{df}{d\eta} = (1 + \alpha f) \frac{dg}{d\eta},$$

where

$$\alpha = \frac{F'(\xi_0)}{\cos \phi}, \quad \beta = \frac{F(\xi_0) G'(\xi_0)}{\cos \phi G(\xi_0)},$$

$$\gamma = \frac{d(\cos^{1/2} \phi)}{d\xi} \frac{F(\xi_0)}{\cos^{3/2} \phi}.$$

These equations differ from (11) and (12) only in the additional term with coefficient γ and in the definitions of α and β . If we put $\phi = 0$, we obtain (11) and (12). The boundary conditions (13) still hold. From the numerical solution of these equations we obtain

$$g(\infty) = h(\alpha, \beta, \gamma)$$

so that instead of (14), we have the nonlinear Ekman condition

$$(\cos \phi)^{1/2} G = F h(\alpha, \beta, \gamma). \quad (20)$$

For the circular pipe,

$$\cos \phi = (1 - 4x^2)^{1/2}.$$

The solution of (19) gives another equation connecting F and G ,

$$G \left(1 + R \frac{dF}{dx} \right) = -\frac{1}{2} (1 - 4x^2)^{1/2}, \quad (21)$$

and the solution of (20) and (21) can be obtained by the methods described here, but the additional parameter increases the complexity of the calculation and the number of values of h which have to be pre-calculated and stored. Solu-

FIG. 5. Circular pipe: axial velocity on the horizontal diameter.

FIG. 6. Circular pipe —, theory; +, experimental values of Ref. 1. For the experiments, E was about 10^{-4} .

tions were obtained for various values of R [note that (20) and (21) are independent of E] and the calculations extended over the whole range of x from $-\frac{1}{2}$ to $+\frac{1}{2}$, i.e., the interior and the layers were not distinguished. The results are shown in Fig. 5. Solutions could be found which satisfied the conditions at the left-hand wall but not at the right. No solutions could be found for which the reverse was true, so the inner viscous layer must be present on the right-hand wall.

The same difficulties about the determination of h which were present before appear in the extension to inclined Ekman layers. The restriction $\alpha \geq -1$ is still present, but values of h for α not satisfying this condition were not required in the solution of (21). No values for h could be found if the parameters satisfied an inequality of approximately the form $\beta + \gamma > \alpha + 1$ and this limited the range of values of R for which solutions could be found to $R < 0.34$. However, there seems no reason to suppose that this heralds a breakdown of the flow, but rather, as for the rectangular pipe, complexity of the Ekman layer equations which the present method cannot include.

In Fig. 6, the results, normalized to give a maximum value of the velocity equal to 1, are compared with the experimental results of Benton and Boyer.¹ From their paper it is difficult to obtain precise values of R but they seem to lie in the range 0.15 to 0.3. The theoretical prediction that the velocity would not tend to zero on one side, except in a very thin viscous layer, seems to be confirmed and the theory and experiment are in reasonable agreement for $x > 0$. On the other side of the pipe, however, the nonlinear theory predicts that the velocity should be reduced from its linear value, whereas the experiment shows values higher than the linear ones. The reasons for this discrepancy are

not clear. Since the theory requires $R \gg E^{1/10}$ and such a condition was hardly met by the experimental values, viscous effects near $x = -\frac{1}{2}$ may be significant. The flow in the test section might not have been fully developed if measurements were made too near the upstream end of the pipe. Centrifugal effects might have caused a distortion of the velocity profile, if the ratio between the pipe radius and the distance of the pipe axis from the axis of rotation were not sufficiently small. Not enough information is available to quantify these possible explanations of the discrepancy between the theoretical and experimental values.

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