Betatron Oscillations in the Synchrotron*

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Betatron oscillations in the University of Michigan synchrotron have been investigated by exciting these oscillations with a transverse electric field. The results indicate that the betatron oscillation is composed of a group of component frequencies which are separated by the frequency of synchrotron phase oscillation. The suggested explanation of the observed splitting is the modulation of the betatron oscillation frequency by the synchroton oscillations.

THE present investigation was undertaken to determine experimentally the frequency of the radial betatron oscillations in the University of Michigan straight section synchrotron. The technique consists of perturbing the electrons with a transverse rf electric field of variable frequency, and measuring those which excite the betatron oscillations and blow up the beam. The unexpected result of the experiment is that not one, but a series of equally spaced frequencies can knock out the beam. The explanation, suggested by H. R. Crane, is that the betatron oscillation is frequency modulated. The spacing of the component frequencies would then be constant and equal to the modulating frequency. The separation of these components is in fact found to be the synchrotron oscillation frequency and it therefore appears certain that the synchrotron oscillations modulate the betatron oscillation frequency, splitting it into a band spectrum in which the separation of the components is equal to the synchrotron oscillation frequency. The following is a discussion of the experiment in greater detail.

The rf electric field which excites the betatron oscillations is set up for about a hundred microseconds between two parallel plates in a straight section of the synchrotron. The electrons pass between the plates and are given a radial kick by the electric field. The plates are 5 inches long, 8 inches wide, and are spaced 5 inches apart. About 200 volts of rf is applied between them. If the radio-frequency is made equal to the betatron oscillation frequency, the rf impulses drive the betatron oscillations until the beam strikes the target or walls of the tube. The destruction of the beam is monitored by a photomultiplier placed near the target and an ionization chamber which detects the final output beam. The frequency of the rf oscillator can be continuously varied by condenser tuning, which is done from the control room by means of a selsyn. The frequency is measured by mixing the pulsed rf signal with the output of a cw oscillator, displaying the result on a scope and tuning the cw oscillator for zero beat, and then measuring the cw oscillator frequency with a precision frequency meter. Frequencies of 25 megacycles are reproducible to a few kilocycles by this technique.

When the region of the betatron oscillation frequency is explored the structure of the oscillation quickly becomes apparent. Oscillations are excited by a band of over seven frequencies, the central ones causing the largest oscillations and the most beam knockout, while the side bands have progressively less effect. These resonances have widths which are of the order of half their spacing. Table I gives a list of these resonance frequencies.

The synchrotron phase oscillation frequency is calculated from an expression given by Blachman and Courant[1] and is also listed on Table I. To confirm this indication that the spacing is the synchrotron oscillation frequency, two parameters are varied: the energy of the electrons and the rf accelerating voltage. The spacings are measured and compared with the predictions of the Blachman and Courant formula. Results are presented in Table II.

TABLE I. Radial oscillation resonance frequencies and their separations.

Resonance frequency Mc	Spacing kc	
24.196		Rf accelerating
	204	
24.400		voltage $V_0 = 1250$ volts
	228	
24.628		Beam energy $E = 6.8$ Mev
	212	
24.840		Orbital frequency
	210	
25.050		$f_0 = 32.0$ Mc
	226	
25.276		Calculated value of synchrotron oscillation frequency
	182	
25.452		
Average spacing 210 kc		220 kc

Variation of the voltage between the plates changes the amount of beam knocked out, but does not change the spacing. Therefore, from the above results it appears clear that the spacing is the synchrotron phase oscillation frequency.

These results can be accounted for by a frequency modulation of the betatron oscillations by the phase oscillations in the synchrotron. The mechanism of this modulation can be understood by considering first separately and then together the effects of the guide field and the radio-frequency acceleration on the beam trajectories.

The synchrotron beam oscillates in the guide field in the radial (r) and vertical (z) directions with fre-

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[1] N. M. Blachman and E. D. Courant, Rev. Sci. Instr. **20**, 596 (1949).

TABLE II. Measured and calculated spacing of the radial oscillation resonance frequencies for different values of electron energy, E, and rf voltage, V_0.

E Mev	V_0 volts	Measured spacing kc	Calculated spacing kc
6.8	1250	210	220
6.8	575	134	141
14.4	1250	150	152

quencies which are determined by the fall off index, n, of the magnetic field. The radial frequency is given by

$$\omega_r=\omega_c(1-n)^{\frac{1}{2}}f$$

and the vertical frequency by

$$\omega_z=\omega_c\sqrt{n}f,$$

where ω_c is the circular frequency of the electrons in the machine and f is a factor which accounts for the straight sections. Superimposed upon this motion, which is characteristic of the guide field alone, are the radial phase oscillations induced by the radio-frequency acceleration of the beam. The frequency of this motion is determined by the applied rf voltage, the beam energy, and the rate of rise of the magnetic field, and is typically a much lower frequency than either of the betatron frequencies. The changing radius from the phase oscillations results in a modulation of ω_r. This modulation occurs for two reasons. Since the electrons are relativistic and their velocity constant, $v\sim c$, a change in the radius of the orbit causes a change in ω_c and hence in ω_r. In addition, any change of the field index, n, with radius causes a change in ω_r.

Since ω_r is modulated it can be expressed as

$$\omega_r=\omega_0+\Delta\omega_0\cos\omega_s t$$

with $\omega_s\ll\omega_0$, where ω_0 is the radial betatron frequency with no phase oscillations, $\Delta\omega_0$ the maximum frequency change due to the phase oscillations, and ω_s is the phase oscillation frequency.

An experiment devised to measure ω_r will not detect a single frequency but a band of frequencies characteristic of a frequency-modulated signal. If the amplitude of the radial betatron oscillation, A_r, is assumed to be sinusoidal (a close approximation) then

$$\begin{aligned}A_r&=A_0\sin\int_0^t\omega_r dt\\&=A_0\sin\left[\omega_0 t+\Delta\omega_0\int_0^t\cos\omega_s t dt\right]\\&=A_0\sin\left[\omega_0 t+\frac{\Delta\omega_0}{\omega_s}\sin\omega_s t\right].\end{aligned}$$

Expanding this expression

$$\begin{aligned}A_r=A_0\bigg\{&J_0\left(\frac{\Delta\omega_0}{\omega_s}\right)\sin\omega_0 t+J_1\left(\frac{\Delta\omega_0}{\omega_s}\right)[\sin(\omega_0+\omega_s)t\\&-\sin(\omega_0-\omega_s)t]+J_2\left(\frac{\Delta\omega_0}{\omega_s}\right)[\sin(\omega_0+2\omega_s)t\\&+\sin(\omega_0-2\omega_s)t]+\cdots+\cdots\bigg\},\end{aligned}$$

where the coefficients $J_n(\Delta\omega_0/\omega_s)$ are Bessel functions of order n. These coefficients have decreasing but appreciable values out to the frequency $\omega_0\pm(\Delta\omega_0+\omega_s)$.

Thus the frequency modulation of the betatron oscillations by the phase oscillations produces a band spectrum, whose spacing is the phase oscillation frequency and whose extent is the limit of the betatron frequency swing. This is precisely the experimental finding. The spacing is the synchrotron phase oscillation frequency, as shown in Table II. The extent of the spectrum, 1.2 Mc at 25 Mc, about 5 percent, is approximately equal to the variation of the betatron oscillation frequency expected due to aperture restrictions. The useful aperture is about 5 cm, the radius of curvature is 100 cm, and the four straight sections total 300 cm. This allows a 3 percent change in ω_c and hence in ω_r. Any variation of n with radius would also contribute to the change of ω_r.

When a frequency modulated wave is expanded into its component frequencies, the sum of the magnitudes of the expansion coefficients is of the order unity. Therefore, increasing the number of resonant frequencies that make up the betatron oscillation decreases the strength of each of these resonances. In order to blow up the beam, a perturbation which excites one of these resonance frequencies has to act over more revolutions than if it excites an unmodulated betatron oscillation.

The frequency range over which perturbations can cause blowup of a particle is of the order of the variation in the betatron frequency. Although it is possible that a perturbing frequency inside this range could lie between two of these resonant frequencies, it is unlikely that it would remain there for long. Hence the entire region of the swing of the betatron frequency should be kept away from any low order resonance.

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