

# On the classification of vacuum zero Simon tensor solutions in relativity

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The Perjes vector is decomposed on a Frenet basis. Geometric insights into the Perjes classification of zero Simon tensor vacuum solutions to the Einstein field equations are obtained. The behavior of the classification under a Killing vector preserving transform is studied.

## I. INTRODUCTION

The metric for a stationary axis-symmetric space can be written as

$$dS^2 = \lambda(dt + \omega_i d\chi^i) - (h_{ij}/\lambda)d\chi^i d\chi^j. \quad (1)$$

The metric functions can be found by solving the field equations or by transforming from known solutions.

A particularly interesting set of solutions for this metric is the static vacuum metrics with a conformally flat three-space  $(g_{ij})$ . These spaces have a zero York tensor  $Y_{ij}$ ,<sup>1</sup>

$$Y_i^j = 2\epsilon^{jkl}(R_{ik} - \frac{1}{4}Rg_{ik})_{;l}, \quad (2)$$

where  $R_{ik}$  is calculated for  $g_{ik}$ . The perhaps best-known member of this solution set is the Schwarzschild metric with  $g_{ik}$  as the metric for the  $t = \text{const}$ , three-space.<sup>2</sup> A second example is the three-space of timelike Killing trajectories,  $\xi_\mu$ , of the metric described by Eq. (1). This three-space has metric  $h_{ij}$  and will be conformally flat if  $Y_i^j$  due to  $h_i^j$  is zero.

Simon<sup>3</sup> has discussed a complex generalization of the York tensor. The Simon tensor  $C_{ij}$  was constructed to characterize the Kerr solution for Eq. (1) in the same way that the York tensor characterizes the Schwarzschild solution. A vanishing Simon tensor, coupled with the requirement of asymptotic flatness, will single out the Kerr solution to the vacuum field equations for the metric (1).<sup>4</sup> Perjes<sup>5</sup> has shown that the condition of zero Simon tensor alone, includes many interesting solutions that can be divided into three classes.

The purpose of this paper is to examine the transformation properties of the zero Simon tensor vacuum solutions. In particular, we will examine the behavior of the Perjes classes under a transformation that preserves the timelike Killing vector  $\xi_\mu$  of a stationary axis-symmetric space-time.

In the next section we briefly review the Perjes classification of the zero Simon tensor solution. Some new geometric insights into this grouping are obtained. The transformation is discussed in Sec. III.

## II. THE CLASSIFICATION OF ZERO SIMON TENSOR VACUUM SOLUTION

### A. Perjes classification

The timelike Killing vector associated with the metric (1) has a norm  $\lambda$  and a vector twist  $\omega_\mu$  (Ref. 6),

$$\lambda = \xi_\mu \xi^\mu > 0, \quad (3)$$

$$\omega^\mu = \epsilon^{\mu\nu\rho\sigma} \xi_\nu \xi_{\rho;\sigma} / \sqrt{-g} = 2\xi^\mu \xi^\nu_{;\nu}, \quad (4)$$

where  $\xi^{\mu\nu}$  is the dual Killing bivector

$$\xi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \xi_{\rho;\sigma} / 2\sqrt{-g}, \quad (5)$$

and  $\omega^\mu$  describes the rotation of the Killing congruence. This will be made more precise in the next section. This  $\omega^\mu$  is curl-free and can be written as the gradient of a scalar potential, where the gradient is defined in the three-space of Killing trajectories  $H$ ;  $h_{ij}$ ,

$$\omega_i = -D_i \phi, \quad (6)$$

where  $i$  runs over spatial indices and  $D_i$  is the covariant derivative in  $h_{ij}$ . Here  $\omega_i$  can be pulled back to the four-space  $g_{\mu\nu}$  to create  $\omega_\mu$ .<sup>6</sup> The negative sign is added to conform to Perjes' conventions. An Ernst potential can be formed from the norm and scalar twist

$$\tau = \lambda + i\phi. \quad (7)$$

The Simon tensor is defined in terms of the Ernst potential  $\tau$ ,

$$C_i^j = (2\epsilon^{klj}/\lambda^2) [\tau_{i;l} \tau_k - h_{il} h^{mn} \tau_{m;n} \tau_k], \quad (8)$$

where  $\tau_i = \partial_i \tau$  is the gradient of the scalar potential  $\tau$ . For static space-times,  $\phi = 0$ ; the Simon tensor is equivalent to the York tensor.

The vacuum spaces with zero Simon tensor can be classified by using the vector  $G$  (Refs. 5 and 7),

$$G_\mu = (1/2\lambda)\tau_\mu = (1/2\lambda)(\lambda_\mu + i\phi_\mu). \quad (9)$$

We use  $\mu$  as an index of the pulled back functions. Here  $\lambda_\mu$  is related to the normal to the surface  $\lambda = \text{const}$ , and  $\phi_\mu$  is related to the twist associated with that trajectory. The three classes of zero Simon tensor vacuum solutions correspond to (1)  $G \cdot G = 0$ , the null class (2)  $G \times G^* = 0$ , the degenerate class, and (3)  $G \times G^* \neq 0$ , the general class. Perjes identified specific solution sets within each class by the behavior of spin coefficients for a triad  $(l_\mu, \bar{m}_\mu, m_\mu)$  defined on  $h_{ij}$ . The spin coefficient method of Perjes is very useful in identifying the Petrov class of solutions belonging to each set. We found that decomposing  $G$  in terms of a Frenet-Serret tetrad of vectors allowed some new geometric insights into the Perjes classification.

### B. Frenet decomposition of $G_\mu$ —non-null Killing vector

The groups that Perjes uses to classify the zero Simon tensor solutions, differentiate properties of the Killing bivector  $\xi_{\mu;\nu}$  and of the congruence of Killing vectors forming  $h_{ij}$ .

The Killing vector and bivector properties are conveniently described in terms of the orthonormal Frenet tetrad  $(e_{(0)}^\mu, A^\mu, B^\mu, C^\mu)$  with

$$-e_{(0)}^\mu e_{\mu(0)} = A^\mu A_\mu = B^\mu B_\mu = C^\mu C_\mu = -1. \quad (10)$$

The timelike tetrad member is chosen to lie along the Killing vector

$$e_{(0)}^\mu = \xi^\mu / \sqrt{\lambda}. \quad (11)$$

The spatial triad  $(A^\mu, B^\mu, C^\mu)$  are the normal and first and second binormals to the Killing trajectory. The trajectories can be described by three scalars  $k, \tau_1$ , and  $\tau_2$ , the curvature and first and second torsions, respectively. These scalars enter the absolute derivatives of the tetrad,

$$\begin{bmatrix} \dot{e}_{(0)}^\mu \\ \dot{A}^\mu \\ \dot{B}^\mu \\ \dot{C}^\mu \end{bmatrix} = \begin{bmatrix} 0 & k & 0 & 0 \\ k & 0 & \tau_1 & 0 \\ 0 & -\tau_1 & 0 & \tau_2 \\ 0 & 0 & -\tau_2 & 0 \end{bmatrix} \begin{bmatrix} e_{(0)}^\mu \\ A^\mu \\ B^\mu \\ C^\mu \end{bmatrix}, \quad (12)$$

where, for example,  $\dot{A}^\mu = A^\mu{}_{;\nu} e^\nu_{(0)}$

The Frenet formalism is ideally suited to the discussion of Killing vectors and bivectors. Normally a timelike vector derivative is written in terms of acceleration, angular speed, expansion, and shear. Because the Killing bivector is antisymmetric, only the three parameters describing acceleration and angular speed are nonzero. These can easily be identified with the Frenet scalars. The bivector expansion is

$$\begin{aligned} \xi_{\mu;\nu} = & k(-\xi_\mu A_\nu + \xi_\nu A_\mu) + \sqrt{\lambda} \tau_1 (A_\mu B_\nu - A_\nu B_\mu) \\ & + \sqrt{\lambda} \tau_2 (B_\mu C_\nu - B_\nu C_\mu). \end{aligned} \quad (13)$$

The parameters  $k, \tau_1$ , and  $\tau_2$  are constant along the Killing trajectory.<sup>8</sup> The "acceleration" and "angular speed" of the Killing vector  $\xi_\mu$  are<sup>8</sup>

$$\begin{aligned} \dot{\xi}_\mu & \equiv \xi_{\mu;\nu} \xi^\nu = \lambda k A_\mu, \\ \Omega^\mu & = \frac{1}{2} \omega^\mu = -\phi^\mu / 2 = \overset{D}{\xi}{}^{\mu;\nu} \xi_\nu = -\lambda(\tau_1 C^\mu + \tau_2 A^\mu), \end{aligned} \quad (14)$$

$\Omega^\mu$  describes the rotation of the Frenet frame relative to a frame that is Fermi-Walker transported along the trajectory. A static space has both torsions zero. From Eq. (13) we find the bivector norms and products can be parametrized in terms of the Frenet scalars and are given by

$$\begin{aligned} \xi^{\mu;\nu} \xi_{\mu;\nu} & = -2\lambda(k^2 - \tau_1^2 - \tau_2^2), \\ \overset{D}{\xi}{}^{\mu;\nu} \xi_{\mu;\nu} & = -4\lambda k \tau_2. \end{aligned} \quad (15)$$

The vector  $G$  can be written in terms of Frenet vectors,

$$\begin{aligned} G_\mu & = (1/2\lambda)(\lambda_\mu + i\phi_\mu) \\ & = -kA_\mu + i(\tau_1 C_\mu + \tau_2 A_\mu), \end{aligned} \quad (16)$$

$G$  has no component along the first binormal  $B_\mu$ .

The first class of solutions corresponds to  $G \cdot G = 0$ . Using Eq. (16), this is equivalent to

$$k^2 = \tau_1^2 + \tau_2^2, \quad (17)$$

$$k\tau_2 = 0. \quad (18)$$

The first condition requires the Killing bivector have

zero norm. The second condition requires that the Killing bivector be orthogonal to its dual, or that the bivector be simple. These are the conditions for the bivector to be null, as pointed out by Perjés.<sup>4</sup> The first or null class then consists of solutions with null Killing bivectors. Simple bivectors are expressible as a single antisymmetric product. Using Eq. (13) we find

$$\xi_{\mu;\nu} = k\sqrt{\lambda} [A_\mu (e_\nu^{(0)} - B_\nu) - A_\nu (e_\mu^{(0)} - B_\mu)]. \quad (19)$$

The two vectors are  $A_\mu$  and the null vector  $e_\mu^{(0)} - B_\mu$ . A possible Perjés triad  $(l, m, \bar{m})$  for this class is  $(B, (A \pm iC)/\sqrt{2})$ .

The spaces with a null bivector are algebraically special.<sup>9</sup> The restriction to non-null Killing vector restricts the space-time to be Petrov III or N.<sup>10,11</sup>

The second class of zero Simon tensor solutions is characterized by  $G \times G^* = 0$ . Using (15), this can be written as

$$(G \times G^*)_\mu = i2\tau_1 k B_\mu. \quad (20)$$

Normally  $k$  is taken nonzero so that for class 2 solutions there is no first torsion. The bivector is not null. For this class  $G_\mu$  can be written

$$G_\mu = (-k + i\tau_2)A_\mu. \quad (21)$$

This set includes both static ( $\phi_\mu = 0$ ) and stationary ( $\phi_\mu = i\tau_2 A_\mu$ ) solutions. It is interesting that the rotation of the Frenet frame relative to the Fermi transported frame lies along the acceleration. A possible triad for this class is  $(l, m, \bar{m}) = (A, B \pm iC/\sqrt{2})$ .

One could choose  $k = 0$  in this class. The  $G$  vector is

$$G_\mu = i(\tau_1 C_\mu + \tau_2 A_\mu).$$

This subclass seems to be artificial since the only example we were able to find also had  $\tau_1$  and  $\tau_2$  zero. We will see it is mathematically nice and so include it for completeness.

The general class is described by Eq. (20). There are no restrictions on any of the Killing scalars.

The Frenet decomposition makes very clear how  $G$  behaves from class I to II. In addition it also describes the classwise behavior of the bivector. This behavior is especially interesting in the case when the second torsion,  $\tau_2$ , is zero. The first class has bivector norm zero and  $G$  perpendicular to  $l = B$ . The second class has two subclasses. In II a, the bivector norm is negative and  $G$  is along  $l = A$ . In II b, the bivector norm is positive and  $G$  is along  $l = C$ . In all three cases, the bivector is orthogonal to its dual. This is summarized in Table I.

TABLE I. Behavior of Perjés vector and Killing bivector norm for zero second torsion.

Class	$G$	$\xi^{\mu;\nu} \xi_{\mu;\nu}$	$l$	$k$	$\tau_1$	$\tau_2$
I						
$G \times G = 0$	$k(-A + iC)$	0	$B$	$k$	$k$	0
II a						
$G \times G^* = 0$	$-kA$	$-2\lambda k^2 < 0$	$A$	$k$	0	0
II b						
$G \times G^* = 0$	$i\tau_1 C$	$2\lambda \tau_1^2 > 0$	$C$	0	$\tau_1$	0

### C. Frenet decomposition—null Killing vector

The decomposition discussed in the previous section is only valid for non-null Killing vectors,  $\lambda \neq 0$ . If we wish to discuss the behavior of  $G$  on a Killing horizon, a different set of Frenet tetrads must be used because of differences in defining trajectory parameters.

If the Killing vector is null but has a non-null normal, the appropriate tetrad is<sup>12</sup> described by

$$\begin{bmatrix} \dot{\xi}^\mu \\ \dot{A}^\mu \\ \dot{B}^\mu \\ \dot{C}^\mu \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_2 & 0 & k_1 & 0 \\ 0 & k_2 & 0 & k_3 \\ k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi^\mu \\ A^\mu \\ B^\mu \\ C^\mu \end{bmatrix}, \quad (22)$$

with  $\xi^\mu B_\mu = 1 = -A^\mu A_\mu = -B^\mu C_\mu$ , all others zero. In terms of these vectors

$$\begin{aligned} \xi^\mu &= \xi^\mu, \quad \lambda = 0, \quad n^\mu = -k_1 A^\mu, \\ n^\mu n_\mu &\neq 0, \quad \omega^\mu = -k_1 C^\mu, \end{aligned} \quad (23)$$

$n^\mu$  is the normal to the  $\lambda = 0$  Killing horizon and should be compared to  $\lambda^\mu$  in the previous section. The Killing bivector has norm and dual product<sup>12</sup>

$$\xi_{\mu;\nu} \xi^{\mu;\nu} = -4k_1 k_2, \quad \xi_{\mu;\nu} \overset{D}{\xi}^{\mu;\nu} = -4k_1 k_3. \quad (24)$$

For this case the vector  $G$  is defined by

$$G^\mu \equiv -k_1 (A^\mu - iC^\mu), \quad (25)$$

which is clearly of the Perjes null class with no bivector restriction. A metric example of this case is the Kerr solution. The Killing horizon ( $\lambda = 0$ ) is not coincident with the event horizon ( $\lambda^\mu \lambda_\mu = 0$ ) except on the axis of rotation. On the Kerr Killing horizon,  $G^\mu$  is a null vector.

If the Killing vector and the normal are both null, as for example on the Schwarzschild horizon, then a single null vector  $N_\mu$  parametrizes all vectors and we have<sup>12,13</sup>

$$\begin{aligned} \xi^\mu &= N^\mu, \quad \lambda = 0, \quad \lambda^\mu = \epsilon N^\mu, \quad \lambda^\mu \lambda_\mu = 0, \\ \omega^\mu &= \delta N^\mu, \quad \omega^\mu \omega_\mu = 0, \quad G = \gamma N^\mu, \quad G^\mu G_\mu = 0. \end{aligned} \quad (26)$$

For this case the classes merge.

### III. TRANSFORMATIONS

The transformations we want to consider are transformations among the three classes of zero Simon tensor solutions which preserve the Killing vector. We consider only non-null Killing vectors. This transformation has been described by Geroch.<sup>6</sup> It generalizes the work of Ehlers<sup>14</sup> and Harrison.<sup>15</sup> The transform is a projective transform on the complex function  $i\tau$ ,

$$i\tau' = (a i\tau + b)/(c i\tau + d), \quad (27)$$

with  $a, b, c, d$  as constants. The transform is performed in the space  $h; h_{ij}$  of Killing trajectories with the requirement

$$h_{ij}' = h_{ij}. \quad (28)$$

Under the transformation the Killing scalars transform as<sup>16</sup>

$$\begin{aligned} \phi' &= \{[a\phi - b][d - c\phi] - ac\lambda^2\}/[(d - c\phi)^2 + \lambda^2 c^2], \\ \lambda' &= \lambda(ad - bc)/[(d - c\phi)^2 + \lambda^2 c^2]. \end{aligned} \quad (29)$$

Under this transform, the zero Simon tensor is preserved. We wish to find the effect of this transform on the vector  $G$ . Take the covariant derivative of Eq. (29) in the Killing space. Pulling back to the metric space one finds

$$\begin{aligned} \lambda'_{;\mu} &= \lambda_\mu \cos \alpha + \phi_\mu \sin \alpha, \\ \phi'_{;\mu} &= \phi_\mu \cos \alpha - \lambda_\mu \sin \alpha, \end{aligned} \quad (30)$$

with

$$\begin{aligned} \cos \alpha &= [(d - c\phi)^2 - c^2 \lambda^2]/[(d - c\phi)^2 + c^2 \lambda^2], \\ \sin \alpha &= 2\lambda c(d - c\phi)/[(d - c\phi)^2 + c^2 \lambda^2]. \end{aligned} \quad (31)$$

Using these vectors to form  $G$  we have

$$G'_\mu = (\lambda/\lambda') e^{-i\alpha} G_\mu, \quad (32)$$

where  $G'$  differs from  $G$  only by a scaled phase factor. Class is clearly preserved by this transformation. For example, this transform will take the Schwarzschild metric into the Taub-Nut<sup>17</sup> space-time. We would expect the Schwarzschild and Taub-NUT spaces to be of the same Perjes class II.

In conclusion, we have decomposed the Perjes  $G$  vector onto a Frenet basis and discussed the Frenet scalar relations in each class. We find also that Perjes class is preserved under a transform that preserves the Killing vector.

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