Instability of stratified flows as a result of resonance

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The stability of stratified flows consisting of a middle layer of homogeneous fluid in linear shear flow and contiguous upper and lower stratified layers of constant velocities is considered. The density and velocity are continuous throughout. It is shown that there are infinitely many pairs of neutral modes with the phase velocity c_r in the range of the velocity which, in general, are not represented by the stability boundary, but which merge as the parameter N (the inverse of the Froude number squared) increases. As N increases further the coalesced neutral modes become modes with complex c, one of which is unstable. The instability may be considered to be caused by the resonance of the original pair of separate modes. In addition, instability of stratified flows is considered for general density and velocity distributions without a layer of constant density and linear velocity, and it is concluded that in that case the modes with complex c on the unstable side of a stability boundary do not continue into any neutral normal modes on the stable side.

I. INTRODUCTION

Miles[1] and Howard[2] showed that if the Richardson number for parallel flows of an inviscid stratified fluid is everywhere greater than 0.25, the flow must be stable. Later, Yih[3] showed that if the density $\bar{\rho}$ decreases monotonically and the velocity U increases monotonically upward, and if (i) $(\bar{\rho}U')'$ and $(ln\bar{\rho})''$ are positive throughout, the primes indicating differentiation with respect to the vertical distance y, or (ii) U'' and $(ln\bar{\rho})''$ are negative throughout, the flow is stable, even if the Richardson number is not everywhere (or even nowhere) greater than 0.25.

The study of stability of stratified flows is then reduced to the study of cases where the sufficient conditions for stability found by Miles, Howard, and Yih do not lead to the conclusion of stability. Calculations by Miles[4] for cases in which U varies monotonically and has a point of inflection fall into this category, and give neutral curves (stability boundaries) which are multivalued. This multivaluedness was explained by Yih[3] in connection with flows in which there is a point where $\bar{\rho}' = U'' = 0$. The calculations of Miles[4] were carried out with a good deal of virtuosity and understanding them requires some knowledge of the theory of hypergeometric functions. And yet neither Miles nor Yih explained exactly what happens to the wave velocity c as the neutral curves cross from the unstable to the stable side.

In this paper we consider a class of flows characterized by (i) a middle layer with constant density and linear velocity, (ii) a stably stratified upper layer with a constant velocity, (iii) a stably stratified lower layer with a constant velocity, and (iv) the density and the velocity are continuous throughout. We discuss the stability of this flow in a general way, and reach the important conclusions that instability is the result of a kind of resonance, that there are infinitely many modes which, in the neutral cases at least, are characterized by the number of internal zeros, and that for the flows considered here gravity is always destabilizing.

It seems that the conclusions to be drawn for the class of flows being considered will lead to a better understanding of the stability of stratified flows, for which the neutral curves, such as those given by Miles,[4] seem to defy any intuitive interpretation. Numerical results are given for exponential stratifications in the upper and lower layers, with the simplification afforded by the Boussinesq approximation.

Stability of stratified flows, in general, not subject to condition (i) is then considered, and it is shown that neutral normal modes which are continuations of complex-conjugate unstable and damped modes across the stability boundary do not exist.

II. THE MEAN FLOW

We shall consider a two-dimensional flow consisting of three layers. The indices 1, 2, and 3 are assigned to the lower, middle, and upper layers, respectively, and the depths of the layers will be denoted by d_1 d_2 and d_3. Cartesian coordinates x and y will be used, with y measured in the direction of the vertical, and with the origin at the midpoint of the middle layer. For convenience let

$$d_2 = 2d.$$

Then, the velocity in the middle layer is Vy/d, the velocity in the upper layer is V, and that in the lower layer is $-V$. The density in the middle layer is constant and is denoted by ρ_0. The mean density $\bar{\rho}$ in the upper or the lower layer is such that $\bar{\rho}' < 0$, but is otherwise unspecified until the numerical example is discussed.

III. THE DIFFERENTIAL SYSTEM

The differential equation governing stability is well known. However, in formulating the interfacial boundary conditions we need to use some of the equations leading to it; therefore, we give a very brief derivation of it. Let u and v be the components of the velocity perturbation, and let the perturbations in pressure and in density be denoted by p and ρ, respectively. Then, the linearized Euler's equations of motion are, with subscripts indicating partial differentiation,

$$\bar{\rho}(u_t + Uu_x + vU') = -p_x, \tag{1}$$

$$\bar{\rho}(v_t + Uv_x) = -p_y - g\rho. \tag{2}$$

The equation of continuity is, since the fluid is assumed incompressible,

$$u_x + v_y = 0, \tag{3}$$

which allows the use of a stream function ψ, in terms of which

$$u = \psi_y, \qquad v = -\psi_x. \tag{4}$$

The linearized equation of incompressibility is

$$\rho_t + U\rho_x + v\bar{\rho}' = 0. \tag{5}$$

Let

$$\psi = f(y) \exp[ik(x - ct)].$$

Then, from (4) and (5) we have

$$(u, v, \rho) = \left(f', -ikf, \frac{1}{U - c}\bar{\rho}'\right)\exp[ik(x - ct)], \tag{6}$$

and (1) gives

$$p = \bar{\rho}(cf' - Uf' + U'f) \exp[ik(x - ct)]. \tag{7}$$

Substitution of (6) and (7) into (2) gives

$$(\bar{\rho}f')' - \left(\frac{(\bar{\rho}U')'}{U - c} + k^2\bar{\rho} + \frac{g\bar{\rho}'}{(U - c)^2}\right)f = 0. \tag{8}$$

Let U and c be measured in units of a velocity scale V, $\bar{\rho}$ in units of a constant density ρ_0, y in units of a length scale d, and let

$$\alpha = kd.$$

Then f is measured in units of Vd, and (8) can be written as

$$(\bar{\rho}f')' - \left(\frac{(\bar{\rho}U')'}{U - c} + \alpha^2\bar{\rho} + \frac{N\bar{\rho}'}{(U - c)^2}\right)f = 0, \tag{9}$$

with (note that N is not the Brunt–Väisälä frequency)

$$N = gd/V^2. \tag{10}$$

For the mean velocity distribution specified in Sec. II, the interfacial conditions are, since U and $\bar{\rho}$ are continuous, (i) the continuity of f, (ii) the continuity of p. In dimensionless terms, the mean velocity is given by

$$U = 1 \text{ for } y \geq 1, \qquad U = y \text{ for } |y| \leq 1,$$
$$U = -1 \text{ for } y \leq -1. \tag{11}$$

The interfacial conditions are

$$f_1(-1) = f_2(-1),$$
$$(c + 1)f_1'(-1) = (c + 1)f_2'(-1) + f_2(-1), \tag{12}$$
$$f_2(1) = f_3(1),$$
$$(c - 1)f_3'(1) = (c - 1)f_2'(1 + f_2)(1). \tag{13}$$

The boundary conditions at the rigid boundaries are

$$f_1(-1 - a) = 0, \qquad f_3(1 + b) = 0, \tag{14}$$

where

$$a = d_1/d, \qquad b = d_3/d. \tag{15}$$

IV. THE RESONANCE THEORY OF INSTABILITY

Although Kelvin's[5] solution for the stability (Helmholtz instability) of a vortex sheet between two fluids of different densities already indicated the importance of resonance, it was Taylor[6] who first explicitly recognized the resonance of two wave trains in the shear flow of a stratified fluid to be the cause of instability. However, as far as resonant forcing is concerned, Taylor was discussing the flow of three homogeneous layers only. We shall, for the U specified in Sec. II, develop this resonance theory generally, for all modes, for the U and $\bar{\rho}$ specified in Sec. II. (Note that $\bar{\rho}$ is not specified in the bottom and top layers, except that $\bar{\rho}' < 0$.) When a wave train propagating against the part of the fluid moving with the greater velocity and another propagating with the part of the fluid moving with the smaller velocity have the same velocity in space, they can force each other and cause instability. Thus, the coalescence of two distinct neutral modes signifies the beginning of instability. We consider this mutual forcing to be a kind of resonance. We shall use the constant density ρ_0 in the middle layer to be the density scale, so that the dimensionless density in the middle layer is 1, or

$$\bar{\rho} = 1 \qquad \text{for } |y| \leq 1. \tag{16}$$

First, we note that at the interfaces $\bar{\rho}$ and U are continuous but U' is discontinuous. Integration of (9) in the Stieltjes sense across the interfaces produces the second conditions in (12) and (13), so that these are *natural* conditions, and the Sturm–Liouville theory can be applied to (9) and (14), without concern about (12) and (13), so long as $|c| \neq 1$, so that (9) is not singular. We now apply that theory.

We consider real values of c with $|c| < 1$, for Howard's[2] semicircle theorem tells us that the neutral modes contiguous with the unstable ones must have their c satisfying $|c| \leq 1$, and Miles[1] has ruled out the extreme values of U as possible values for c.

First, we note that for $N = 0$ (zero gravity) there may be one or two positive values of α^2 which enable a solution of (9) to satisfy the boundary conditions and interfacial conditions, since the term in (9) containing $(\rho U')'(U - c)^{-1}$, considered as a generalized function at the interfaces, has, for the U specified in Sec. II, the effect of making the coefficient of f in (9) positive. There can be no more than two modes for $N = 0$, the first having no internal node and the second only one. For if there were a third mode it would, according to the Sturm–Liouville theory, have two internal nodes, and either one of them would be in the bottom and top layers, or both would be in the middle layer. In either case the eigenfunction f in (9) would have two nodes in *one* of the three layers (remembering that f is zero on the solid boundaries), and this is quite impossible, since $N = 0$, and the coefficient of f in (9) is now always negative between these nodes, because no interface is crossed. We are mainly interested in the neutral curves in the N-α plane. All that we have said above means that, for the $\bar{\rho}$ and U specified in Sec. II, there may be one or two values of α (positive by choice) at which the neutral curves intersect the axis $N = 0$.

For higher modes we can always increase N. With the understanding that n may have to be greater than 2 for small α^2, for any given α and for a specified number $(n-1)$ of internal zeros of the eigenfunction, we can always choose an N and two corresponding c's so that the boundary conditions (14) are satisfied by f, the solution of (9), for we can make $-N\bar{\rho}'/(U-c)^2$ for the bottom or the top layers as large as we please by choosing $|c|$ very near 1. Let the N so chosen be denoted by N_n, and the two corresponding c's be denoted by c_{n1} and c_{n2}, the n always indicating the mode. Then, c_{n1} and c_{n2} are functions of N_n.

Still fixing n and α at any given value, we now increase N_n continuously. The Sturm–Liouville theory applied to (9) then leads to the conclusion that at a sufficiently large N_n no real c_n with $|c_n| < 1$ can exist. At some N_n the two eigenvalues c_{n1} and c_{n2} must coalesce and for greater values of N_n the eigenvalues of c must become complex. When c_{n1} coalesces with c_{n2}, the pair of values (α, N_n) must be on the stability boundary in the α-N plane, or the neutral curve. The coalescence of c_{n1} with c_{n1} (to make c_n a double eigenvalue) is a sort of resonance, for it occurs when two wave trains have the same wave velocity in a shear flow.

Note that for any given α, $n > 2$, and an N_n however small, we can always choose c_n such that

$$1 \gg 1 - |c_n|^2 > 0,$$

in order to satisfy (9) and (14). There are two values, c_{n1} and c_{n2}, for c_n. As n increases, c_{n1} is nearer and nearer 1 and c_{n2} nearer and nearer -1. As N_n is allowed to increase, c_{n1} will coalesce with c_{n2} for a value of N_n, as already explained. Thus, for any α and any integral value of n, at least from 3 onward, there is a positive N_n for which the flow is neutrally stable. That is to say, for any α there are infinitely many eigenvalues for N for neutral stability, and, correspondingly, infinitely many neutral modes.

The fact that for any n the value of N_n must increase to attain complex values of c_n shows that gravity is always destabilizing for the specified U and $\bar{\rho}$.

For more general distributions of U and $\bar{\rho}$ the situation is complicated by the following facts:

(a) If $\bar{\rho}'$ and U'' are not zero at the place where $U = c$, (9) is singular.

(b) Then, if the Richardson number at $U = c$ is less than 0.25, the eigenfunction, if one exists, must be one or the other of the two independent solutions of (9).

(c) As N is allowed to increase, there will be a limit for N beyond which the Richardson number is everywhere greater than 0.25, if $\bar{\rho}' < 0$ everywhere.

Item (c) indicates that if $\bar{\rho}' < 0$ everywhere there cannot be infinitely many neutral modes. There may be a few, or only one such mode, or none. The role of gravity will no longer always be stabilizing, as indicated by the solutions of Miles.[4]

V. AN EXAMPLE

For an example, we shall let

$$a = 2 = b,$$

and

$$\bar{\rho} = \begin{cases} \rho_0 \exp[-\beta(y-1)] \text{ for the top layer,} \\ \rho_0 \text{ for the middle layer,} \\ \rho_0 \exp[-\beta(y+1)] \text{ for the bottom layer,} \end{cases}$$

where y is in units of d, the half depth of the middle layer. Then, with U still specified by (11), and with f_1, f_2, and f_3 for f in the bottom, middle, and top layers, respectively, (9) has the forms

$$f_1'' - \beta f_1' - \left(\alpha^2 - \frac{\beta N}{(1+c)^2}\right) f_1 = 0, \tag{17a}$$

$$f_2'' - \alpha^2 f_2 = 0, \tag{18}$$

$$f_3'' - \beta f_3' - \left(\alpha^2 - \frac{\beta N}{(1-c)^2}\right) f_3 = 0. \tag{19a}$$

The interfacial conditions are still (12) and (13), and (14) now becomes

$$f_1(-3) = 0, \qquad f_3(3) = 0. \tag{20}$$

The differential system consisting of (12), (13), and (17a), (18), (19a), and (20) define an eigenvalue problem. For given α, β, and N, one can determine c from this system.

For simplicity we shall adopt the Boussinesq approximation and write (17a) and (19a) as

$$f_1'' - \left(\alpha^2 - \frac{\beta N}{(1+c)^2}\right) f_1 = 0, \tag{17b}$$

$$f_3'' - \left(\alpha^2 - \frac{\beta N}{(1-c)^2}\right) f_3 = 0. \tag{19b}$$

With the Boussinesq approximation, we can show that if c is an eigenvalue, so is $-c$. This is shown by making the transformation

$$\hat{y} = -y, \qquad \hat{f}_1(\hat{y}) = f_3(y), \qquad \hat{f}_3(\hat{y}) = f_1(y),$$
$$\hat{f}_2(\hat{y}) = f_2(y). \tag{21}$$

Substituting (21) into the system consisting of (12), (13), (17b), (18), (19b), and (20), and then dropping the circumflexes, we regain that system, except that c is replaced by $-c$. Thus, if c is an eigenvalue, so is $-c$.

This fact makes it clear that for any neutral mode (not necessarily corresponding to a stability boundary) and in the notation of the last section,

$$c_{n1} = -c_{n2}. \tag{22}$$

Thus, for any mode, to obtain the relationship between α

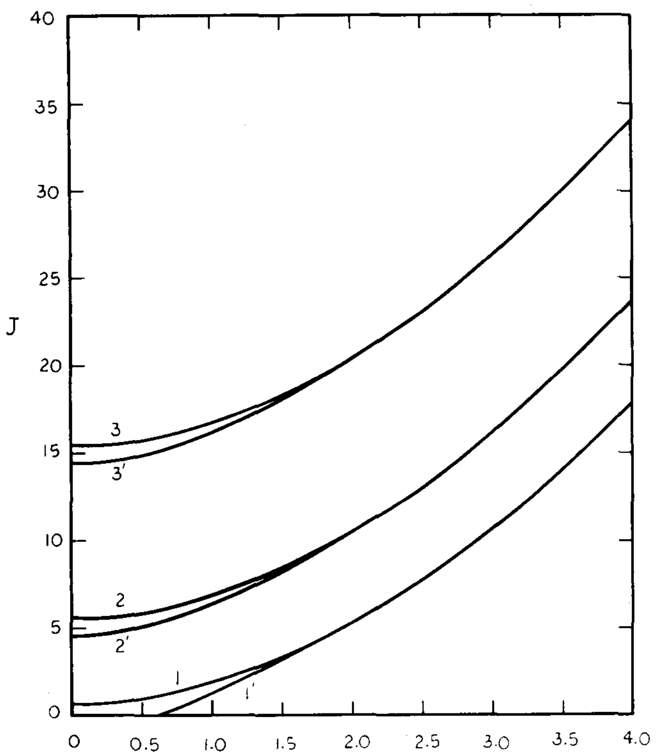

FIG. 1. Stability boundaries in the α–J plane for various modes. The nth mode for an odd eigenfunction is indicated by the number n, and the nth mode for an even eigenfunction is indicated by n'.

and βN (which are the only two parameters left aside from c) at the stability boundary, which corresponds to a double root in c, we can simply put c equal to zero, which is the value that an eigenvalue for c of multiplicity 2 must take. This simplifies matters a great deal. Putting c equal to zero and solving (17b), (18), and (19b) with interfacial conditions (12) and (13) and boundary conditions (20), we obtain the results,

$$(\alpha^2 - J)^{1/2} \coth 2(\alpha^2 - J)^{1/2} = 1 - a \tanh\alpha, \tag{23}$$

and

$$(\alpha^2 - J)^{1/2} \coth 2(\alpha^2 - J)^{1/2} = 1 - \alpha \coth\alpha, \tag{24}$$

representing two families of the stability boundary, with

$$J = \beta N. \tag{25}$$

Equation (23) corresponds to $f_2(y)$ being an even function of y and (24) corresponds to $f_2(y)$ being odd. Figure 1 shows the variation of J with α on the stability boundary. The curve for the nth mode for (24) is indicated by the number n. Only three odd modes are shown. The curves for the nth mode for (23) are indicated by n'. Again, only three modes are shown.

All the conclusions of the last section are borne out. There are infinitely many modes, gravity is always destabilizing for the assumed U and $\bar{\rho}$, and resonance is the cause of instability. At a point in the α-J plane above the curves 1 and 1′ but below 2′, there are two unstable modes, two damped modes, and infinitely many neutral modes. The enumeration of modes at other points is similar.

Note that although for $N = 0$, Eqs. (17b) and (19b) have the same form as (18), and all these have $\exp(\pm\alpha y)$ for solutions, the boundary conditions (20) can be satisfied because there are two interfaces at which U' is discontinuous, and (12) and (13) must be applied.

For each mode J increases with α. For any given α there are infinitely many values for J, but for any J there are only a finite number of values for α. As expected, increasing α stabilizes the flow.

In addition to the modes separated by the stability boundaries (23) and (24), there are two other infinite sets of real c values outside of the range of U. These are the neutral modes not contiguous to unstable modes, discussed by Yih.[3]

VI. GENERAL DISCUSSION

So far we have discussed only the density and velocity distributions specified in Sec. II, and given an example in Sec. V, for a further specified density distribution. The arguments in Sec. IV clearly remain valid even if there are no regions of constant velocities, provided there is a region of linear velocity and constant density, and provided there are neutral modes, with their c or c_r in this region, which coalesce in the manner described in Sec. IV. Of course, if there are no regions of constant velocity, there cannot be infinitely many unstable modes because there cannot be infinitely many cases of coalescence.

We now turn to the general case, in which there is no region of linear velocity and constant density. First, we shall now use the usual definition of the Richardson number

$$J = -g\bar{\rho}'/\bar{\rho}U'^2, \tag{26}$$

where all quantities on the right-hand side are dimensional. This definition is consistent with (25) for the density and velocity distributions considered in Secs. IV and V, and is the general definition of the Richardson number for any density and velocity distributions. Miles[4] has shown that a singular neutral mode cannot exist if the local value of J at $y = y_c$ where $U = c$, denoted by J_c, is greater than 0.25. We shall first dispose of the case in which the local J is equal to 0.25. By the method of Frobenius we obtain, in that case, the following two independent solutions of (9):

$$f_1 = (y - y_c)^{1/2} w_1, \tag{27}$$

where

$$w_1 = 1 + A(y - y_c) + \cdots,$$
$$A = \left[(1 + J)\frac{(\bar{\rho}U')'}{\bar{\rho}U'} - \frac{J\bar{\rho}''}{\bar{\rho}'} + \gamma(\ln\bar{\rho})'\right]_c, \qquad \gamma = \tfrac{1}{2}, \tag{28}$$

and

$$f_2 = f_1 \ln(y - y_c) - [2A + (\ln\bar{\rho})_c'](y - y_c)^{3/2} \times [1 + B(y - y_c) + \cdots]. \tag{29}$$

In (27) to (29), y is now supposed to be dimensionless, the prime indicates differentiation with respect to the dimensionless y, the subscripts c indicate that the quantity involved is evaluated at $y = y_c$, and B is a constant.

The Reynolds stress is defined by

$$\tau = -\bar{\rho}\overline{uv},$$

where the bar over uv means space average over x. In terms of f,

$$\tau = \frac{\rho_0 V^2}{2} \alpha (f'f^*)_i \exp(2\alpha c_i t), \tag{30}$$

in which the asterisk denotes the complex conjugate and the subscript i means the "imaginary part of." Suppose now that a singular neutral mode exists for $J_c = 0.25$, then the solution for f is

$$f = C_1 f_1 + C_2 f_2,$$

in which C_1 and C_2 are constants. If $C_2 \neq 0$, then one can easily show that τ suffers a jump when y crosses y_c. Now, f and f^* are also two independent solutions of (9) if c is real and C_1 and C_2 are not both real. If C_1 and C_2 are both real, then f and f^* are equal for $y > y_c$ and complex conjugate for $y < y_c$. In case C_1 and C_2 are not both real, f and f^* are independent solutions of (9), and the quantity

$$(f'f^*)_i = \frac{1}{2i}(f'f^* - ff'^*), \tag{31}$$

being proportional to the Wronskian of independent solutions of (9), is constant so long as the critical point is not crossed. Since it is zero at both solid boundaries (because $f = 0$ there), it cannot afford a jump at y_c if the velocity is monotonic. In case C_1 and C_2 are both real, the quantity (31) is zero for $y > y_c$ since f is real for $y > y_c$, whereas f and f^* are still two independent solutions for $y < y_c$, and the same argument applies. Thus, for $J_c = 0.25$ we can only have solution f_1, since a nonzero C_2 would give rise to a jump in the Reynolds stress. As we shall see presently, the form of f_1 in (27) is the limiting form of the solutions of (9) for $J_c < 0.25$, as J_c approaches 0.25. Thus, we need discuss only the case $J_c < 0.25$, and can treat the case $J_c = 0.25$ as a limiting case for $J_c < 0.25$.

Concentrating on the case $J_c < 0.25$, we recall that the solutions of (9) for real c in the range of U, given by Miles[4] by use of the Frobenius method, are

$$f_\pm = (y - y_c)^{(1\pm\nu)/2} w_\pm,$$

in which

$$w_\pm = 1 + A(y - y_c)/(1 \pm \nu) + \cdots, \tag{32}$$

with A given by (28), but with $\gamma = (1 \pm \nu)/2$ therein, and with

$$\nu = (1 - 4J_c)^{1/2}, \qquad J_c = J(y_c).$$

If $\bar{\rho}'$ and U'' are not both zero, then a look at (9) convinces one that it is singular if c is within the range of U, and the eigenfunction, if there is one, represents a singular neutral mode. Miles[4] showed that there are always unstable modes contiguous to a singular neutral mode, but he did not say what happens to the unstable modes as one crosses a stability boundary into the stable side. We now direct our attention to this question.

Miles[4] assumed U and $\bar{\rho}$ to be analytic, and showed that for a singular neutral mode the solution of (9), with vanishing f on the solid boundaries, must be either f_+ or f_-, never a combination of the two. This is to say, since y_c is not at the boundary, if f_+ is the solution considered (the argument for f_- is identical), then w_+ must vanish at both boundaries, where $y = y_1$ and $y = y_2$, respectively. Thus, for any given U and $\bar{\rho}$, and for a fixed α,

$$w_+(y_1, c, N) = 0 \tag{33}$$

and

$$w_+(y_2, c, N) = 0. \tag{34}$$

Let us assume U and $\bar{\rho}$ to be analytic, and consider a stability boundary in the α-N plane. For any singular neutral mode (33) and (34) must both be satisfied. They can at most be satisfied by discrete pairs of values of c (real) and N, for (33) and (34) are independent, and not the same relationship between c and N. Hence, when there is a genuine singular neutral mode, it is contiguous to unstable modes (or more precisely, to modes with complex c), but not contiguous to any other singular neutral modes. In other words, the modes with complex c cannot be continued into any neutral modes on the stable side of the stability boundary, where no contiguous normal modes exist. Thus, the significance of all normal-mode analysis of the stability of stratified flows depends on the implicit assumption that when no normal modes exist the flow is stable provided the density is statically stable. These points have not been recognized before by previous investigators.

It is appropriate to mention here that, in general, a given pair (α, N) which lies in the unstable region of the α-N plane can correspond to real values of c outside of the range of U, the existence of which has been discussed by Yih.[3]

The simultaneous solution of (33) and (34) gives discrete pairs of values for c and N. The number of such pairs may be infinite if there are regions of constant velocity, in which case there must, of course, be a limit point for c within the range of U.

Summarizing the results in this and foregoing sections, we state the following results in general terms:

(i) If there is a region of constant density and linear velocity between two stratified regions of constant but different velocities, such that U and $\bar{\rho}$ are continuous, there are infinitely many unstable modes each of which are contiguous to a neutral mode with c within the range of U. As the stability boundary is crossed, the modes with complex values of c (one of which corresponds to instability) are continued into two neutral modes with c real and equal to c_1 and c_2, with

$$c_1 < c_0 < c_2,$$

where c_0 is the c on the stability boundary. In this case instability is entirely due to resonance, and there are neutral modes with their c's within the range of U which are not contiguous to unstable modes. Of course, such modes are not really singular. Therefore, this result does not contradict Miles' statement[4] that singular neutral modes are contiguous to unstable modes.

Note also that if there is a region of constant density and linear velocity, but no regions of constant velocity, there may be a finite number of neutral modes on the stability boundary, with c in this region, resulting from coalescence of pairs of neutral modes with their c's in that region.

(ii) If there are no regions of constant density and linear velocity, there exist no neutral modes contiguous with the modes with complex c as the stability boundary is crossed and the region of stability entered, except the one right on the stability boundary.

(iii) In the α-N plane, aside from the eigenvalues of c divided by the stability boundaries when the resonance theory applies, or those whose existence is bounded by stability boundaries when it does not apply, any pair of values (α, N) may correspond to one or more nonsingular modes with real values of c outside of the range of U.

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