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Time-Dependent One-Speed Albedo Problem for a Semi-Infinite Medium

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A Laplace transformation technique is used to determine the neutron distribution in a semi-infinite medium which has been irradiated by a neutron pulse. The result is given in terms of known solutions of Milne's problem and of the steady-state albedo problem, which in turn are expressed by aid of Case's X-function. Simple asymptotic approximations, valid for $t \gg 1$, are deduced from the exact result.

I. INTRODUCTION

It is well known that time-dependent transport problems with given initial values can be formally converted to steady-state problems by Laplace transformation. In simple cases the transformed equation can be solved rigorously, e.g., by the singular eigenfunction method of Case.[1-3] Then the solution of the time-dependent problem is constructed by inverse Laplace transformation.

The indicated method has been used by Bowden[4] for a problem with slab geometry, the general aspects of which problem have been clarified previously by Lehner and Wing.[5] A slightly different approach has been used by Case[1] for an infinite medium with a pulsed plane source. It seems worthwhile to extend these investigations also to the semi-infinite medium, in which case several explicit results can be deduced.

We restrict our attention to the one-speed equation with isotropic scattering and seek the neutron distribution everywhere in an infinite half-space following irradiation of the surface with a monodirectional pulse of neutrons at $t=0$. The appropriate equation (using units in which $\sigma = v = 1$) is[1]

$$\psi(x, \mu, t; \mu_0) + \frac{\partial \psi}{\partial t} + \mu \frac{\partial \psi}{\partial x} = \frac{c}{2} \int_{-1}^{1} \psi(x, \mu', t; \mu_0)\, d\mu', \tag{1a}$$

where $x \geq 0$, $\mu_0 > 0$, and the boundary and initial conditions are

$$\psi(0, \mu, t; \mu_0) = \mu_0^{-1} \delta(\mu - \mu_0)\delta(t) \quad \text{for} \quad \mu > 0, \tag{1b}$$

$$\psi(x, \mu, t; \mu_0) \to 0 \quad \text{for} \quad x \to \infty, \tag{1c}$$

and

$$\psi(x, \mu, t; \mu_0) = 0 \quad \text{for} \quad t < 0.$$

We shall also be interested in the distribution

$$\psi^*(x, \mu, t) = \int_0^1 \psi(x, \mu, t; \mu_0)\mu_0\, d\mu_0, \tag{2}$$

produced by a pulsed isotropic incident distribution.

Finally, we shall need the values of the neutron densities and net currents, defined by

$$\rho(x, t; \mu_0) = \int_{-1}^{1} \psi(x, \mu, t; \mu_0)\, d\mu, \tag{3}$$

$$j(x, t; \mu_0) = \int_{-1}^{1} \psi(x, \mu, t; \mu_0)\mu\, d\mu, \tag{4}$$

(and similarly for ρ^*, j^*). For convenience the factor 2π has been omitted here, which can be justified by saying that ψ represents the angular density integrated over the azimuth.

Certain general properties of the solution are immediately apparent. First, we notice that the pulse initiates some transient discontinuities in the neutron distribution. Evidently $\psi = 0$ for $x > t$, since the neutrons enter the medium with a speed which is unity in the present notation. Moreover, a term $\delta(x - \mu_0 t)\delta(\mu - \mu_0)e^{-t}$, describing the distribution of the uncollided neutrons, is contained in ψ. However, all such singularities die out exponentially, and ψ becomes a smooth function for $t \gg 1$.

Second, according to a reciprocity theorem,[6] the following relation for the angular density of the reflected neutrons must hold

$$\psi(0, -\mu, t; \mu_0) = \psi(0, -\mu_0, t; \mu), \qquad \mu > 0. \tag{5}$$

Finally, for an absorbing medium ($c < 1$), we expect that the decay of the neutron distribution is governed mainly by the true absorption rate, i.e.,

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[1] K. M. Case, Ann. Phys. (N. Y.) **9**, 1 (1960).

[2] K. M. Case, *Recent Developments in Neutron Transport Theory* (Michigan Memorial Phoenix Project, 1961).

[3] K. M. Case and P. F. Zweifel, *Neutron Transport Theory* (to be published).

[4] R. L. Bowden, thesis, Virginia Polytechnic Institute (Report TID 18 884, 1963). See also: R. L. Bowden and C. D. Williams, J. Math. Phys. **5**, 1527 (1964).

[5] G. M. Wing, *An Introduction to Transport Theory* (J. Wiley & Sons, Inc., New York, 1962).

[6] L. M. Biberman and B. A. Veklenko, Zh. Eksperim. i Teor. Fiz. **39**, 88 (1960) [English transl.: Soviet Phys.—JETP **12**, 64 (1961)].

ψ should be roughly proportional to $e^{-(1-c)t}$. After an appropriate substitution is made, Eq. (1a) shows that

$$\psi^{(c)}(x, \mu, t; \mu_0) = ce^{-(1-c)t}\psi^{(1)}(cx, \mu, ct; \mu_0), \qquad (6)$$

where the value of c is indicated by a superscript. Hence it is sufficient to study the problem for a nonabsorbing medium ($c = 1$), and therefore the subsequent discussion will be limited to this case only.

Following Lehner and Wing[5] and Bowden,[4] we multiply both sides of (1a)–(1c), where now $c = 1$, by $e^{(1-s)t}dt$, and integrate from 0 to ∞. The integral converges for Re $(s) > 1$, and the transform

$$\psi_s(x, \mu; \mu_0) = \int_0^\infty \psi(x, \mu, t; \mu_0)e^{(1-s)t}\, dt \qquad (7)$$

is found to obey the equation

$$s\psi_s(x, \mu; \mu_0) + \mu \frac{\partial \psi_s}{\partial x} = \frac{1}{2}\int_{-1}^{1} \psi_s(x, \mu'; \mu_0)\, d\mu', \qquad (8a)$$

with the boundary conditions

$$\psi_s(0, \mu; \mu_0) = \mu_0^{-1}\delta(\mu - \mu_0) \quad \text{for} \quad \mu > 0, \qquad (8b)$$

and

$$\psi_s(x, \mu; \mu_0) \to 0 \quad \text{for} \quad x \to \infty. \qquad (8c)$$

From $\psi_s(x, \mu; \mu_0)$ the solution of the time-dependent problem will be computed by inverse Laplace transformation,

$$\psi(x, \mu, t; \mu_0) = \lim_{\omega\to\infty} \frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} \psi_s(x, \mu; \mu_0)e^{-(1-s)t}\, ds, \qquad (9)$$

where $\gamma > 1$. However, before carrying out this inverse transformation it seems advisable to modify it in the usual way by shifting and bending the path of integration as far as possible to the left in the complex s-plane. In order to be able to do this we first must check the analyticity properties of $\psi_s(x, \mu; \mu_0)$ as a function of s. We shall start with explicit expressions for this function.

II. PROPERTIES OF THE TRANSFORM OF THE SOLUTION

According to Eqs. (8a)–(8c), the function $\psi_s(x,\mu;\mu_0)$ coincides with the solution of the steady-state albedo problem, normalized to unit ingoing net current, for a semi-infinite medium with a macroscopic total cross section s and a macroscopic scattering cross section equal to unity. This problem has been solved by Case, and we can copy his results, at least for real s. The only novelty encountered with the present problem lies in the necessity of performing an analytical continuation to complex values of s. We shall postpone this task temporarily, and start with the assumption that s is real and > 1.

Besides $\psi_s(x, \mu; \mu_0)$ we shall need later on the solution $\psi_s(x, \mu)$ of Milne's problem, which is defined by an equation like (8a), and the boundary conditions

$$\psi_s(0, \mu) = 0 \quad \text{for} \quad \mu > 0, \qquad (10a)$$

$$\psi_s(x, \mu) < O(e^{sx}) \quad \text{for} \quad x \to \infty. \qquad (10b)$$

In both cases we shall be interested also in the neutron densities and net currents, which will be denoted by $\rho_s(x; \mu_0)$, $\rho_s(x)$ and $j_s(x; \mu_0)$, $j_s(x)$, respectively. We normalize the solution of Milne's problem to unit emerging net current: $j_s(0) = -1$.

All these quantities can be expressed in terms of Case's X-function, or equivalently, in terms of Chandrasekhar's H-function,[7] $H(\mu, s) = [(1 - s^{-1})^{\frac{1}{2}}(\nu_0 + \mu)X(-\mu, s)]^{-1}$. In Case's notation the formulas for $\psi_s(x, \mu; \mu_0)$, etc., are[2,3,7,8]

$$\psi_s(x, \mu; \mu_0) = \frac{1}{(\nu_0 + \mu_0)X(-\mu_0, s)} \times \left\{\frac{4sX(-\nu_0, s)}{\nu_0\Lambda_s'(\nu_0)}\phi_{s+}(\mu_0)\phi_{s+}(\mu)e^{-sx/\nu_0} + \int_0^1 \frac{(\nu_0 + \nu)X(-\nu, s)}{\nu\Lambda_s^+(\nu)\Lambda_s^-(\nu)}\phi_{s\nu}(\mu_0)\phi_{s\nu}(\mu)e^{-sx/\nu}\, d\nu\right\}, \qquad (11)$$

$$\psi_s(0, -\mu; \mu_0) = [2(s - 1)(\mu + \mu_0)(\nu_0 + \mu_0) \times X(-\mu_0, s)(\nu_0 + \mu)X(-\mu, s)]^{-1}, \qquad \mu \geq 0, \qquad (12)$$

$$\rho_s(0; \mu_0) = [(1 - s^{-1})^{\frac{1}{2}}\mu_0(\nu_0 + \mu_0)X(-\mu_0, s)]^{-1}, \qquad (13)$$

$$j_s(0; \mu_0) = [(\nu_0 + \mu_0)X(-\mu_0, s)]^{-1}, \qquad (14)$$

$$\psi_s(x, \mu) = \frac{1}{2(1 - s^{-1})\nu_0^2}\left[\frac{1}{X(\nu_0, s)}\phi_{s+}(\mu)e^{-sx/\nu_0} + \frac{1}{X(-\nu_0, s)}\phi_{s-}(\mu)e^{sx/\nu_0}\right] - \frac{1}{2s}\int_0^1 \frac{X(-\nu, s)}{\Lambda_s^+(\nu)\Lambda_s^-(\nu)}\phi_{s\nu}(\mu)e^{-sx/\nu}\, d\nu, \qquad (15)$$

$$\psi_s(0, -\mu) = [2(s - 1)(\nu_0^2 - \mu^2)X(-\mu, s)]^{-1}, \qquad \mu \geq 0, \qquad (16)$$

$$\rho_s(0) = [(1 - s^{-1})^{\frac{1}{2}}\nu_0]^{-1}. \qquad (17)$$

The following functions appear in the above formulas:

$$X(z, s) = \frac{1}{1 - z}\exp\left[\frac{1}{2\pi i}\int_0^1 \ln\frac{\Lambda_s^+(\nu)}{\Lambda_s^-(\nu)}\frac{d\nu}{\nu - z}\right] \qquad (18)$$

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[7] S. Chandrasekhar, *Radiative Transfer* (Oxford University Press, London and New York, 1950).
[8] I. Kuščer, Can. J. Phys. **31**, 1187 (1953).

(with the integrand $= 0$ at $\nu = 0$),

$$\Lambda_s(z) = 1 - (z/s)\tanh^{-1}(1/z) \tag{19a}$$

(defined in the complex plane, cut along $-1 < z < 1$),

$$\Lambda_s^{\pm}(\nu) = \lambda_s(\nu) \pm \pi i\nu/2s \quad \text{for} \quad -1 < \nu < 1, \tag{19b}$$

$$\lambda_s(\nu) = 1 - (\nu/s)\tanh^{-1}\nu, \tag{20}$$

$$\pm\nu_0(s) = \text{roots of } \Lambda_s(\nu_0) = 0,$$

$$\Lambda_s'(\nu_0) = \nu_0 s^{-1}(\nu_0^2 - 1)^{-1} - \nu_0^{-1}, \tag{21}$$

$$\phi_{s\pm}(\mu) = \frac{\nu_0}{2s}\frac{1}{\nu_0 \mp \mu}, \tag{22}$$

$$\phi_{s\nu}(\mu) = \frac{\nu}{2s} P\frac{1}{\nu - \mu} + \lambda_s(\nu)\,\delta(\nu - \mu), \tag{23}$$

where P indicates that we have to take the Cauchy principal value of any integral over ν or μ of the expression $1/(\nu - \mu)$ following that symbol. The integral in (11), with the two singularities of the integrand merging when $\mu \to \mu_0$, has to be understood in the same sense as with the orthogonality relation

$$\int_{-1}^{1} \phi_{s\nu}(\mu)\phi_{s\nu'}(\mu)\mu\, d\mu = \nu\Lambda_s^{+}(\nu)\Lambda_s^{-}(\nu)\,\delta(\nu - \nu'), \tag{24}$$

used in full-range developments.[1-4] It can then be seen that the right-hand side of Eq. (11) contains the discrete term $\mu_0^{-1}\delta(\mu - \mu_0)e^{-sx/\mu_0}$, corresponding to the uncollided neutron beam.

The neutron densities and net currents, belonging to (11) and (15), follow immediately if we observe that $\phi_{s\pm}$ and $\phi_{s\nu}$ are normalized to unit density, and that the corresponding net currents are $\pm(1 - s^{-1})\nu_0$ and $(1 - s^{-1})\nu$, respectively.

We may introduce the "extrapolation distance" $q(s)$ and another parameter $Q(s)$ by

$$-X(\nu_0, s)/X(-\nu_0, s) = e^{2q/\nu_0}, \tag{25}$$

$$-X(\nu_0, s)X(-\nu_0, s) = \frac{\Lambda_s'(\nu_0)}{2\nu_0(1 - s^{-1})} = \frac{1}{Q^2\nu_0^6(1 - s^{-1})^2}, \tag{26}$$

with the purpose of expressing $\rho_s(x)$ in a shorter form

$$\rho_s(x) = Q\nu_0 \sinh\frac{sx + q}{\nu_0} - \frac{1}{2s}\int_0^1 \frac{X(-\nu, s)}{\Lambda_s^{+}(\nu)\Lambda_s^{-}(\nu)} e^{-sx/\nu}\, d\nu. \tag{27}$$

When $s \to 1$ one should use the well-known approximation

$$\nu_0 \approx [3(s - 1)]^{-\frac{1}{2}}, \tag{28}$$

which leads to $Q(1) = 3$, whereas $q(1) = 0.71045$.

Let us now turn to complex values of s. By retracing the derivation of Eqs. (11)–(17) one verifies that they remain valid so long as s is such that $\Lambda_s(z)$ has a pair of zeros. The condition for this to happen is, according to Bowden,[4] that s belongs to a certain region S_i of the complex plane, as shown by Fig. 1. This region is the conformal map, produced by the function $s = \nu_0 \tanh^{-1}(1/\nu_0)$, of the (say) right-hand half of the complex plane of ν_0, cut as mentioned before. Hence the boundary C of S_i is the (double) conformal map of half of this cut.

The analytic behavior of $\psi_s(x, \mu; \mu_0)$ inside the region S_i of the complex s-plane is linked to the properties of $\nu_0(s)$. This is the inverse of the previously mentioned function $s = \nu_0 \tanh^{-1}(1/\nu_0)$, and its values can be read from the quoted figure. We see there that the point $s = 1$ is a branch point of $\nu_0(s)$, as shown also by the approximation (28). Hence, if we want ν_0 to be uniquely determined for $s \in S_i$, a cut has to be drawn in the s-plane, most conveniently to the left of that point. If we chose ν_0 to be the particular root which is positive for $s > 1$, then $\mathrm{Re}(\nu_0) \geq 0$ in the whole cut region S_i.

Expression (11) shows that inside S_i the function $\psi_s(x, \mu; \mu_0)$ is regular in s, except for the branch cut ($0 \leq s \leq 1$) due to $\nu_0(s)$. The reason why this cut is inherited by $\psi_s(x, \mu; \mu_0)$ is that, by definition, only one of the discrete eigenfunctions, $\phi_{s+}(\mu)e^{-sx/\nu_0}$, is involved in the expansion (11). Consequently, when s approaches the branch cut from above or below, two different limits ψ_s^+ and ψ_s^- are obtained, involving the negative and positive imaginary ν_0 in (11), respectively. Since ψ_s^+ and ψ_s^- both are solutions of Eqs. (8a, b), the difference $\psi_s^+ - \psi_s^-$ is a solution of the corresponding homogeneous problem, i.e., of Milne's problem. (For Milne's problem $s = 1$ is no branch point because going around this point merely interchanges the two discrete terms in (15) and leaves the sum unchanged.) Taking account of the value (14) of the net current at the surface we find that

$$\begin{aligned}\psi_s^+(x, \mu; \mu_0) - \psi_s^-(x, \mu; \mu_0) &= -2i\,|\nu_0|\,[(|\nu_0|^2 + \mu_0^2)X(-\mu_0, s)]^{-1}\psi_s(x, \mu) \\ &= -4i(1 - s)\,|\nu_0|\,\psi_s(0, -\mu_0)\psi_s(x, \mu), \quad 0 \leq s \leq 1.\end{aligned} \tag{29}$$

If s is in the external region S_e (Fig. 1), the situation is different because $\Lambda_s(z)$ then has no zero and

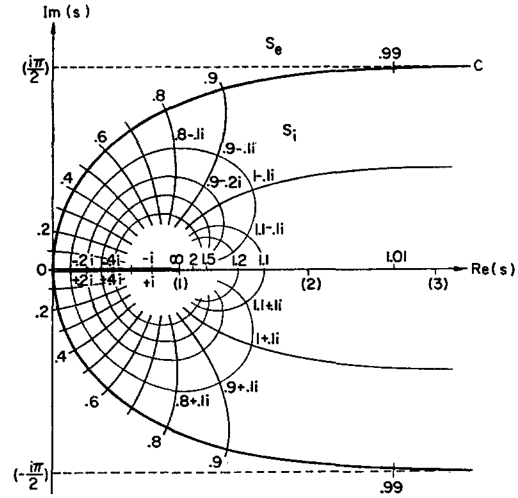

FIG. 1. Values of $\nu_0(s)$ plotted in the complex plane of s. The values of s are given in parentheses.

the corresponding discrete term in the expansion is missing. By the use of $X_0(z, s) = (1 - z)X(z, s)$ instead of $X(z, s)$ the following formulas are obtained for this case:

$$\psi_s(x, \mu; \mu_0) = \frac{1}{X_0(-\mu_0, s)} \times \int_0^1 \frac{X_0(-\nu, s)}{\nu \Lambda_s^+(\nu) \Lambda_s^-(\nu)} \phi_{s\nu}(\mu_0)\phi_{s\nu}(\mu) e^{-sx/\nu}\, d\nu, \qquad (30)$$

$$\psi_s(0, -\mu; \mu_0) = [2(s - 1)(\mu + \mu_0) \times X_0(-\mu_0, s)X_0(-\mu, s)]^{-1}, \qquad \mu \geq 0. \qquad (31)$$

Similarly as for $s \in S_i$, we now conclude from (30) that $\psi_s(x, \mu; \mu_0)$ is regular in s also for $s \in S_e$, Re $(s) \geq 0$. However, we are still uncertain about what happens when s crosses the boundary C separating the two regions.

One way to assure the analyticity of $\psi_s(x, \mu; \mu_0)$ across C would be to extend the existence theorems, worked out by Lehner and Wing for the slab case,[4,5] to the semi-infinite medium. An alternative method, chosen in the following, consists in the comparison of the limits of the explicit expressions (11) and (30), when s approaches the line C from one or the other side. First however, we have to insert a discussion about the X-function, which itself is discontinuous at $s \in C$.

It can be inferred from the definition (18) that the change of $X(z, s)$, as s crosses C, is expressed by

$$[(\nu_0 - z)X(z, s)]_{\substack{\text{inside}\\ \text{limit}}} = [(1 - z)X(z, s)]_{\substack{\text{outside}\\ \text{limit}}}. \qquad (32)$$

Thus, $(\nu_0 - z)X(z, s)$ for $s \in S_i$ and $(1 - z)X(z, s)$ for $s \in S_e$ represent one and the same analytic function, which we may denote by $X_0(z, s)$, if the definition is adapted as follows:

$$X_0(z, s) = \begin{cases} (1 - z)X(z, s) & \text{for } s \in S_e, \\ (\nu_0 - z)X(z, s) & \text{for } s \in S_i, \end{cases} \qquad (33)$$

with $X(z, s)$ given by (18).

It may be mentioned that the analyticity of $X_0(z, s)$ in both variables is obvious from a complex representation, which in a different form has been given by Chandrasekhar,[7] and which also readily ensues from the above definition:

$$\ln X_0(z, s) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{-i\infty}^{i\infty} \ln \frac{\Lambda_s(z')}{\Lambda_s(\infty)} \frac{dz'}{z' - z} + \ln \frac{\Lambda_s(z)}{\Lambda_s(\infty)}, & \text{Re } (z) > 0 \\ \dfrac{1}{2\pi i} \displaystyle\int_{-i\infty}^{i\infty} \ln \frac{\Lambda_s(z')}{\Lambda_s(\infty)} \frac{dz'}{z' - z}, & \text{Re } (z) < 0, \end{cases} \qquad (34)$$

with $\Lambda_s(\infty) = 1 - s^{-1}$. For fixed s the function $X_0(z, s)$ has no singularity in the z-plane, cut along $(0, 1)$, and it has one simple zero at $z = \nu_0(s)$, Re $(\nu_0) \geq 0$, only if $s \in S_i$. The zero disappears by crossing the cut when s crosses the boundary C. Vice versa, for fixed z there is no singularity in the s-plane, cut along $(0, 1)$, and for Re $(z) > 0$ there is only one zero at $s = z \tanh^{-1} (1/z)$. The zero disappears by crossing the cut when z crosses the imaginary axis.

We return now to the problem of the behavior of $\psi_s(x, \mu; \mu_0)$ at $s \in C$. For the case $x = 0$, it is immediately clear, in view of Eq. (33), that (31) is an analytical continuation of (12). For $x > 0$, an apparent difficulty arises from the discontinuity of the individual terms in (11) and (30), when s crosses C. Moreover, certain terms have poles at those values of s which make $\nu_0(s)$ equal to μ or μ_0. However, a closer inspection proves that all these singularities cancel each other, so that $\psi_s(x, \mu; \mu_0)$ is, in fact, continuous across C, and consequently regular in the whole right-hand half-plane of s, cut along $0 \leq s \leq 1$. This is what we wanted to know.

The tedious term-by-term comparison of (11) with (30) can be avoided by transforming both expressions into a unique complex representation, from which the analyticity in s is evident:

$$\psi_s(x, \mu; \mu_0) = \mu_0^{-1}\delta(\mu - \mu_0)e^{-sx/\mu_0} - \frac{1}{2sX_0(-\mu_0, s)} \frac{1}{2\pi i} \times \int \frac{X_0(-z, s)}{\Lambda_s(z)} \frac{1}{(z - \mu)(z - \mu_0)} e^{-sx/z}\, dz. \qquad (35)$$

The integration over z is carried out along a contour which starts and ends at $z = 0$, with $\mathrm{Re}\,(z/s) \geq 0$ at $z \to 0$, and embraces the branch cut $0 < z < 1$, as well as the pole $z = \nu_0$ of the integrand.

We see, by the way, that the discrete term in (11) is due to the residue of the integrand in (35) at $z = \nu_0$, and that the expressions (12), (31) stem from the residue at $z = \mu$.

III. FINAL FORM OF THE SOLUTION

The above conclusions permit us to deform the integration path in (9) as shown by Fig. 2. Thereby and by the use of relation (29) the integral in (9) is put into a more convenient form,

$$\psi(x, \mu, t; \mu_0) = \frac{2}{\pi}\int_0^1 (1 - s)\,|\nu_0(s)|\,\psi_s(0, -\mu_0)\psi_s(x, \mu)e^{-(1-s)t}\,ds + \lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{-i\omega}^{i\omega}\psi_s(x, \mu; \mu_0)e^{-(1-s)t}\,ds. \quad (36)$$

This, with the expressions (16), (15), and (30) substituted, represents the final result. Expressions for $\rho(x, t; \mu_0)$ and $j(x, t; \mu_0)$ follow immediately.

For $x = 0$ a further simplification is possible because the expression (31) can be analytically continued to $\mathrm{Re}\,(s) < 0$, which for $x > 0$ was impossible, because of the factor $e^{-sx/\nu}$ in the integrand in (30). Now the integration path can be bent still further to the left, and we end up with a closed loop encircling the branch cut. This means that the last term in (36) drops out for $x = 0$, so that

$$\psi(0, -\mu, t; \mu_0) = \frac{2}{\pi}\int_0^1 (1 - s)\,|\nu_0(s)| \times \psi_s(0, -\mu_0)\psi_s(0, -\mu)e^{-(1-s)t}\,ds, \quad \mu \geq 0. \quad (37)$$

The validity of the reciprocity relation (5) is clearly demonstrated. The values of $X(-\mu, s)$ involved in $\psi_s(0, -\mu)$ through Eq. (16) can be taken from graphs presented by Bowden.[4]

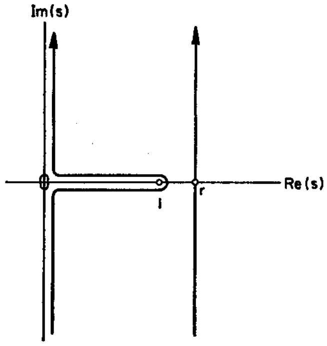

FIG. 2. Integration paths for inverse Laplace transformation of $\psi_s(x, \mu; \mu_0)$.

The neutron density and the net current at the surface of the medium are obtained from (37) by substituting the factor $\psi_s(0, -\mu)$ of the integrand by the expression (17) and by $j_s(0) = -1$, respectively, and by adding the contribution due to the incident neutrons, $\rho_{\mathrm{inc}}(0, t; \mu_0) = \mu_0^{-1}\delta(t)$, $j_{\mathrm{inc}}(0, t; \mu_0) = \delta(t)$. Especially simple formulas follow for the case of isotropic angular distribution of the incident neutrons:

$$\rho^*(0, t) = \delta(t) + t^{-1}e^{-t/2}I_1(\tfrac{1}{2}t) \quad (38a)$$

$$= \delta(t) + \frac{1}{4}\left[1 - \frac{1}{2}t + \frac{5}{32}t^2 - \cdots\right] \quad (38b)$$

$$= (\pi t^3)^{-\frac{1}{2}}\left[1 - \frac{3}{4}t^{-1} - \frac{15}{32}t^{-2} + O(t^{-3})\right], \quad (38c)$$

$$j^*(0, t) = \tfrac{1}{2}\delta(t) - \pi^{-1}\int_0^1 2(1 - s)\,|\nu_0(s)|\,e^{-(1-s)t}\,ds \quad (39a)$$

$$= \tfrac{1}{2}\delta(t) - \tfrac{1}{3}(1 - \ln 2) + O(t) \quad (39b)$$

$$= -(3\pi t^3)^{-\frac{1}{2}}[1 - (27/20)t^{-1} + O(t^{-2})]. \quad (39c)$$

The formula for $\rho^*(0, t)$ has been reduced to an expression containing the modified Bessel function I_1 by aid of the substitution $s = \frac{1}{2}(1 - \cos\theta)$, which leads to Poisson's integral representation for this function.[9]

The initial values of the reflected angular density could be computed from Eq. (37) by substituting $t = 0$. However, an easier way is to expand the previously mentioned closed-loop integration path into a very large circle, instead of shrinking it onto the branch cut. Observing that $X_0(z, s) = 1 + O(s^{-1})$ for $s \to \infty$, as can easily be shown, we obtain, using (31),

$$\psi(0, -\mu, 0; \mu_0) = [2(\mu + \mu_0)]^{-1}, \qquad \mu \geq 0. \quad (40)$$

This angular density is entirely due to neutrons scattered only once, as one can infer directly from the transport equation.

IV. DISCUSSION

The above results closely resemble those obtained by Bowden for the slab problem. The main difference is that in the latter case two discrete terms, involving the factors $e^{\pm sx/\nu_0}$, enter a development analogous to (11). Therefore the function $\psi_s(x, \mu; \mu_0)$ for the slab needs no branch cut, but has instead a finite number of poles at certain "critical" values of s inside the interval $0 < s < 1$. The poles fill

[9] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II.

up this interval more and more densely as the thickness of the slab is increased.

Thus in the slab case the integral over the branch cut in (36) is replaced by a sum over the residues. Actually, Eq. (36) can be deduced as a limit from Bowden's result [Ref. 4, Eq. (5.12)]. This is done by proving that the factor $(2/\pi)(1-s)\,|\nu_0(s)|$ of the first integrand in (36) is equal to the limit of the product of the pole number density and a normalization factor.

The individual terms of the mentioned sum, just as the integrand of the branch-cut term in (36), can be pictured by standing waves decaying at various rates, slower than e^{-t}. Each wave corresponds in the slab case to a solution of the critical problem, and in the present case to a solution of Milne's problem for a multiplying medium.

The last term in (36), when $\psi_s(x, \mu; \mu_0)$ is developed according to Eq. (30), represents a sum over a continuous family of traveling waves,[10] all decaying like e^{-t}, i.e., with a decay time equal to the mean time between collisions of a neutron. Only ingoing waves, with speeds ν ranging from 0 to 1, are present in the case of a semi-infinite medium, whereas waves propagating in both directions are included in the slab solution. As shown by Eq. (37) those waves do not contribute to the angular density of the neutrons reflected by a semi-infinite medium.

In view of the fast decay rate of the traveling waves we may say that their sum describes the transient effects mentioned in the introduction. Actually this sum contains the uncollided beam term $\delta(x-\mu_0 t)\delta(\mu-\mu_0)e^{-t}$, since the Laplace transform of this term, $\mu_0^{-1}\delta(\mu-\mu_0)e^{-sx/\mu_0}$, is contained in $\psi_s(x, \mu; \mu_0)$.

On the other hand, one expects the branch-cut term in (36) alone to describe the behavior of ψ for large values of t, so that this term represents an asymptotic approximation. Some simplification, consistent with this kind of approximation, can be achieved by using (28) and by substituting $\psi_s(0, -\mu_0) \approx \psi_1(0, -\mu_0)$, and, for small x only, $\psi_s(x, \mu) \approx \psi_1(x, \mu)$. An expression results, which contains the integral $\int_0^1 (1-s)^{\frac{1}{2}} e^{-(1-s)t}\, ds$. For $t \gg 1$ it is permissible to shift the lower limit of this integral to $-\infty$. Then, with (16), the expression simplifies to

$$\psi(x, \mu, t; \mu_0) \approx (\tfrac{4}{3}\pi t^3)^{-\frac{1}{2}}[X(-\mu_0, 1)]^{-1}\psi_1(x, \mu). \quad (41)$$

Approximations for ρ, j, and for $\psi(0, -\mu, t; \mu_0)$ follow in a simple way upon application of the formulas (17), and (16), with (28).

In a similar way, by substituting the asymptotic part of (15) into (36), we arrive at a different asymptotic approximation, valid for $t \gg 1$, $x \gg 1$. Let us write down only the expression for the neutron density, which follows from the asymptotic part of (27):

$$\rho(x, t; \mu_0) \approx 3(\tfrac{4}{3}\pi t^3)^{-\frac{1}{2}}[X(-\mu_0, 1)]^{-1}[x + q(1)] \times \exp\{-\tfrac{3}{4}t^{-1}[x+q(1)]^2\}. \quad (42)$$

Tables of $H(\mu, 1) = \sqrt{3}/X(-\mu, 1)$ and of $\rho_1(x)$, needed for the evaluation of $\psi(0, -\mu, t; \mu_0)$ and $\rho(x, t; \mu_0)$, according to the approximations (41) and (42), are available.[7,11]

Various refined asymptotic approximations could be conceived by making less crude substitutions for the functions involved in the exact expressions. For instance, we observe that the factor $(1-s)^{\frac{1}{2}}e^{-(1-s)t}$ of the first integrand in (36) is zero at $s = 1$. Hence it seems advisable to approximate the remaining (finite) factor $(1-s)^{\frac{1}{2}}\,|\nu_0(s)|\,\psi_s(0, -\mu_0)\psi_s(x, \mu)$ by its value at s slightly below 1, rather than at $s = 1$. We may require that this procedure should be correct if the latter factor were a linear function of s. We find then that for $t \gg 1$ the appropriate value of s is $1 - \frac{3}{2}t^{-1}$. This, with (15) and (16), has to be inserted into

$$\psi(x, \mu, t; \mu_0) \approx (\pi t^3)^{-\frac{1}{2}}(1-s)^{\frac{1}{2}}\,|\nu_0(s)|\,\psi_s(0, -\mu_0)\psi_s(x, \mu), \quad (43)$$

which is valid for $t \gg 1$, $x \ll t^{\frac{1}{2}}$, as one can show. The improvement of (43) over (41) can be judged from the fact that the first two terms in (38c) and (39c) follow from (43), whereas only the first term is obtained from (41).

It should be mentioned that the approximations (41)–(43) can be deduced also without knowing the exact result, solely by considerations based upon the diffusion equation and upon a reciprocity theorem. Such a derivation, though not rigorous, has the advantage of being amenable to generalizations to anisotropic scattering and to energy-dependent problems.

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[10] A. M. Weinberg and E. P. Wigner, *The Physical Theory of Neutron Chain Reactors* (University of Chicago Press, Chicago, 1958), p. 235.

[11] C. Mark, Phys. Rev. **72**, 558 (1947).