Structure of the 12j and 15j coefficients in the Bargmann approach

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(Received 19 November 1973; final revised manuscript received 4 March 1974)

Generating functions of the 12j and 15j angular momentum recoupling coefficients are computed explicitly in the Bargmann formalism. Symmetry properties are deduced therefrom. A geometrical Möbius strip representation (originally due to Ord-Smith for the 12j case), which can be generalized to all n, suggests a $4n$-fold symmetry for the $3nj$ coefficients ($n \geq 4$).

I. INTRODUCTION

The structure of the angular momentum 9j coefficient[1-3] has been studied in the Bargmann approach.[4,5] It is the purpose of this note to extend some of the considerations to higher $3nj$ coefficients.

(A) The generating functions for the 12j and 15j coefficients are explicitly computed in the Bargmann scheme. It is a tribute to the powerful Bargmann lemmas on the Laplacian integrals[4] that those seemingly complicated $6n$-fold integrals can in fact be systematically carried out. Thus in principle the generating functions for the $3nj$ coefficients are computable for arbitrary n in the Bargmann approach. Alternatively, the generating functions can also be found in the algebraic recursive scheme of Schwinger.[3] For $n=4$ and 5, they have been verified; the answers are essentially the same apart from a difference in an over-all phase factor.[6]

(B) Symmetry relations of the $3nj$ coefficients ($n=4, 5$) are here deduced on the basis of the explicit knowledge of their generating functions. They turn out to confirm the $4n$-fold symmetry ($n=4, 5$). For $n=4$, this was first discussed by Ord-Smith[7] using (i) a geometrical Möbius strip picture which incorporates the basic 3j triangular relations and (ii) an reduction formula (attributed to J. P. Elliott) of the 12j coefficient as a sum over products of four Racah coefficients.

The Möbius strip picture can be properly generalized to all n. (There is a slight technical difference between even or odd n cases.) Thus a basic $4n$-fold symmetry is expected to hold for arbitrary n. The situation may be summarized as in Table I. Lower order coefficients (for various reasons such as looser structure) are seen to possess larger symmetry. We find it gratifying that for $n \geqslant 4$, the symmetry for the $3nj$ coefficients becomes more systematic. [Note, however, the remark (b) below].

(C) Explicit expressions for the 12j and 15j coefficients can be extracted from their generating functions. However, in view of the excessively large numbers of summations involved [namely, $(2^{n+1} - 1 - 3n)$—fold], we shall not write them down here. The reduction formulas[6,10] of $3nj$ coefficients in terms of $3(n-1)j$ coefficients on one hand, and in terms of the Racah coefficients on the other, are probably more useful in practice.

The following remarks are made in view of the extensive work on the theory of angular momentum by A. P. Jucys *et al.*,[10] although the present undertaking is entirely independent of their approach.

(a) Jucys *et al.* have adopted a graphical method of their own; they were able to do calculations with the aid of their graphical method. Our emphasis, however, is on the explicit calculation of the generating functions.

(b) There is a proliferation in the definition of the $3nj$ coefficients. As the number of j's goes up, there are obviously various different recoupling schemes. Thus Jucys *et al.* have defined several kinds of $3nj$ coefficients. The ones we discussed here in this paper, the canonical ones, correspond to what they call *the first kind.* We shall not be concerned with those other than the first kind here.

(c) We have independently rediscovered a set of recursion formulas for the $3nj$ coefficients (i) in terms of $3(n-1)j$ coefficients and (ii) in terms of 6j coefficients.[6] These are known to Jucys *et al.* The basic $4n$-fold symmetry is also implicit in their work. However, we wish to emphasize that the methodology used are quite different, especially in regard to the symmetry. Our emphasis in this paper is to carry out the explicit calculation of the generating functions. From what we learn from the previously known cases, we adopt the viewpoint that all the symmetry of the $3nj$ coefficients is contained in the generating functions. The symmetry should be transparent and unambiguous in the Bargmann approach. What we have found is that (i) from our study of the generating functions comes the basic $4n$-fold symmetry ($n=4, 5$); (ii) the symmetry operations can be transcribed to those on a suitably defined Möbius strip; and (iii) this geometrical picture and the $4n$ fold symmetry is obviously valid for arbitrary $n \geqslant 4$.

II. THE 12j COEFFICIENT

A. Definition

In analogy with the previously discussed $n \leqslant 3$ cases,[4,5] we express the 12j coefficient (which is the recoupling coefficient involved in adding five angular momenta to a total j, or adding six angular momenta to zero) in terms of sums of products of eight 3j coefficients. We adopt the following labeling in Eq. (1) for the twelve j's, which is a slight modification of that of Ord-Smith[7]:

$$\{12j\} \equiv \left\{\begin{array}{ccccccccc} j_{30} & & j_{01} & & j_{12} & & j'_{23} & & \\ & j_{00} & & j_{11} & & j_{22} & & j_{33} & \\ j'_{30} & & j'_{01} & & j'_{12} & & j'_{23} & & \end{array}\right\} \qquad (1)$$

TABLE I.

$3nj$ recoupling coefficients		Symmetry relations
$n=2$	6j Racah	144 [Refs. 8, 4]
$n=3$	9j	72 [Refs. 2, 9, 5]
$n=4$	12j	16 [Ref. 7]
$n \geq 4$	$3nj$ general	$4n$ [this work and Ref. 10]

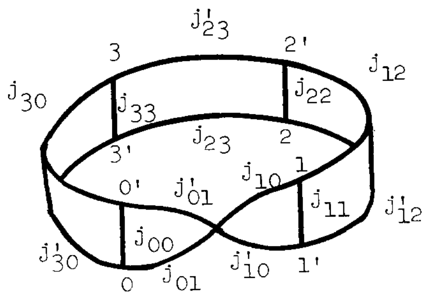

FIG. 1. Möbius strip representation for the 12j coefficient.

$$=\sum_{m,m'}\begin{pmatrix} j_{30} & j_{00} & m'_{01}\\ m_{30} & m_{00} & j'_{01}\end{pmatrix}\begin{pmatrix} j'_{01} & j_{11} & j'_{12}\\ m'_{01} & m_{11} & m'_{12}\end{pmatrix}\begin{pmatrix} j_{12} & j_{22} & j'_{23}\\ m_{12} & m_{22} & m'_{23}\end{pmatrix}$$

$$\times\begin{pmatrix} j_{23} & j_{33} & j_{03}\\ m_{23} & m_{33} & m_{03}\end{pmatrix}\begin{pmatrix} m'_{30} & m_{00} & j_{01}\\ j'_{30} & j_{00} & m_{01}\end{pmatrix}\begin{pmatrix} m_{01} & m_{11} & m_{12}\\ j_{01} & j_{11} & j_{12}\end{pmatrix}$$

$$\times\begin{pmatrix} m'_{12} & m_{22} & m_{23}\\ j'_{12} & j_{22} & j_{23}\end{pmatrix}\begin{pmatrix} m'_{23} & m_{33} & m_{30}\\ j'_{23} & j_{33} & j_{30}\end{pmatrix}, \tag{2}$$

where

$$j'_{pq}\equiv j_{qp},\quad m'_{pq}\equiv m_{qp} \tag{3a}$$

except

$$j'_{01}\equiv j'_{10},\qquad j_{01}\equiv j_{10}\qquad \text{(even-}n\text{ rule)},$$
$$m'_{01}\equiv m'_{10},\qquad m_{01}\equiv m_{10}. \tag{3b}$$

It is clear that there are triangle relations governing in each of the eight 3j coefficients. In the present notation, each 3j factor calls for a set of consecutive triplet indices ($p-1\,q$, pq, $p+1\,q$) or ($pq-1$, pq, $pq+1$), ($p,q=0,1,\ldots,n-1$, mod n). It is convenient to label a set of vertices p, p' accordingly. This results in the Möbius strip representation[7] (see Fig. 1). Note that the index convention is as follows: (i) j_{pq} connects from vertices p to q; (ii) j gets primed if the first index is primed; the prime on the second index is suppressed [except for those for t_{pq}, see Eq. (10)]; (iii) rules (3a) and (3b) are to be obeyed for even n cases [cf. Eq. (22) for odd n]

B. Generating function of the 12j coefficients

The generating function is defined as follows:

$$G^{(12)}(t,t')\equiv\sum_{k,k'}N_4^{-1}\{12j\}\prod_{p,q}t_{pq}^{k_{pq}}t'^{k'_{pq}}_{pq}, \tag{4}$$

where the normalization factor is given by

$$N_4\equiv\left[\prod_{p=0}^{3}(J_p+1)!(J'_p+1)!\Big/\Big(\prod_{p,q}k_{pq}!\,k'_{pq}!\Big)\right]^{1/2}. \tag{5}$$

For a triplet of indices ($p-1, p, p+1$), we define

$$J_p\equiv j_{p\,p-1}+j_{pp}+j_{p\,p+1}=\sum_q j_{pq}, \tag{6a}$$

$$J'_p\equiv j_{p-1\,p}+j_{pp}+j_{p+1\,p}=\sum_q j_{qp}=\sum_q j'_{pq}, \tag{6b}$$

and

$$k_{pq}\equiv J_p-2j_{pq}, \tag{6c}$$

$$k'_{pq}\equiv J'_p-2j'_{pq}. \tag{6d}$$

In a manner which is perfectly parallel to the known cases $n\leqslant 3$,[4,5] the generating function can be converted into the following integral:

$$G^{(12)}(t,t')=\int d\mu_{24}(\zeta)\exp\Big(\sum_{p=0}^{3}(D_p+\overline{D}'_p)\Big), \tag{7}$$

where

$$d\mu_N(\zeta)\equiv\pi^{-N}\exp(-\overline{\zeta}\cdot\zeta)\,d^N\zeta,\quad \zeta\equiv\xi+i\eta,$$
$$D_p\equiv t_p\times\xi_p\cdot\eta_p,\qquad p=0,1,\ldots,3, \tag{8a}$$
$$\overline{D}'_p\equiv t'_p\times\overline{\xi}'_p\cdot\eta'_p \tag{8b}$$

denote 3×3 determinants formed by components of the indicated 3-vectors. The components of t_p are labeled by the triplets ($t_{p\,p-1}$, t_{pp}, $t_{p\,p+1}$); likewise for t'_p. For ξ_p and η_p, a distinction has to be made involving the index 0, namely for $l\neq0$, ξ, η, have components labeled by ($l-1\,l$, ll, $l\,l+1$); likewise for $\overline{\xi}'_l$ and $\overline{\eta}'_l$. On the other hand, for $p=0$, the components are

$$\xi_0\equiv(\xi_{03},\xi_{00},\overline{\eta}'_{01});\qquad \eta_0\equiv(\eta_{03},\eta_{00},-\overline{\xi}'_{01}),$$
$$\overline{\xi}'_0\equiv(\overline{\xi}'_{03},\overline{\xi}'_{00},-\eta_{01}),\ \ \overline{\eta}'_0\equiv(\overline{\eta}'_{03},\overline{\eta}'_{00},\xi_{01}). \tag{9}$$

This complication comes about because two of the 3j coefficients in Eq. (1) (namely those involving the 0 and 0′ vertices) appear in a mixed conjugate fashion. In (8b) and (9), we have

$$\xi'_{pq}\equiv\xi_{qp},\quad \eta'_{pq}\equiv\eta_{qp} \tag{10}$$

while $t'_{pq}\equiv t_{pq'}$ are distinct from t_{qp}.

The 24-fold integration in (7) can be carried out in four steps. The calculation is straightforward with the aid of the Bargmann lemmas on the Laplacian integrals.[4] A slight extension leads to the following formula which turns out to be quite useful[6]:

$$\int d\mu_3(\xi)\,d\mu_3(\eta)\exp(t\times\xi\cdot\eta+t'\times\overline{\xi}\cdot\overline{\eta}+c\cdot\xi+d\cdot\overline{\eta})$$
$$=(1-t\cdot t')^{-1}\exp[(t\times c\cdot d)\,(1-t\cdot t')^{-1}]. \tag{11}$$

The final answer for the generating function (7) is

$$G^{(12)}(t,t')=(1-a_1-a_2-a_3-a_4-a_5)^{-2}, \tag{12}$$

where

$$a_1=\hat{t}_{01}\hat{t}_{32}+\hat{t}_{10}\hat{t}_{23}+\hat{t}_{30}\hat{t}_{21}+\hat{t}_{03}\hat{t}_{12},$$
$$a_2=t_{00}t_{11}(t'_{22}t'_{32}t_{30}+t'_{33}t'_{23}t_{21})+t_{11}t'_{22}t_{33}t'_{01}t_{03}$$
$$+t_{22}t'_{33}t_{00}t_{12}t'_{10}-t_{pq}\longleftrightarrow t'_{pq},$$
$$a_3=-\hat{t}_{00}\hat{t}_{12}\hat{t}_{32}-\hat{t}_{11}\hat{t}_{23}\hat{t}_{03}+\hat{t}_{22}\hat{t}_{30}\hat{t}_{10}+\hat{t}_{33}\hat{t}_{01}\hat{t}_{21},$$
$$a_4=t'_{00}t_{10}t'_{12}t_{21}t'_{23}t_{32}t'_{30}+t'_{11}t'_{21}t_{23}t'_{32}t_{30}t'_{03}t_{01}$$
$$+t'_{22}t'_{32}t_{30}t'_{03}t_{01}t_{10}t'_{12}+t'_{33}t'_{03}t_{01}t_{10}t'_{12}t_{21}t'_{23}$$
$$-t_{pq}\longleftrightarrow t'_{pq},$$
$$a_5=\hat{t}_{00}\hat{t}_{11}\hat{t}_{22}\hat{t}_{33}-\hat{t}_{00}\hat{t}_{11}\hat{t}_{23}\hat{t}_{32}-\hat{t}_{11}\hat{t}_{22}\hat{t}_{03}\hat{t}_{30}$$
$$+\hat{t}_{22}\hat{t}_{33}\hat{t}_{01}\hat{t}_{10}-\hat{t}_{33}\hat{t}_{00}\hat{t}_{12}\hat{t}_{21}+\hat{t}_{30}\hat{t}_{03}\hat{t}_{12}\hat{t}_{21}$$
$$-\hat{t}_{01}\hat{t}_{10}\hat{t}_{23}\hat{t}_{32}. \tag{13}$$

with

$$\hat{t}_{pq}\equiv t_{pq}t'_{pq}. \tag{14}$$

C. Consistency check

Setting one of the appropriate angular momentum to be zero should reduce the 12j coefficient to a 9j coefficient, and this implies that $G^{(12)}$ should reduce to $G^{(9)}$, which is known.[5] Our expression (12) satisfies this test. [To be precise, there are some sign difference among some of the corresponding terms and this is attributed to a difference in the choice of phase in going from $3nj$ to $3(n-1)j$ coefficients.]

D. Symmetry (even n case)

(a) Define the operation $P_{\updownarrow}^{(t)}$ which carries $t_{pq} \longleftrightarrow t'_{pq}$ and the operation $P_{\updownarrow}^{(k)}$ which carries $k_{pq} \longleftrightarrow k'_{pq}$

$$\tag{15}$$

It is easily verified that the generating function $G^{(12)}(t,t')$ is invariant under $P_{\updownarrow}^{(t)}$. From Eq. (4), it follows that the 12j coefficient is invariant under $P_{\updownarrow}^{(k)}$ which carries $j_{pq} \longleftrightarrow j'_{pq}$.

(b) Define the operation $P_{\leftrightarrow}^{(t)}$ = permutation $\binom{0123}{1032}$ among the t_{pq} indices. Likewise $P_{\leftrightarrow}^{(k)}$ among the the k_{pq} indices. (16)

$G^{(12)}(t,t')$ is readily seen to be invariant under $P_{\leftrightarrow}^{(t)}$. This implies that the 12j coefficient is invariant under $P_{\leftrightarrow}^{(k)}$.

(c) Define the operation $P_{\&}^{(t)}$ that carries

$$t_{pq} \to t_{p-1\,q-1},$$
$$t'_{pq} \to t'_{p-1\,q-1}$$

except

$$t_{1q} \to -t_{0\,q-1}, \quad q \neq 1,$$
$$t_{11} \to t_{00}, \tag{17a}$$
$$t'_{1q} \to t'_{0\,q-1},$$

and $P_{\&}^{(k)}$ that carries

$$k_{pq} \to k'_{p+1\,q+1}, \qquad k'_{pq} \to k_{p+1\,q+1} \tag{17a}$$

except

$$k_{0q} \to k_{1\,q+1}, \quad k'_{0q} \to k'_{1\,q+1}. \tag{17b}$$

Then $G^{(12)}(t,t')$ is invariant under $P_{\&}^{(t)}$. The 12j coefficient is left unchanged apart from a phase:

$$P_{\&}^{(k)}\{12j\} = (-1)^{2j_{11}}\{12j\}. \tag{18}$$

In terms of the Möbius strip picture, the above three operations correspond to the following:

$P_{\updownarrow}^{(t)}$: up—down symmetry of the Möbius strip: two fold symmetry,

$P_{\leftrightarrow}^{(t)}$: left—right symmetry of the Möbius strip: two fold symmetry,

$P_{\&}^{(t)}$: moving the "twist" between p and $p+1$ vertices: n-fold symmetry

Thus the combined symmetry is $4n$-fold ($n \geq 4$). (19)

The fact that the 6j and 9j coefficients in fact posses larger symmetry than the basic $4n$-fold symmetry discussed here might be attributed to the looser structure of their corresponding Möbius networks. (We emphasize the lines rather than the surface.) For $n \leq 3$ (i.e., with at most three vertical lines), it is possible to interchange the roles of horizontal and vertical lines, thereby resulting in enlarged symmetry. We claim that this is no longer possible for a Möbius network with four (or more) vertical lines.

III. THE 15j COEFFICIENT

A. Definition

Parallel to the discussion of the 12j case, we take

$$\{15j\} \equiv \begin{Bmatrix} j'_{40} & & j'_{01} & & j_{12} & & j'_{23} & & j_{34} & \\ & j_{00} & & j_{11} & & j_{22} & & j_{33} & & j_{44} \\ j_{40} & & j_{01} & & j'_{12} & & j_{23} & & j'_{34} & \end{Bmatrix} \tag{20}$$

$$= \sum_{m,m'} \begin{pmatrix} m'_{40} & m'_{00} & m'_{10} \\ j'_{40} & j'_{00} & j'_{10} \end{pmatrix} \begin{pmatrix} m'_{01} & m'_{11} & m'_{21} \\ j'_{01} & j'_{11} & j'_{21} \end{pmatrix}$$
$$\times \begin{pmatrix} m'_{12} & m'_{22} & m'_{32} \\ j'_{12} & j'_{22} & j'_{32} \end{pmatrix} \begin{pmatrix} m'_{23} & m'_{33} & m'_{43} \\ j'_{23} & j'_{33} & j'_{43} \end{pmatrix} \begin{pmatrix} m'_{34} & m'_{44} & m'_{04} \\ j'_{34} & j'_{44} & j'_{04} \end{pmatrix}$$
$$\times \begin{pmatrix} j_{40} & j_{00} & j_{10} \\ m_{40} & m_{00} & m_{10} \end{pmatrix} \begin{pmatrix} j_{01} & j_{11} & j_{21} \\ m_{01} & m_{11} & m_{21} \end{pmatrix} \begin{pmatrix} j_{12} & j_{22} & j_{32} \\ m_{12} & m_{22} & m_{32} \end{pmatrix}$$
$$\times \begin{pmatrix} j_{23} & j_{33} & j_{43} \\ m_{23} & m_{33} & m_{43} \end{pmatrix} \begin{pmatrix} j_{34} & j_{44} & j_{04} \\ m_{34} & m_{44} & m_{04} \end{pmatrix}, \tag{21}$$

where

$$j'_{pq} \equiv j_{qp}, \quad m'_{pq} \equiv m_{qp}$$
$$(\text{odd-}n\text{ rule}), \quad p = 0, 1, \ldots, 4. \tag{22}$$

The remarks following Eq. (3) for the 12j coefficient apply here also with Eq. (22) replacing Eq. (3). The Möbius strip picture for the 15j is shown in Fig. 2.

B. Generating function of the 15j coefficient

As an obvious generalization from Eq. (4), we have

$$G^{(15)}(t,t') \equiv \sum_{p,q} N_5^{-1}\{15j\} \prod_{p,q} t_{pq}^{k_{pq}} t'^{k'_{pq}}_{pq} \tag{23}$$

where

$$N_5 \equiv \left[\prod_{p=0}^{4} (J_p+1)!(J'_p+1)! \Big/ \left(\prod_{p,q} k_{pq}!\, k'_{pq}!\right)\right]^{1/2} \tag{24}$$

with the k, k', J, J' defined as in Eq. (6) now for $p,q = 0,1,\ldots,4$. As before, Eq. (23) is converted into the following integral:

$$G^{(15)}(t,t') = \int d\mu_{15}(\xi)\, d\mu_{15}(\eta) \exp\left(\sum_{p=0}^{4} (D_p + \bar{D}'_p)\right), \tag{25}$$

where D_p and $\bar{D}'_p$ are defined as in (8) now for p

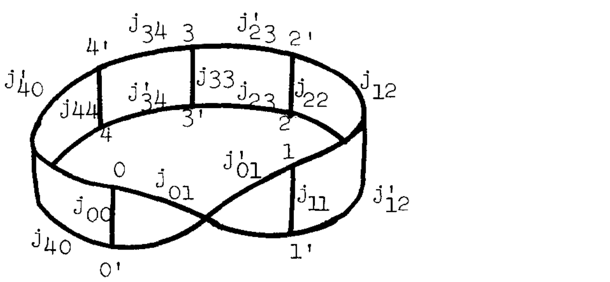

FIG. 2. Möbius strip representation for the 15j coefficient.

$=0, 1, \ldots, 4$, t_p having components $(t_{p-1\,p}, t_{pp}, t_{p+1\,p})$ (mod 5), labeled by a set of triplet indices. Likewise for ξ_p and η_p. All this is quite parallel to the $12j$ case except that the even-n complication (9) is absent here. Furthermore, ξ'_{pq}, η'_{pq} satisfy Eq. (10).

After performing the 30-fold integration in (25), the final answer reads:

$$G^{(15)}(t, t') = (1 - b_1 - b_2 - b_3 - b_4)^{-2}, \tag{26}$$

where b_i consists of polynomials of degree $2(i+1)$ in t and t', namely

$$b_1 = \sum_{p=0}^{4} \hat{t}_{p\,p+1}\hat{t}_{p-1\,p-2}$$

$[\hat{t}_{pq}$ defined in (14)],

$$b_2 = \sum_{p=0}^{4} [\hat{t}_{pp}\hat{t}_{p+1\,p+2}\hat{t}_{p-1\,p-2} + (t'_{pp}t_{p+1\,p+1} \times t'_{p+2\,p+2}t_{p+3\,p+3}t'_{p-1\,p}t_{p-1\,p-2} + t \longleftrightarrow t')],$$

$$\begin{aligned} b_3 &= \sum_{p=0}^{4} \{(\hat{t}_{pp}\hat{t}_{p+1\,p+1} - \hat{t}_{p\,p+1}\hat{t}_{p+1\,p})\hat{t}_{p+2\,p+3}\hat{t}_{p-1\,p-2} \\ &\quad + [t'_{pp}(t_{p+1\,p+1}t_{p+2\,p+1}t'_{p+2\,p+3} \\ &\quad + t_{p+2\,p+2}t_{p+1\,p+2}t'_{p+1\,p})\,t_{p+3\,p+2}t'_{p-2\,p-1} \\ &\quad \times t_{p-1\,p-2}t'_{p-1\,p} + t \longleftrightarrow t']\}, \end{aligned}$$

$$\begin{aligned} b_4 &= \sum_{p=0}^{4} \hat{t}_{pp}(\hat{t}_{p+1\,p+1}\hat{t}_{p+2\,p+2} - \hat{t}_{p+1\,p+2}\hat{t}_{p+2\,p+1}) \\ &\quad \times \hat{t}_{p-2\,p-1}\hat{t}_{p-1\,p-2} - \prod_{p=0}^{4} [\hat{t}_{pp} + (t'_{p\,p+1}t_{p+1\,p} + t \longleftrightarrow t')]. \end{aligned} \tag{27}$$

C. Consistency check

The statement made under Sec. IIC for the $12j$ case is valid also for the $15j$ case.

D. Symmetry (odd-n case)

(a) Define the operation $P_{\updownarrow}^{(t)}$ which carries $t_{pq} \longleftrightarrow t'_{pq}$; correspondingly for $P_{\updownarrow}^{(k)}$: $k_{pq} \longleftrightarrow k'_{pq}$. It is obvious that $G^{(15)}(t, t')$ is invariant under (28). This implies that the $15j$ coefficient is invariant under $P_{\updownarrow}^{k}$.

(28)

(b) Define the operation $P_{\leftrightarrow}^{(t)} =$ permutation $\binom{0\,1\,2\,3\,4}{1\,0\,4\,3\,2}$ on t_{pq} (recall $t'_{pq} \equiv t_{pq'}$). Correspondingly for $P_{\leftrightarrow}^{(k)}$ on k_{pq}. We have $G^{(15)}(t, t')$ invariant under $P_{\leftrightarrow}^{(t)}$, thus the $15j$ coefficient is invariant under $P_{\leftrightarrow}^{(k)}$.

(29)

(c) Define $P_{\circlearrowleft}^{(t)}$:

$$t_{pq} \to t'_{p-1\,q-1}, \quad t'_{pq} \to t_{p-1\,q-1} \text{ (mod 5)} \tag{30}$$

and $P_{\circlearrowleft}^{(k)}$:

$$k_{pq} \to k'_{p+1\,q+1}, \quad k'_{pq} \to k_{p+1\,q+1}. \tag{31}$$

Since $G^{(15)}(t, t')$ is invariant under (30), we have the invariance of $\{15j\}$ under (31). The remark following Eq. (18) holds here for $n = 5$.

IV. CONCLUDING REMARKS

What we have done is to demonstrate by explicit calculations that the study of the properties of higher-order $3nj$ angular momentum recoupling coefficients can be carried out in principle for all n. The algebraic complexities, though increasing rapidly with n, turn out still to be controlable. Extraction of the explicit expansion forms for the $3nj$ coefficients are in principle possible from the generating functions.

The $3nj$ coefficients ($n \geq 4$) are seen to possess a $4n$-fold symmetry. Visualization of some of the structural properties of $3nj$ coefficients are greatly enhanced with the aid of a geometric Möbius network representation.

Note added in proof: For graphical method for angular momentum, see also E. El Baz and B. Castel, *Graphical Methods of Spin Algebras* (Dekker, New York, 1972).

*Based in part on a dissertation submitted by C. S. Huang in partial fulfillment of the requirements for the Ph. D. degree at the University of Michigan, 1973 (unpublished).

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