

CHAPTER 12

The Role of Mathematics in Education for Democracy

Mathematics enables us to fly to the moon, track our genetic codes, create beautiful music, design our cars, build our houses, and contact others around the world almost instantaneously. However, mathematics, that abstract language which helps us to access the relationships in our physical universe(s), is rarely invoked in the service of preparing young people for democratic participation. Deborah Ball and Hyman Bass take on the challenge of situating the highly revered, somewhat mystical discipline of mathematics as a key contributor to concepts of democracy.

Ball and Bass address one of the enduring concerns in schooling—how to relate meaningful classroom experiences to greater public purposes, specifically in mathematics classrooms. They hint at how analytical tools can be used to critique public policies and social problems and reveal how the historical development of mathematical thought can contribute to cultural and intercultural understanding. However, their key insight is that mathematics instruction can embrace, uphold, and promote the norms, skills, and dispositions of democracy. Using a classroom discussion as example, they imagine how young people might develop competence (and joy) in problem solving, democratic dialogue, and consensus building. They conclude, optimistically, that teaching and teachers can change the ways we perceive and use the resources mathematics offers to enrich our democratic reasoning.

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*The Role of Mathematics in Education for Democracy*¹

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In this chapter, we argue that mathematics—and mathematics instruction—has a special role to play in education for democracy. This is an argument apart from, although not at odds with, the importance of mathematical literacy for all students. It is different from but compatible with the urgent need to redress serious inequities and disparities in both opportunity and achievement. It is abundantly clear that efforts are needed to improve every student's access to and development of usable mathematical literacy, including the skills for everyday life, preparation for the increasing mathematical demands of even relatively nontechnical workplaces, and resources for continued mathematical study. Indeed, the need for collective commitment to this goal has never been greater.² In addition, however, we claim that mathematics has a special role to play in educating young people for participation in a pluralistic democratic society. Making it possible for mathematics to play this role in a democratic education depends on how it is taught. Needed is instruction that uses the special resources that mathematics itself holds for reaching these broader societal aims.

One way in which mathematics teaching can help to build the resources for a pluralistic society is through the development of tools for analysis and social change.³ Mathematics offers tools to examine and analyze critically the deep economic, political, and social inequalities in our society, for studying crucial societal problems, and for considering a host of issues that can be understood and critiqued using quantitative tools. For example, who voted in the last election and why? How does the Electoral College shape whose votes count most in a presidential election? How do our income and inheritance tax laws shape the distribution of wealth and access to fundamental resources? How does our system of school funding shape the quality of education that different children in our country receive? Developing and using the mathematical skills that enable young people to engage in social analysis and improvement is one way in which mathematics can contribute to the development of a diverse democracy.

A second way in which mathematics teaching can play a role in education for democracy is as a setting for developing cultural knowledge and appreciation, important resources for constructive

participation in a diverse society. Mathematics represents an ancient and remarkable set of cultural achievements and engagements. As such, the historical development of mathematical ideas and methods offers a medium for studying history and culture and their intersections in domains of human activity as diverse as architecture, art, music, science, and religion. Mathematics, because of its universality, offers opportunities for young people to learn about their own cultural heritage and that of others. For example, what systems of counting and recording were developed and used by different peoples? How does sophisticated mathematics manifest itself in the craft of artisans? Such learning is crucial for developing the understanding and appreciation of diverse traditions, values, and contributions, and for ways to notice, respond to, and use them. Such learning is also crucial for developing a sense of one's own cultural identity and membership, both for oneself and as a participant in the broader cultural milieu.

But a third way that mathematics teaching can support the development of democratic goals—the one on which we focus here—is through the skills and norms of mathematical practice itself. In other words, we argue that it is not just the content and history of mathematics and its tools that contributes to democratic goals, but the very nature of mathematical work. Mathematics instruction, we claim, can offer a special kind of shared experience, a facility with productive collective work that is so essential to the realization of democratic ideals.

How so? Consider that mathematics is centrally about problem solving, and about discovering and proving what is true. As a discipline, mathematics offers powerful tools for abstracting and generalizing from particularities, hence offering a special experience in collective action out of diverse experience. Mathematics offers a singular context where everyone is working in a new space and where diversity is not an obstacle but a resource. Reasoning about the parity of numbers does not depend on resources that advantage the more privileged. Producing a number line representation or explaining the equation for a circle takes place in a universal mathematical context, not a context that advantages some and excludes others. Still, diverse perspectives are crucial for leveraging work: Alternative interpretations and representations of a problem can often serve to open a path to its solution; sometimes a novel metaphor, diagram, or context can crack a difficult part of a problem. At the same time, diversity is structured and supported by common disciplinary language, norms, and practices. The rules for operating in that context can be made transparent and explicit, usable by all. Terms must be precisely defined and used in commonly understood

ways. Disagreements are resolved not by shouting or by plurality, but by reasoned arguments calling on skills that can be taught and learned. Decisions such as whether 0 is even or odd, how to interpret the meaning of $3/4$, whether $5/5$ is greater or less than $4/4$, or whether a solution to a particular problem is valid are subject to mathematical reasoning, not governed by simple desire or power. Thus mathematics is a context in which conflict is both common and rationally resolved, with allegiance only to the ideas. Moreover, mathematical reasoning is a powerful practice that can be learned; it is not an innate talent.

How Can Mathematics Contribute to Education for Democracy? A Visit to a Classroom

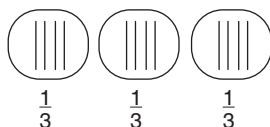
To make our argument concrete, we turn now to an example from an elementary classroom.⁴ This is a third grade class in which norms of mathematical reasoning and respect for others' ideas have been cultivated. In addition, these eight- and nine-year-old pupils have been taught specific mathematical and relational skills needed for the work in which we see them engaged.

The problem on which they are working is to figure out how many crayons are in three-quarters of a box of a dozen crayons. From a mathematical perspective, this question is designed to focus the students' attention on the unit (in this case, one dozen crayons) and on the meaning of the quantity three-fourths. Learning fractions is one of the more challenging topics of the elementary curriculum, and one that is crucial for pupils' later success in algebra. Students must make a major shift from the domain of whole numbers where the unit is clear (52 means fifty-two of something) to fractions, which are inherently multiplicative and where the unit must be established. This is a major shift for students, and learning fractions often presents significant challenges. For students to figure out how many crayons are in three-quarters of a box of a dozen crayons demands that they reason about three-fourths of twelve.

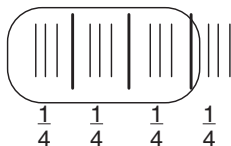
The children are discussing the size of the groups when 12 is divided into fourths. The teacher calls on Sean,⁵ who begins the discussion by showing how he got four as the answer. Midway through his explanation, however, he "revises" what he is saying and instead shows how he arrived at three as the answer. Other students agree and several observe that the "bottom number" signifies "how many groups you have to make." In other words, three-*fourths* means to divide the unit (in this case, 12) into four equal parts. At this point in the discussion, a girl named Riba asks, "Shouldn't one fourth have four in it?" (i.e., rather

than three). Rather than reply to her query, the teacher tells the class that this is a good question and asks them what they think about this.

Sean volunteers to explain and draws lines on the board:



Pointing at his diagram, he says, “This is one group of four. There’d be—there’s only three groups, so one of those groups makes it, uh—a third—and then these groups each are four. There’s four groups, and—



one of these groups would be a *fourth*.”

The teacher asks Riba if she is following Sean’s explanation. He makes a second try. This time he draws a rectangle and divides it into four equal parts and shades in one part. “This would be a fourth.” Then he draws a second, similar, rectangle, and divides it into three equal parts. “One of these would be a third.”

Riba and the other children are watching. After a few moments Riba says, “I still disagree because this one fourth, I’m just—I’m not—well, I’m not sure that I really disagree. I’m saying, um, if you call it one fourth, could you—be four in a group?” The teacher directs Riba to Sean’s explanation, and asks her to restate what he is saying.

Riba: He’s saying one fourth is—um, four groups.

Teacher: If you cut something into four groups and one of them is one fourth. And he showed you two different pictures of fourths. Can you find the two pictures? Where’s one?

Teacher: Okay. Where are the fourths?

Riba: Right here.

Teacher: There’s one. Where’s his other picture of fourths?

Riba: Right there.

Teacher: Okay. And both times he cut something into four parts and called one of them *one-fourth*. What about other people in the class? Keith?

Keith: Um, I, I think Riba disagrees because she think like—she thinks one—the reason why she thinks one-fourth should have four, um, crayons in it because—because of the four in the um—um—

Teacher: In the bottom number?

Keith: Yeah.

Teacher: When she look—when Riba looks at the four, she thinks . . . thinks this is telling her how many to put in a group. And Sean is thinking it means how many groups to make. Now let's think about this for a minute.

In the next few minutes, the teacher gets the students to consider a familiar fraction—one-half—and to use their familiarity with it to leverage a more general sense of the meaning of three-fourths. Riba announces that she wants to explain what she is thinking. “Three fourths is like three groups of four.” Another student, Ofala, nods.

At this point, Sean raises his hand, and asks, “Can we vote?”

This is a critical moment in the discussion. Up to this point, the students have been focusing on the ideas. What does the 4 in $\frac{3}{4}$ mean? How are one-third and one-fourth different? Using mathematical tools—drawings, language, argument—these young learners are working to arrive at common understanding. Mathematics rests on shared definitions, meanings, and ways of establishing conclusions. In this way, mathematics differs from arenas of human activity where individuals are free—even encouraged—to develop their own ideas, interpretations, and ways of working. The importance of the collective in mathematics is a special educational resource. This confusing problem they are debating is one on which they must agree. They must move from their different perspectives to a common one, and they must do so using the rules and tools of mathematical practice rather than personal or idiosyncratic ones whose effectiveness would depend on power or personal persuasion. Mathematics both requires and depends on common ideas and practices; it does not submit to individual domination or privilege.

Sean, our third-grade protagonist, does recognize this need for common meanings in mathematics. His suggestion that the class vote shows just how clearly he appreciates this. What he has yet to learn, however, is the proper means for establishing general agreement.

The teacher asks Sean to explain how voting would help to resolve the disagreement about the different interpretations of one-fourth. “Well, I just wanted to see how many people . . . will think that my answer is correct, raise their hand, and how many people think Riba's answer is correct, raise their hand.” The teacher probes further: “What would that do if we saw that?” Sean explains: A majority vote would prove which answer is wrong.

Sean's proposal about voting surprises his classmates, several of whom are shaking their heads vigorously. The teacher asks Keith to explain what he is thinking, and he says, firmly, “Just because, like—just

because somebody agrees more with one person, doesn't mean that they're *right*." Many other students have their hands up.

- Teacher: Other people want to comment on that—about voting and deciding what's right? Tembe, what do you think?
- Tembe: I agree with Keith because if we voted, maybe the, the answer might be wrong, and the people who didn't vote—less people who voted—might be right.
- Teacher: Hmm. Daniel?

Daniel, a student who is just learning English, is emphatic: "Yeah, I agree because—um—like—when some people say what's one plus one, and one person might say two and most of the persons might say three, and . . . [so] I—I agree with Keith." The teacher asks whether anyone thinks that voting would settle the question of which answer was correct. Hearing a chorus of "no" she presses: "Then if voting doesn't help, then how *can* you tell if something's right?" she asks the class.

- Tori: Figure it out.
- Teacher: Sheena, how can we tell something's right, if voting doesn't work?
- Sheena: Well, all you have to do is try and figure it out yourself, and if you think you got the right answer, then maybe you should, um, discuss it with somebody, and maybe they might be able to change your mind if the answer's wrong.

Mathematics as a Context for Learning to Reconcile Differences

In this episode, the children confront a fundamental disagreement within the content and have an opportunity to learn how such a disagreement can be resolved. Because mathematics has agreed-upon practices for reaching consensus, in particular for certifying knowledge, mathematics instruction can deliberately help young people learn the value of others' perspectives and ideas, as well as how to engage in and reconcile disagreements. Mathematics instruction can be designed to help students learn that differences can be valuable in joint work, and that diversity in experience, language, and culture can enrich and strengthen collective capacity and effectiveness. Students can also learn that mathematics is not an arena in which differences are resolved by voting. There are legitimate areas of social life in which differences are managed in this way, but the study of mathematics is not among them. In a democratic society, how disagreements are reconciled is crucial. But mathematics offers one set of experiences and norms for doing so, and other academic studies and experiences provide others. In literature, differences of interpretation need not be reconciled. In mathematics,

consensus matters. In this way, mathematics contributes to young people's capacity for participation in a diverse society in which conflicts are not only an inescapable part of life, but their resolution, in disciplined ways, is a major source of growing new knowledge and practice.

The Role of Instruction in Using Mathematics for Democratic Education

Our readers may be skeptical. Thinking back to their own days with school mathematics, readers may remember experiences in which the book or the teacher carried the authority for knowledge, and students simply accepted ideas proffered by those regarded as "smart." The democratically rich technology of mathematics—the practices and tools of mathematical reasoning—remained mostly invisible and untaught except for some often ritualized processes practiced in high school geometry.

Mathematical reasoning comprises a set of practices and norms that are collective, not merely individual or idiosyncratic, and rooted in the discipline. Making mathematics reasonable entails making it subject to, and the result of, such reasoning. That an idea makes sense to you is not the same as reasoning toward knowledge that is common, shared by others. Our argument in this chapter is based on a mathematical perspective on classroom learning. Much has been written about constructivist theories of learning and their implications for instruction. Indeed, "constructivism" has been one of the most dominant—and most multiply interpreted—theories in mathematics education. Our research analyzes classroom mathematics learning and teaching in light of ideas about the construction of knowledge that are rooted in mathematics as a discipline. When students are at work in a mathematics class, we see them as constructing mathematical knowledge.⁶ Looking at the development of students' knowledge in this way highlights the fundamentally *mathematical* nature of their work.⁷ As students explore problems, make and inspect claims, and seek to prove their validity, we see that even young children engage in substantial forms of mathematical reasoning and make use of mathematical resources.

In addition, our conception of teaching is founded on three specific guiding principles: (1) the integrity of the discipline; (2) the centrality of taking student thinking seriously; and (3) the classroom collective as an intellectual community.⁸ First is the principle of drawing from mathematics as a discipline in intellectually sound and honest ways.⁹ Second is that teaching demands a sensitivity to and responsiveness to students'

ideas, interests, lives, and trajectories. Teachers must strive to hear their students, to work with them as they investigate and interpret their worlds. Respecting students means attending to who they are, and what they bring, as well as helping them grow beyond their present capabilities, interests, and aspirations. Finally, the teaching in which we are interested aims to create a classroom community in which differences are valued, in which students learn to care about and respect one another, in which intellectual consensus is negotiated using mathematical norms, and in which commitments to a just, democratic, and rational society are embodied and learned.¹⁰ Care and respect for others includes listening, hearing, and being able to represent others' ideas, even those with which you disagree. Respect also means taking others' ideas seriously, appraising them critically and evaluating their validity. In this essay, as we consider the role of mathematics in education for democracy, we consider mathematical reasoning as being about producing more than individual conviction: it is about how mathematical knowledge comes to be public and usable by the collective.

So then, specifically, how might instruction be designed to serve both mathematical and democratic ends? Three components are involved. One component lies in what students are asked to work on, a second in how the work is conducted, and a third with the teacher as a model of the collective attitude and reasoned practice of mathematics as a discipline. Consider first what students work on: the mathematical tasks designed or selected. Tasks that serve to develop common skills, language, and practices offer ways that can help to build the common skills needed for class work on mathematics. Also useful are tasks that yield to alternative representations or approaches, so that students' understanding of the material is deepened through the different ways in which their classmates see the ideas. Although it is valuable to use mathematical tasks that profit from others' interpretations, such tasks should not, however, depend unfairly on unevenly distributed cultural experience or knowledge.

Take, for example, two different tasks, each designed to help students develop an understanding of fractions as numbers. The first task asks students to compare the numbers $\frac{4}{4}$ and $\frac{4}{8}$; the second asks, "Mr. Good offers you $\frac{4}{4}$ of one pizza or $\frac{4}{8}$ of a second one. Which would you rather have?" In the first case, students can use many different methods to resolve the question—diagrams, the number line, pictures. In the second, students may bring views about actual pizza, or their own preferences or norms about sharing, that will bring out other extra-mathematical considerations. Discussion of the "correct" answer

to the second problem would be uneven because reasons other than mathematical ones are clearly legitimate. For example, it is acceptable for students to argue that their decision about which pizza to select would depend on what kind each one is (cheese, pepperoni, etc.). It is acceptable to answer that they do not like to be selfish and so would choose $\frac{4}{8}$ of a pizza; it is also possible to choose on the basis of whether or not one *likes* pizza. None of these is subject to the explicit structures of mathematical reasoning that provide the opportunity for civilized resolution of disagreement.

Teaching matters, too. How mathematical tasks are used is crucial for whether or not their potential is realized in classrooms. If not carefully structured and guided, cognitively complex tasks can degrade to simple routine problems, and problems ripe with opportunity for reasoning and representation can become procedural.¹¹ Similar vigilance is needed in order for tasks to serve as contexts for the development of democratic skills and dispositions. Such vigilance is centered on cultivating attention to and respect for others' mathematical ideas. Students would need to develop a consistent stance of civility with one another, a stance based on intellectual interest and respect, not mere social politeness or "niceness." This would require learning to listen carefully to others' ideas, and checking for understanding before disagreeing. Other skills, norms, and practices of collective mathematical work include giving credit to others' ideas—referring to ideas by their authors' names, for example—and critiquing ideas, not people, using the tools and practices of the discipline. Students would work to seek agreement on meanings and solutions, drawing on past shared experiences, definitions, ideas, and agreements about meaning, and they would use and contribute to one another's ideas in a collective effort to solve and understand the mathematics and the problems on which they are working. Important to our argument is that the skills and practices that are central to mathematical work are ones that can contribute to the cultivation of skills, habits, and dispositions for participation in a diverse democracy. This requires doing mathematics in public, where practice in democratic discourse is possible.

Making mathematical ideas public in useful ways entails moving individuals' ideas into the collective space. When individual students offer ideas or solutions, these can often become no more than a collection of bilateral exchanges between one student and the teacher. We mean something different. For students to take note of and use one another's ideas requires deliberate effort. First, students' contributions must be comprehensible to their peers. This requires that students

speaking loudly enough to be heard and that students learn to listen closely to others' talk. Teachers may simply direct students to "speak up" and tell others to listen closely. But teachers may also have to assist students to articulate their ideas in ways that are both audible and understandable. They may have to ask individual students to repeat what they have said, or ask them questions about what they are saying. They may need to do this to help make what individual students are saying more explicit, so their contributions do not remain private, vague, half-developed, and weakly articulated statements to which others cannot usefully respond. Making ideas public entails helping make them accessible for others' consideration. Once ideas are more clearly expressed, teachers can ask students to respond directly to another student's point, may ask students to explain what a classmate has said, or may ask them whether they can articulate how a classmate reached a conclusion. As students' ideas become regular sources of the class's work, students will both speak more clearly and ask one another to speak more audibly.

Teachers also play an important role in *modeling* the use of others' ideas and of public mathematical knowledge, of using language carefully. Teachers can make references to "Lucy's method" or remark on uses of established ideas: "How is this related to Sean's conjecture?" They can expect connections to public knowledge by asking questions like "Are you using the definition that we agreed on for even numbers?" or "How does what we figured out about multiplying by 10 or 100 or 1,000 help with this problem?"

For mathematics instruction to contribute to the building of a socially just and diverse democracy, more than care with curriculum and teaching is required—more even than committed teachers, however sensitive to and skillful in working toward these aims they may be. Accomplishing this end would require significant change in teachers' education and professional development—no small task. Responsible for helping prepare young people for life in society, teachers must be comfortable with the discipline of mathematical practice. The instruction they provide must be able to take advantage of its fundamentally democratic toolkit for using and mediating differences of view. On the one hand, these differences are crucial to solving problems, as people with different perspectives or ideas bring various resources to the task. For this to work in school, teachers must be skillful at teaching students to be respectful of and interested in others' perspectives, and show them how to use those perspectives in the service of collective problem solving. In the episode above, Sheena was aware that "conferring" with other people could help her solve problems. And the teacher was

guiding the students to consider Riba's question in order to get clearer about the meaning of the 3 and the 4 in $3/4$. On the other hand, since consensus matters, the tools of mathematical practice—for instance, defining, representing, comparing, and reconciling—provide structure for resolving discrepancies. Teachers oriented to this task can use mathematics to teach norms of civilized and respectful disagreement, regulated by mathematical principles rather than personal will. In this third grade class, the teacher helped students understand that voting was not a vehicle for deciding how to interpret $3/4$. Instead, the number $3/4$ has a meaning derived from shared knowledge and language and norms about precision and interpretation. Common ground in mathematics is reached not by plurality but by building on prior established knowledge, careful use of language, and disciplinary practices of reasoning.

What Does Mathematical Practice Offer Education for Democracy?

Our argument is this: Mathematics, with its commitment to common ground and its welcoming of diverse and imaginative perspectives, together with its extensive set of tools for establishing consensus, can help to develop the skills and dispositions and values crucial to the development of a diverse democracy. Using mathematics education to advance democratic capacities would expand the educational resources that formal schooling can deploy. More experiences with efforts to solve complex problems, for which diverse perspectives and ideas were crucial, could provide graduates of our schools with systematic training in listening closely to others' contributions and studying others' diagrams or models. More opportunities to resolve disagreements using the tools of mathematical argument could develop the capacity for reasoned debate and an appreciation of its value.

On one hand, of course, many crucial social issues depend on quantitative reasoning and evidence. Rational, respectful argument is more possible when the capacity for quantitative claims and the norms for the inspection of the reasonableness of such claims are shared. On the other hand—and this is our central argument—the experience of disciplined argument not based on personal privilege, power, or majority rule can help build the skills needed for managing disagreement in fair and disciplined ways. Experience with the value of diverse perspectives and the habit of attending to, critiquing, and learning from others' ideas is also a crucial resource for life in a diverse democracy. Mathematics is special in the opportunities it can provide for young people to develop the orientations, values, and skills necessary for democracy. Adults who

had experienced these sorts of mathematics learning might be better prepared for community deliberations and decision making, better oriented to respectful attention to others' ideas, and more open to the power of disciplined argument. Realizing the potential of mathematics to achieve these educational ends depends on instructional practices that draw on the practice of mathematics itself and that deliberately develop the skills and environment for such work by children in school.

Making mathematics a resource for democracy will require learning. These practices are not natural in our society or our schools. They are not ones that teachers or their students can do simply because it is suggested that they do so. If students are to experience mathematics through disciplined practices of reasoning, then teachers must have opportunities to develop the knowledge and practice necessary to make mathematics reasonable in school and to make it a force and a resource in educating young people for participation in a democratic society.

NOTES

1. This paper draws on an earlier related paper: D. L. Ball, I. M. Goffney, and H. Bass, "The Role of Mathematics Instruction in Building a Socially Just and Diverse Democracy," *The Mathematics Educator* 15, no. 1 (2005): 2–6.

2. C. E. Malloy, "Democratic Access to Mathematics through Democratic Education," in *Handbook of International Research in Mathematics Education*, ed. L. D. English (Mahwah, NJ: Lawrence Erlbaum Associates, 2002), 17–25; R. Moses and C. Cobb Jr., *Radical Equations: Math Literacy and Civil Rights* (Boston: Beacon Press, 2001).

3. R. Gutiérrez, "(Re)Defining Equity: The Importance of a Critical Perspective," in *Diversity, Equity, and Access to Mathematical Ideas*, ed. P. Cobb and N. Nasir (New York: Teachers College Press, 2007), 37–59; E. Gutstein, "'And That's Just How It Starts': Teaching Mathematics and Developing Student Agency," *Teachers College Record* 109 (2007): 420–48; V. Walkerdine, *The Mastery of Reason: Cognitive Development and the Production of Rationality* (London: Routledge, 1988).

4. This episode is drawn from data collected over an entire school year in a third grade class during 1989–90 under a National Science Foundation grant to the first author and Magdalene Lampert. Daily records were made of Ball's third grade (and Lampert's fifth grade) mathematics classes, including videotapes and audiotapes of lessons, photocopies of students' work, teacher's journals and plans, tests, quizzes, and homework, and the mathematics problems and tasks on which the students worked.

5. All names are pseudonyms, standardized across published analyses of these data, and selected to be culturally similar to the children's real names. For example, Riba was from Egypt, and her pseudonym was selected from among similar Egyptian girls' names.

6. Magdalene Lampert has written about similar resonances between the practices of knowing mathematics in school and in the discipline: See M. Lampert, "When the Problem is Not the Question and the Answer is Not the Solution," *American Educational Research Journal* 27 (1990): 29–63; M. Lampert, "Practice and Problems in Teaching Authentic Mathematics," in *Effective and Responsible Teaching: The New Synthesis*, ed. F. Oser, D. Andreas, and J. Patry (San Francisco: Jossey-Bass, 1992), 295–314.

7. D. L. Ball, J. Lewis, and M. Thames, "Making Mathematics Work," *A Study of Teaching: Multiple Lenses, Multiple Views, Journal for Research in Mathematics Education (Monograph Series No. 18)*, ed. A. Schoenfeld (Reston, VA: National Council of Teachers of Mathematics, 2008). D.L. Ball and H. Bass, "Making Mathematics Reasonable in School." In *Research Compendium for the Principles and Standards for School Mathematics*, ed. G. Martin (Reston, VA: National Council of Teachers of Mathematics, 2003), 3–14.

8. D. L. Ball and H. Bass, "Making Believe: The Collective Construction of Public Mathematical Knowledge in the Elementary Classroom," in *Constructivism in Education: Yearbook of the National Society for the Study of Education*, ed. D. C. Phillips (Chicago: National Society for the Study of Education, 2000), 193–224.

9. D. L. Ball, "With an Eye on the Mathematical Horizon: Dilemmas of Teaching Elementary School Mathematics," *Elementary School Journal* 9, no. 4 (1993): 373–97; D. L. Ball and H. Bass, "Interweaving Content and Pedagogy in Teaching and Learning to Teach: Knowing and Using Mathematics," in *Multiple Perspectives on the Teaching and Learning of Mathematics*, ed. J. Boaler (Westport, CT: Ablex, 2000), 83–104; D. L. Ball and H. Bass, "Making Believe," 193–224; J. Bruner, *The Process of Education* (Cambridge: Harvard University Press, 1960); M. Lampert, "When the Problem Is Not the Question and the Answer Is Not the Solution," 295–314; M. Lampert, "Practice and Problems in Teaching Authentic Mathematics," 295–314; M. Lampert, *Teaching Problems* (New Haven, CT: Yale University Press, 2001).

10. John Dewey, *Democracy and Education* (Chicago: University of Chicago Press, 1916); J.J. Schwab, "Education and the State: Learning Community," in *Great Ideas Today* (Chicago: Encyclopedia Britannica, 1976), 234–71.

11. M. K. Stein, B. Grover, and M. Henningsen, "Building Student Capacity for Mathematical Thinking and Reasoning: An Analysis of Mathematical Tasks Used in Reform Classrooms," *American Educational Research Journal* 33 (1996): 455–88.