FINDING THE CONVEX HULL OF A SIMPLE POLYGON IN LINEAR TIME

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Summary

A new linear algorithm for finding the convex hull of a simple polygon is given. Based on the original idea by Sklansky (7), our version is easy to understand. Adopting the form of CH-POL by Toussaint and Avis (9), the presentation is concise. As shown in the Appendix, a PASCAL implementation of the algorithm itself is only half a page long.

In the paper, we define a "zipper" as a non-self-intersecting, concave chain. Choosing an extreme vertex of the polygon as the initial zipper, we update it by classifying a vertex of the given polygon by one of three cases. Case 1: vertex of the given polygon is added to the zipper. Case 2: vertex of the given polygon is not added to the zipper. Case 3: zipper vertex is deleted. We show that, after a complete traversal of the given polygon, the zipper thus constructed is the convex hull.

Abstract

Though linear algorithms for finding the convex hull of a simply-connected polygon have been reported, not all are short and correct. A compact version based on Sklansky's original idea (7) and Bykat's counter-example (8) is given. Its complexity and correctness are also shown.

Keywords: Convex hull, linear algorithm, computational geometry

1. **Introduction**

There have been many reports on a linear algorithm for finding the convex hull of a simple polygon. Certain versions were prone to counter-examples. In particular, a recent version by Ghosh and Shyamasundar (1) turned out to be incorrect (2,3). Ideally, an algorithm should be not only correct but also easy to implement. McCallum and Avis (4), for example, reported a version using two stacks. Lee (5) used one stack but the algorithm itself was two pages long. Recently, Graham and Yao (6) reported a compact algorithm that is said to be similar in spirit to Lee's version. Both (5) and (6) included two types of pocket test. In this paper, we present a version employing only one pocket test.

Perhaps, the simplest version is still the one presented by Sklansky (7) in 1972. After a counter-example by Bykat (8), sufficiency condition was established by Toussaint and Avis (9) in 1982 and by Orlowsky (10) in 1983. Almost concurrently, Sklansky gave a modified version (11) but it was later shown to be incorrect by Toussaint and El Gindy (12). Our search for a simple, concise, and correct linear convex hull algorithm traces the following path. For simplicity, we adopt the ideas from the original version by Sklansky (7). For conciseness, we follow the form of CH-POL by Toussaint and Avis (9). For correctness, we use the notion of a pocket (or lobe) as in Graham and Yao (6) (or Lee (5)) with Bykat's counter-example (8) in mind.

2. **Preliminaries**

Let P be a simple polygon with n vertices. Each vertex V_i, i = 0, 1, 2, ..., (n-1), is represented by its X and Y coordinates, (X_i,Y_i). Let V_0 be the vertex with the minimum Y coordinate. If two or more vertices are tied then we choose among them the vertex with the minimum X coordinate as V_0. Starting from V_0 and traversing the boundary B(P) of P in the clockwise order, we label the jth vertex from V_0 as V_i, where i is j modulo n. These vertices in sequence are maintained as a circular doubly linked list. Throughout this paper we assume the following:

(1) The boundary B(P) of a simple polygon P is traversed in the clockwise order from V_0.

(2) No three consecutive vertices are colinear.

Definition 2.1: $L(P_i,P_j)$ denotes a directed line segment joining two points P_i and P_j in the direction from P_i to P_j.

Definition 2.2: An edge $E(V_i,V_{i+1})$ of P is a directed line segment $L(V_i,V_{i+1})$ joining two adjacent vertices V_i and V_{i+1} on B(P). A chain $C(V_i,V_j)$ is a sequence of edges $E(V_i,V_{i+1})$, $E(V_{i+1},V_{i+2}),\ldots,$ $E(V_{j-1},V_j)$ on B(P) in the clockwise order.

Definition 2.3: A vertex V_i of P is extreme if V_i cannot be expressed as a convex combination of other vertices in P, i.e., V_i is extreme if and only if

$$V_i \neq \sum_{j\neq i} \alpha_j V_j, \quad \sum_{j\neq i} \alpha_j = 1, \quad \text{and} \quad \alpha_j \geq 0.$$

Definition 2.4: The convex hull CH(P) of P is the smallest convex polygon containing P.

Definition 2.4 necessarily implies that every vertex of CH(P) is an extreme vertex of P. Hence, one way to find CH(P) is to discard all non-extreme vertices. To characterize a non-extreme vertex, we employ the notion of a pocket.

Definition 2.5: A pocket PKT(V_i,V_j) is one or more regions bounded by L(V_i,V_j) and C(V_i,V_j) such that all points in C(V_i,V_j) are on or to the right of L(V_i,V_j).

We state an interesting property of a pocket due to Graham and Yao (6).

Lemma 2.1: Let V_r be in a PKT(V_i,V_j). If V_r is neither V_i nor V_j, then V_r is not an extreme vertex of P.

3. Property of Zipper

A pocket PKT(V_i,V_j) is said to be maximal with respect to C(V_0,V_q) if C(V_i,V_j) is not contained in another pocket PKT(V_k,V_m), where $o \leq i < j \leq q$, and $o \leq k < m \leq q$. Let an ordered list (Z_0,Z_1,Z_2,...,Z_j) be the sequence of all vertices in C(V_0,V_q) such that PKT(Z_i,Z_{i+1}), $o \leq i < j$, is maximal with respect to C(V_0,V_q). The sequence of line segments (L(Z_0,Z_1),L(Z_1,Z_2),..., L(Z_{j-1},Z_j)) is said to be a zipper ZPR(V_0,V_q) as illustrated in Figure 3.1.

‹Insert Figure 3.1›

In this section, we show that ZPR(V_0,V_q) is concave and non-self-intersecting. Our first lemma forms the basis for showing this property. In its proof and in all subsequent discussions, we use the following notations.

V_{q+1}:	=	the most recently visited vertex in P.
V_q:	=	the previous (counter-clockwise) vertex of V_{q+1} in P.
Z_j:	=	the vertex that is most recently added into ZPR(V_0,V_q).
Z_{j-1}:	=	the previous vertex of Z_j in ZPR(V_0,V_q).
V_*:	=	the previous vertex of Z_j in P.

Lemma 3.1: Let ZPR(V_0,V_q) = (L(Z_0,Z_1), L(Z_1,Z_2,),..., L(Z_{j-1},Z_j)) and $o < q < n$. Any vertex V_k in the chain C(Z_r,V_q) must be to the right of L(Z_i,Z_{i+1}), $o \leq i < r \leq j$, if $V_k \neq Z_{i+1}$.

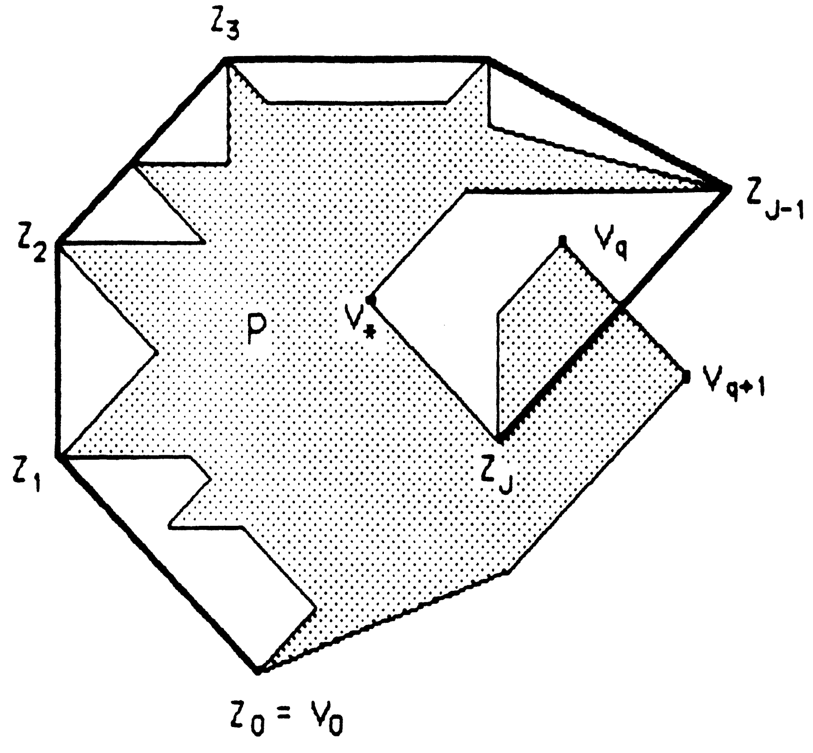


Figure 3.1 : Vertices of a polygon P and $ZPR(V_0, V_q)$

[Proof] The proof will be by the induction on the subscript i of a zipper vertex Z_i in $ZPR(V_0,V_q)$. Let Z_{-1} be a point on the horizontal line containing Z_0 such that Z_{-1} lies to the right of Z_0. Let L_i be the line containing $L(Z_{i-1},Z_i)$, $i=0,1,2,\ldots,j$. L_i partitions the plane into two half planes. Let LHP_i be the half plane to the left of $L(Z_{i-1},Z_i)$ and RHP_i be the other.

i = 0: Since V_0 is extreme, V_0 coincides with Z_0. By the way in which V_0 is chosen, the Y coordinate of V_0 is not greater than the Y coordinate of any other vertex in P. Therefore, $C(V_0,V_q)$ cannot pass through LHP_0. Now, RHP_0 is partitioned by L_1 into two regions, $RHP_0 \cap LHP_1$ and $RHP_0 \cap RHP_1$. We need to show that $C(Z_1,V_q)$ cannot be in $RHP_0 \cap LHP_1$. Suppose that some vertices in $C(Z_1,V_q)$ are in $RHP_0 \cap LHP_1$. Let W be the vertex in $C(Z_1,V_q)$ such that $C(Z_0,W)$ is to the right of $L(Z_0,W)$. Clearly, $PKT(Z_0,W)$ contains $C(Z_0,Z_1)$, which contradicts the maximality of $PKT(Z_0,Z_1)$. Suppose that the lemma is true for $i = m-1 < j-2$.

i=m: We need to show that $C(Z_{i+1},V_q)$ cannot be in $R = [\bigcap_{p=0}^{m} RHP_p] \cap LHP_{m+1}$ as shown in Figure 3.2

‹Insert Figure 3.2›

Suppose that some vertices in $C(Z_{m+1},V_q)$ are in R. Let W be the vertex in $C(Z_{m+1},V_q)$ such that $C(Z_m,W)$ is to the right of $L(Z_m,W)$. $PKT(Z_m,W)$ contains $C(Z_m,Z_{m+1})$, which contradicts the maximality of $PKT(Z_m,Z_{m+1})$.

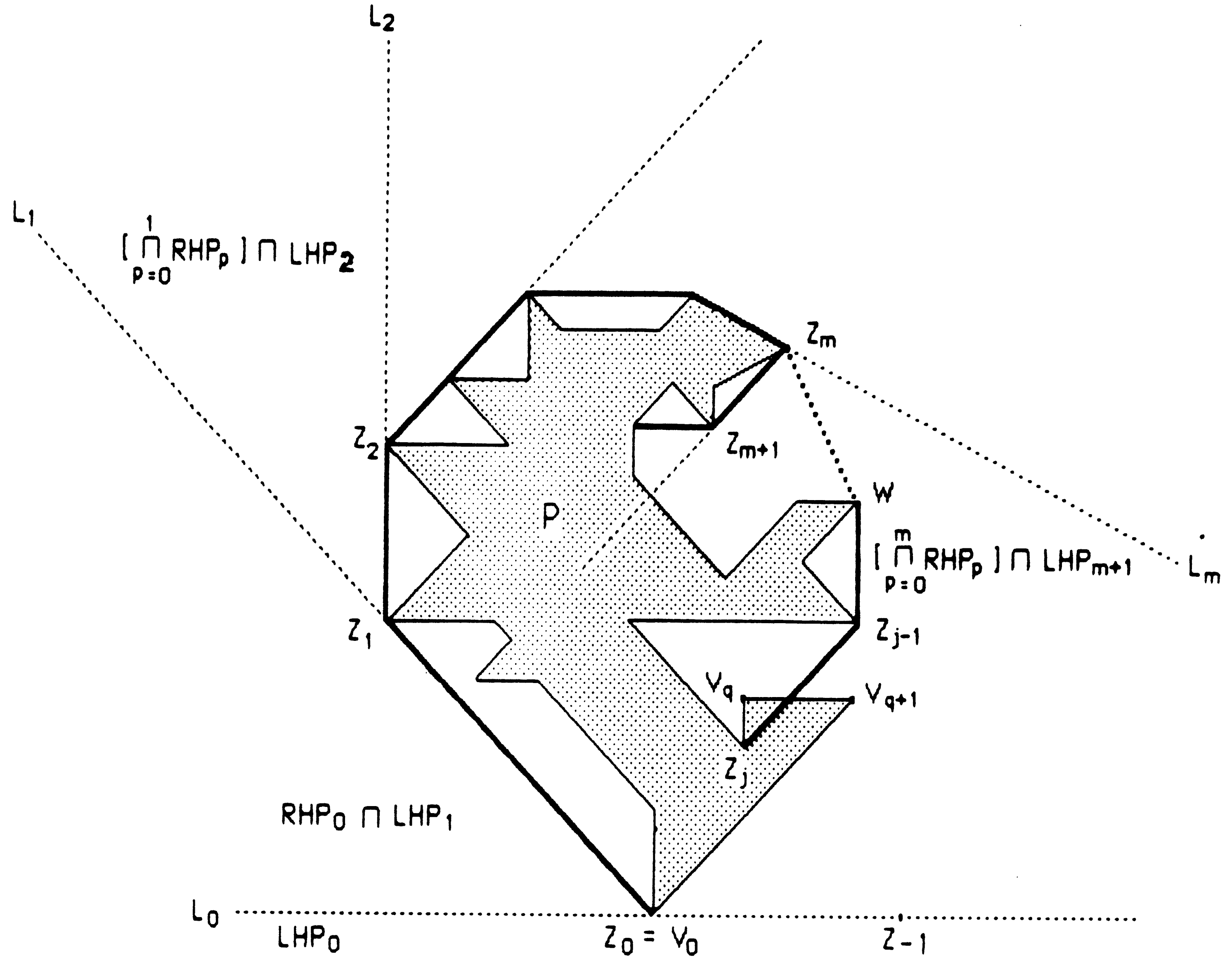


Figure 3.2 : $C(V_{m+1}, V_q)$ cannot be in $[\bigcap_{p=0}^{m} RHP_p] \cap LHP_{m+1}$.

As illustrated in Figure 3.3, the property described in Lemma 3.1 does not necessarily hold true unless V_0 is an extreme vertex of P. We next state the lemmas characterizing a ZPR(V_0,V_q), the proofs of which are direct consequences of Lemma 3.1.

‹Insert Figure 3.3›

Lemma 3.2: Let ZPR(V_0,V_q) = (L(Z_0,Z_1), L(Z_1,Z_2),..., L(Z_{j-1},Z_j)). The internal angle ANGLE (Z_i,Z_{i+1},Z_{i+2}) between two consecutive line segments L(Z_i,Z_{i+1}) and L(Z_{i+1},Z_{i+2}), $o \leq i \leq j-2$, is strictly between zero and 180 degrees.

Lemma 3.3: A ZPR(V_0,V_q) is not self-intersecting.

Finally, we show that a zipper vertex Z_k cannot be in a pocket PKT(Z_i,Z_{i+1}) if $k \neq i$ and $k \neq i+1$. We use this property to update ZPR(V_0,V_q).

Lemma 3.4: Let ZPR(V_0,V_q) be (L(Z_0,Z_1),L(Z_1,Z_2),..., L(Z_{j-1},Z_j)).

$$\text{Then, } Z_k \cap \text{PKT}(Z_i,Z_{i+1}) = \begin{cases} Z_k & \text{if } k=i \text{ or } i+1 \\ \emptyset & \text{otherwise} \end{cases}$$

for all $o \leq i < j$ and $o \leq k \leq j$.

[Proof] Suppose that $Z_k \cap \text{PKT}(Z_i,Z_{i+1}) \neq \emptyset$ for some $k \neq i$ and $k \neq i+1$. Then either P is not simple or V_0 is not an extreme point.

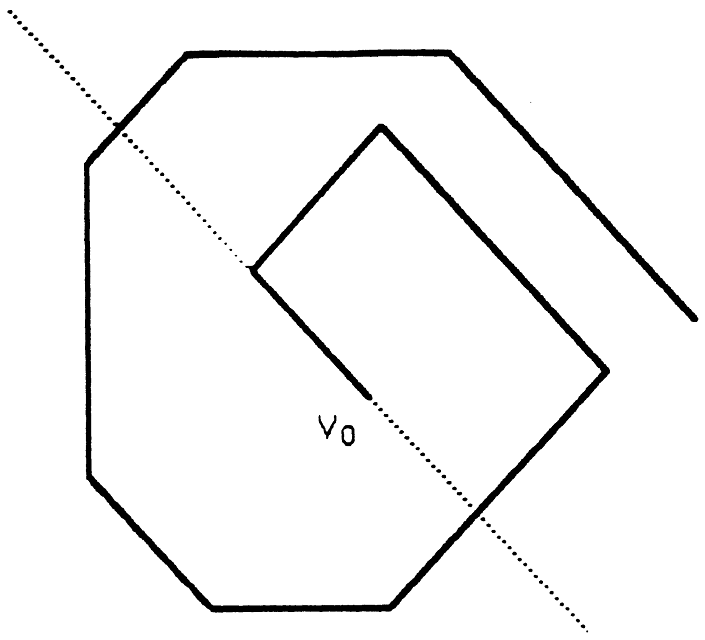


Figure 3.3 : Lemma 3.1 does not hold true if V_0 is not extreme.

4. Updating of Zipper

Consider the relationship between two line segments $L(Z_{j-1},Z_j)$ and $E(V_*,Z_j)$ As illustrated in Figure 4.1, the vertex V_{q+1} can be in any one of the four quadrants formed by the extensions of these two line segments. The quadrants are:

Q1a: to the right of $L(Z_{j-1},Z_j)$ and to the right of $E(V_*,Z_j)$

Q1b: to the right of $L(Z_{j-1},Z_j)$ and to the left of $E(V_*,Z_j)$

Q2a: to the left of $L(Z_{j-1},Z_j)$ and to the right of $E(V_*,Z_j)$

Q2b: to the left of $L(Z_{j-1},Z_j)$ and to the left of $E(V_*,Z_j)$

‹Insert Figure 4.1›

If V_{q+1} is in Q1b, it is also in $PKT(Z_{j-1},Z_j)$. By Lemma 2.1, V_{q+1} and its clockwise vertices in $PKT(Z_{j-1},Z_j)$ can be deleted. Otherwise, we need to show if the existing zipper vertices are to be deleted or kept to advance to V_{q+1}. The following three lemmas as illustrated in Figure 4.2 are useful for the updating of $ZPR(V_0,V_q)$.

‹Insert Figure 4.2›

Lemma 4.1: Let $ZPR(V_0,V_q) = (L(Z_0,Z_1), L(Z_1,Z_2),\ldots., L(Z_{j-1},Z_j))$ and $V_q = Z_j \neq V_0$. All pockets $PKT(Z_i,Z_{i+1})$, $0 \leq i < j$, are maximal with respect to $C(V_0,V_{q+1})$, if V_{q+1} is in Q1a.

[Proof] If $V_{q+1} = V_0$, then $ZPR(V_0,V_q)$ together with $E(V_q,V_{q+1})$ forms a convex polygon since $V_{q+1} = V_0$ and $ZPR(V_0,V_q)$ is concave and non-self-

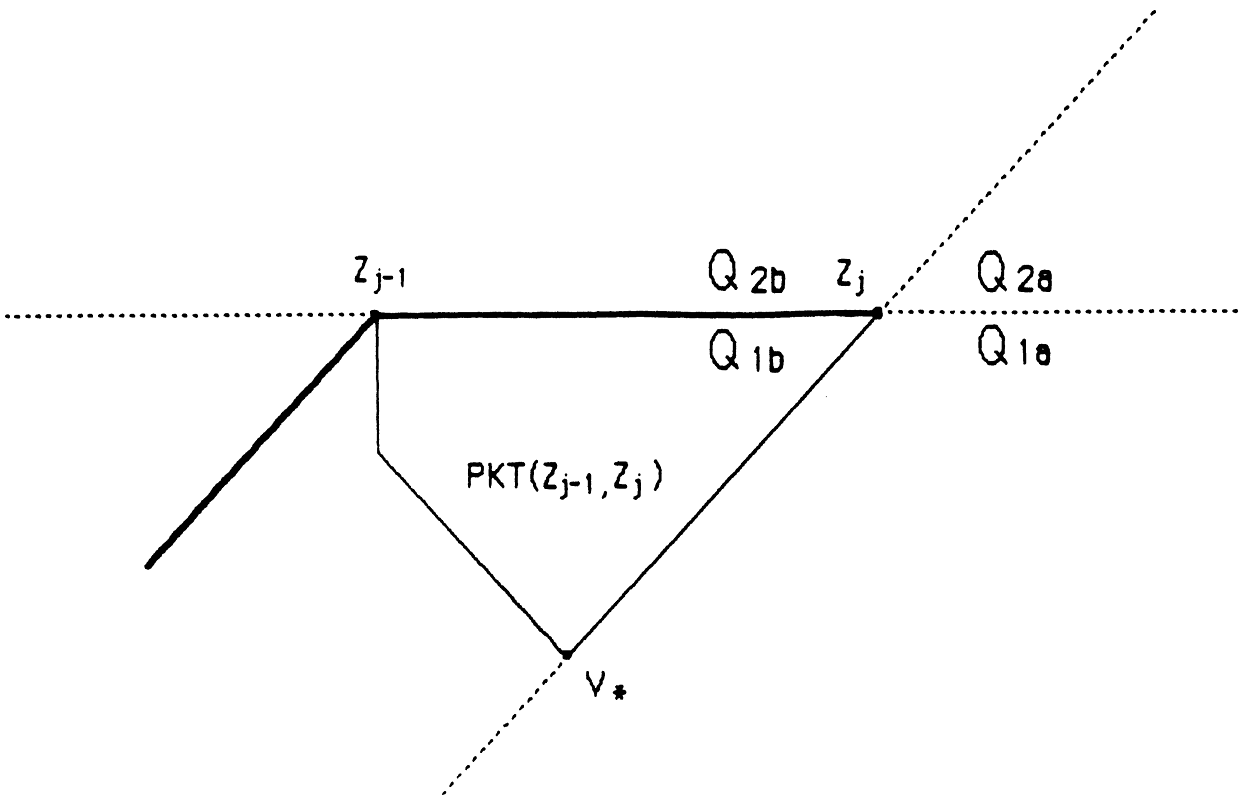
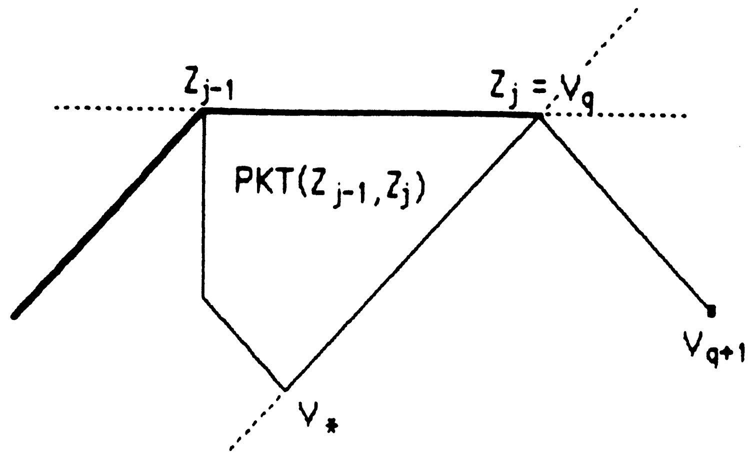
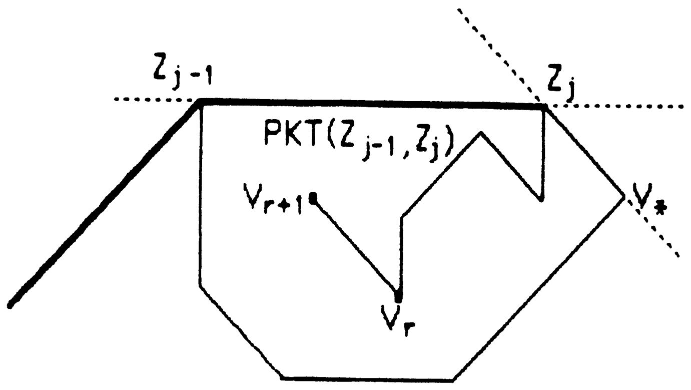


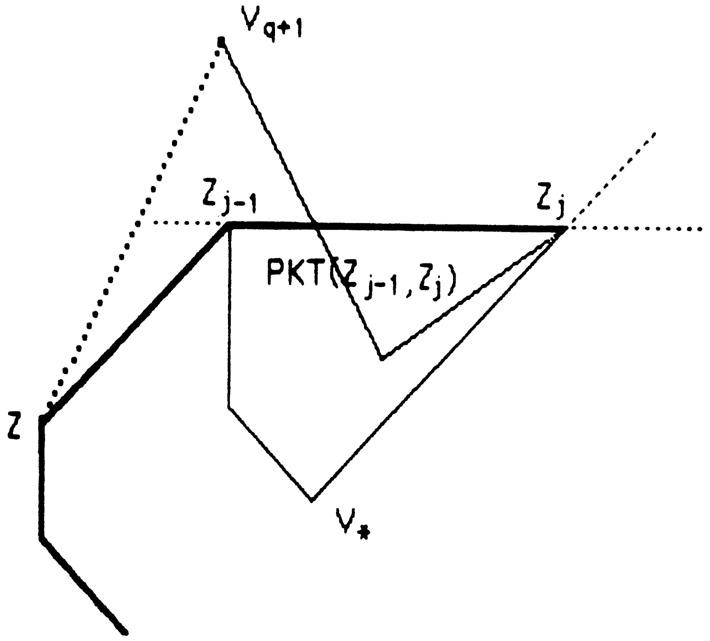
Figure 4.1 : Possible locations of vertex V_{q+1}



(a) Illustration of Lemma 4.1



(b) Illustration of Lemma 4.2



(c) Illustration of Lemma 4.3

Figure 4.2. Updating Zipper Vertices

intersecting. Therefore, the result follows immediately.

Let us consider the case for $V_{q+1} \neq V_0$. Since ZPR(V_0,V_q) implies that PKT(Z_i,Z_{i+1}), $o \leq i < j$, is maximal with respect to C(V_0,V_q), all we need to show is that E(Z_j,V_{q+1}) is PKT(Z_j,V_{q+1}) and is maximal with respect to C(V_0,V_{q+1}). First we show E(Z_j,V_{q+1}) $\cap$ PKT(Z_i,Z_{i+1}) $\neq$ E(Z_j,V_{q+1}) for any $o \leq i < j$. By Definition 2.5, V_{q+1} cannot be in PKT(Z_{j-1},Z_j) since V_{q+1} is in Q1a. From Lemma 3.4, Z_j cannot be in PKT(Z_i,Z_{i+1}) for any $o \leq i < j-1$. Therefore, E(Z_j,V_{q+1}) $\cap$ PKT(Z_i,Z_{i+1}) $\neq$ E(Z_j,V_{q+1}) for any $o \leq i < j$. Finally, there does not exist a vertex V_r in C(V_0,V_*) such that C(V_r,V_{q+1}) and L(V_r,V_{q+1}) form a pocket PKT(V_r,V_{q+1}) since ZPR(V_0,V_q) is concave and P is simple. Hence, the result follows.

<u>Lemma 4.2</u>: Let ZPR(V_0,V_q) = (L(Z_0,Z_1), L(Z_1,Z_2),..., L(Z_{j-1},Z_j)). If C(V_q,V_r), $r>q$, is in PKT(Z_{j-1},Z_j), then V_{r+1} is also in PKT(Z_{j-1},Z_j) unless V_{r+1} is to the left of L(Z_{j-1},Z_j).

[Proof] Since P is simple, C(V_q,V_{r+1}) can get out of PKT(Z_{j-1},Z_j) only through L(Z_{j-1},Z_j).

<u>Lemma 4.3</u>: Let ZPR(V_0,V_q) = (L(Z_0,Z_1), L(Z_1,Z_2),..., L(Z_{j-1},Z_j)). Then PKT(Z_{j-1},Z_j) is <u>not</u> maximal with respect to C(V_0,V_{q+1}), if V_{q+1} is in quadrant Q2a or Q2b.

[Proof] ANGLE(Z_{j-1},Z_j,V_{q+1}) is greater than or equal to 180 degrees since V_{q+1} is in Q2a or Q2b. Since ZPR(V_0,V_q) is concave and non-self-intersecting, there must exist a vertex Z in ZPR(V_0,V_q) such that L(Z,V_{q+1}) and C(Z,V_{q+1}) form a pocket PKT(Z,V_{q+1}). Clearly, PKT(Z,V_{q+1}) contains C(Z_{j-1},Z_j).

5. The Algorithm and Its Analysis

Our linear algorithm for finding the convex hull of a simple polygon P takes V_i, i=o,1,...n-1, as input and constructs a ZPR(V_0,V_q) with vertices Z_j.

Algorithm 5.1

```
Step 0.    $Z_0$ <-- $V_0$, $Z_1$ <-- $V_1$, j <-- 1, q <-- 1.
           while ($V_q \neq V_0$) do;
Step 1.      if $V_{q+1}$ is to the right of $L(Z_{j-1},Z_j)$, then do;
Step 1a.       if $V_{q+1}$ is to the right of $E(V_*,Z_j)$
                 then j <-- j+1, $Z_j$ <-- $V_{q+1}$, q <-- q+1.
Step 1b.       else while ($V_{q+1}$ is on or to the right of $L(Z_{j-1},Z_j)$) do;
                 q <-- q+1
               end
             end
Step 2.      else do;
               while ($Z_j \neq V_0$ and $Z_{j-1}$ is not to the right of $L(Z_j,V_{q+1})$) do;
                 j <-- j-1
               end.
               j <-- j+1, $Z_j$ <-- $V_{q+1}$, q <-- q+1.
             end
           end
Step 3.    Stop.
```

We show the correctness of Algorithm 5.1 with the following lemma.

Lemma 5.1: Algorithm 5.1 constructs ZPR(V_0,V_q) correctly.

[Proof] The proof will be by induction on the number of times Step 1 is reached. Initially, the statement is trivially satisfied by Step 0 of the algorithm. Suppose that the lemma is true when Step 1 is executed m times. Then, there are three cases:

(1) Case 1a: V_{q+1} is in Q1a

(2) Case 1b: V_{q+1} is in Q1b

(3) Case 2 : V_{q+1} is in Q2a or Q2b

Case 1a: V_{q+1} qualifies as a zipper vertex if PKT(Z_j,V_{q+1}) is maximal with respect to C(V_0,V_{q+1}). Since V_{q+1} is in quadrant Q1a, by Lemma 4.1, PKT(Z_j,V_{q+1}) is maximal. Indeed, Step 1a takes V_{q+1} as the new Z_j. Since the correct vertex is added to the zipper the next time Step 1 is reached, the induction holds. Now, Lemma 4.1 requires the precondition that V_q equals Z_j. This precondition is satisfied iteratively after executing Step 1a or Step 2. After executing Step 1b, though $V_q \neq Z_j$, the control must go to Step 2 because V_{q+1} cannot be to the right of L(Z_{j-1},Z_j). Hence, the precondition for Lemma 4.1 is always satisfied.

Case 1b. Because V_{q+1} is in quadrant Q1b, by Definition 2.5, V_{q+1} is in PKT(Z_{j-1},Z_j). Therefore, V_{q+1} should not be a zipper vertex. Furthermore, by Lemma 4.2, all the subsequent vertices in PKT(Z_{j-1},Z_j) should not be in the zipper ZPR(V_0,V_q) either. This is precisely what Step 1b does. Since no zipper vertex is added, the next time Step 1 is reached, ZPR(V_0,V_q) is still correct.

Case 2: Step 2 deletes Z_j since PKT(Z_{j-1},Z_j) is not maximal with respect to C(V_0,V_{q+1}) by Lemma 4.3. The old Z_{j-1} becomes the new Z_j. This process is repeated until either $Z_j = Z_0$ or Z_{j-1} is to the right of L(Z_j,V_{q+1}). At that point PKT(Z_j,V_{q+1}) is maximal with respect to C(V_0,V_{q+1}), because ZPR(V_0,V_{q+1}) is concave and non-self-intersecting. Hence, the lemma is true.

When V_{q+1} coincides with V_0, Step 3 terminates the algorithm, and the lemma is still true by the induction hypothesis.

Since ZPR(V_0,V_q) is concave and non-self-intersecting, it must form a convex polygon P_c containing P if $V_q = V_0$. Since every vertex of P_c is a vertex of P, it is clear that P_c is the smallest convex polygon containing P. By Definition 2.4, P_c must be the convex hull of a simple polygon P.

Theorem 5.1: Algorithm 5.1 finds the convex hull of a simple polygon P with n vertices in O(n) time.

[Proof] The algorithm moves forward, except in Step 2, until V_0 is revisited. Step 2 is executed at most a total of n-3 times.

6. Concluding Remarks

Algorithm 5.1 removes the vertices that cause self-intersection (8) in CH-POL (9). It is shorter than the version by Graham and Yao (6) when both the Left Hull and the Right Hull are taken into account.

Acknowledgement

The authors wish to thank J. D. Wolter and H. C. Lee for their critical reading of the manuscript and their constructive suggestions. J. D. Wolter implemented Algorithm 5.1 in several languages. His version in PASCAL is supplied in the Appendix.

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Appendix

```
program main (input,output);

var     X,Y:      array [0..50] of real;          {coordinates of points}
        V,Z:      array [0..50] of integer;       {polygon and hull}
        q,j:      integer;                        {index into polygon and hull}
        n:        integer;                        {number of vertices}

        i:        integer:                        {loop index}

{ Is point p to the left of Line (a,b)?  }
function left (p,a,b:  integer) :boolean;
begin
        left  := (Y[p] - Y[a])*(X[b] - X[a]) > (X[p] - X[a])*(Y[b] - Y[a]);
end;

{ Is point p to the right of Line (a,b)? }
function right (p,a,b: integer) :boolean;
begin

        right := (Y[p] - Y[a])*(X[b] - X[a]) < (X[p] - X[a])*(Y[b] - Y[a]);
end;

{ Read in the Polygon }
procedure readin;
var     i:        integer;
        W:        array [0..50] of integer;
        mx,my:    real;
        mi:       integer;
begin
        { Read in the number of points }
        repeat
                  write(' Number of points? ');
                  read(n);
        until (n > 3) and (n < 50);

        mx := le38;
        my := le38;

        { While reading in vertices, find an extremal one }
        for i  := 0 to n-1
            do begin
                write(' ',i:3,': ');
                read(X[i],Y[i]);
                W[i]  := i;
                if  (Y[i] < my) or ((Y[i] = my) and (X[i] < mx))
                    then begin
                        mx := X[i];
                        my := Y[i];
                        mi := i;
                    end;
             end;
        { Reorder with an extreme vertex first }
        V[n]:=W[mi];
        for i := 0 to n-1
            do begin
                V[i]  := W[mi];
                mi := (mi + 1) mod n;
            end;
end;

begin
        { Get the polygon, and echo it back }
        readin;
        writeln(' Polygon:');
        for i := 0 to n-1 do
                writeln(' ',1:3,': ',X[V[i]]:10:5,', ',Y[V[i]]:10;5);

        {Step 0}
        q := 1;
        j := 1;
        Z[0] := V[0];
        Z[1] := V[1];

        while (q < n) do
                if right ( V[q+1], Z[j-1], Z[j] )
                    then
                        if right ( V[q+1], V[q-1], V[q] )
                            then begin
                                { Step 1a }
                                j := j + 1;
                                q := q + 1;
                                Z[j] := V[q];
                                end
                            else
                                { Step 1b  }
                                while not left ( V[q+1], Z[j-1], Z[j] ) do
                                        q := q + 1;
                    else begin
                        { Step 2 }
                        while j > 0 and not right ( Z[j-1], Z[j], V[q+1] ) do
                                j := j - 1;
                        j := j + 1;
                        q := q + 1;
                        Z[j] := V[q];
                        end;

        { Print the hull }
        writeln(' Hull:');
        for i := 0 to j-1 do
                writeln(' ',i:3,': ',X[Z[i]]:10:5,',',Y[Z[i]]:10:5);

        end.
```